

PRIVATE
P.R. [Depository Soc.]

SHL
WITHDRAWN





Digitized by the Internet Archive
in 2026

PROCEEDINGS

OF

THE LONDON MATHEMATICAL SOCIETY

SECOND SERIES

VOLUME 5

LONDON :

PRINTED BY C. F. HODGSON & SON,

2 NEWTON STREET, KINGSWAY, W.C. ;

AND PUBLISHED FOR THE SOCIETY BY

FRANCIS HODGSON, 89 FARRINGDON STREET, E.C.

1907

73/42

LONDON :
PRINTED BY C. F. HODGSON & SON,
NEWTON STREET, KINGSWAY, W.C.



C O N T E N T S.

Records of Proceedings at Meetings, pp. iv-ix.

Library [Presents to Library (p. x), Exchanges (p. xi), International Catalogue (p. xii)], pp. x-xii.

Obituary Notices of A. Mannheim (p. xiii) and E. J. Routh (p. xiv), pp. xiii-xx.

Notes and Corrections, p. xxi.

Papers published in the *Proceedings*, pp. 1-490.

Index of Proceedings at Meetings, p. 491.

Index of Papers read and Communications made during the Session 1906-1907, p. 492.

Index of Papers published in this Volume arranged according to the names of Authors, p. 493.

Index according to Subjects, p. 494.

RECORDS OF PROCEEDINGS AT MEETINGS

SESSION NOVEMBER, 1906—JUNE, 1907.

Thursday, November 8th, 1906.

ANNUAL GENERAL MEETING.

Prof. A. R. FORSYTH, President, in the Chair.

Present nineteen members.

Mr. W. F. S. Churchill and, later, Dr. T. Stuart were admitted into the Society.

The Report of the Treasurer (Prof. J. Larmor) was presented by Prof. A. E. H. Love. On the motion of Lieut.-Col. A. Cunningham, seconded by Rev. E. W. Barnes, the Report was received.

Dr. J. G. Leathem was appointed Auditor.

Mr. Grace, as Secretary, reported that the number of members at the beginning of the last Session was 276. During the Session 8 members had been elected, 2 had died, 2 had resigned and the names of 2 had been removed from the List. The number of members at the beginning of the Session was 278. The Accademia di Scienze di Torino had been added to the list of Societies with which the Society exchanges publications.

The Council and Officers for the ensuing Session were elected as follows:—President, Prof. W. Burnside; Vice-Presidents, Sir W. Niven and Prof. A. R. Forsyth; Treasurer, Prof. J. Larmor; Secretaries, Prof. A. E. H. Love and Mr. J. H. Grace; other members of the Council, Dr. H. F. Baker, Mr. A. Berry, Mr. A. L. Dixon, Prof. E. B. Elliott, Dr. J. W. L. Glaisher, Mr. G. H. Hardy, Dr. E. W. Hobson, Prof. H. M. Macdonald, Mr. A. E. Western, Mr. A. Young.

The newly elected President having taken the Chair, Prof. Forsyth read his *Presidential Address* entitled: **“Partial Differential Equations: Some Criticisms and some Suggestions.”*

The following papers were communicated :—

*Harmonic Expansions of Functions of Two Variables : Prof. A. C. Dixon.

The General Solution of Laplace's Equation in n Dimensions : Mr. G. N. Watson.

*On Sub-groups of a Finite Abelian Group : Mr. H. Hilton.

*On Bäcklund's Transformation and the Partial Differential Equation $s = F(x, y, z)$: Mr. J. E. Campbell.

†On the Inversion of a Definite Integral : Mr. H. Bateman.

*On Partial Differential Coefficients and on Repeated Limits in General : Dr. E. W. Hobson.

Thursday, December 13th, 1906.

Prof. W. BURNSIDE, President, in the Chair.

Present twelve members.

Messrs. B. Dé and W. H. Macaulay were elected members.

Prof. Love presented the Report of the Auditor (Dr. J. G. Leathem). On the motion of the President, seconded by Lt.-Col. Cunningham, the Treasurer's Report was adopted, and the thanks of the Society were given to the Treasurer and Auditor.

The following papers were communicated :—

*The Diophantine Equation $x^n - Ny^n = z$: Major P. A. Macmahon.
On the Form of the Surface of a Searchlight Reflector : Mr. C. S. Jackson.

The Potential Equation and others with Function given on the Boundary : Mr. L. F. Richardson (communicated by Mr. A. Berry).

*On the Limits of Real Variants : Mr. J. Mercer (communicated by Dr. E. W. Barnes).

*The Asymptotic Expansion of Integral Functions defined by Generalised Hypergeometric Series : Dr. E. W. Barnes.

* Printed in this volume.

† Printed in *Proceedings*, Ser. 2, Vol. 4.

Thursday, January 10th, 1907.

Prof. W. BURNSIDE, President, in the Chair.

Present fourteen members and four visitors.

Mr. W. P. Milne and Mr. G. S. Le Beau were elected members.

Mrs. A. Stott was admitted into the Society.

Mrs. Stott exhibited a collection of models of three-dimensional sections of the four-dimensional regular hypersolids.

The following papers were communicated :—

*The Uniform Convergence of Fourier's Series : Dr. E. W. Hobson.

*On Hyper-even Numbers and on Fermat's Numbers : Lt.-Col. A. Cunningham.

†A new development of the Theory of Hypergeometric Functions :
Rev. Dr. E. W. Barnes.

*On Partial Differential Equations of the Second Order having
Integral Systems free from Partial Quadratures : Prof. A. R.
Forsyth.

*The Singular Points of certain Classes of Functions of Several
Variables : Mr. G. H. Hardy.

*The Singularities of Functions defined by Taylor's Series : Mr.
G. H. Hardy.

*Asymptotic Approximation to Integral Functions of Zero Order :
Mr. J. E. Littlewood.

*On the Reducibility of Covariants of Binary Quantities of Infinite
Order (Part III.) : Mr. P. W. Wood.

The Forms of the Stream Lines due to the Motion of an Ellipsoid
in Infinite Fluid, Frictionless or Viscous : Dr. T. Stuart.

Thursday, February 14th, 1907.

Sir W. D. NIVEN, Vice-President, in the Chair.

Present thirteen members.

Prof. A. R. Forsyth gave an account (from details kindly supplied by Prof. G. Darboux) of the life and scientific work of the late Colonel A. Mannheim, who was an Honorary Foreign Member of the Society.

* Printed in this volume.

† To be printed in *Proceedings*, Ser. 2, Vol. 6.

The following papers were communicated :—

*On Repeated Integrals : Dr. E. W. Hobson.

*On the Projective Geometry of a Binary Quartic and its Hessian :
Prof. E. B. Elliott.

*On a Formula for the Sum of a Finite Number of Terms of the
Hypergeometric Series : Prof. M. J. M. Hill.

On Groups defined by the Order of Two Generators and the Order
of their Commutator : Prof. G. A. Miller.

Lt.-Col. A. Cunningham made an informal communication :

On Hyper-exponential Numbers.

Thursday, March 14th, 1907.

Sir W. D. NIVEN, Vice-President, in the Chair.

Present thirteen members and a visitor.

Messrs. F. J. W. Whipple, A. Ackermann Teubner, M. T. N. Lyengar
were elected members.

Mr. G. W. Evans-Cross exhibited his “myriometer” calculating
apparatus.

The following papers were communicated :—

Reduction of the Factorisation of Binary Septans and Octans to
the Solution of an Indeterminate Equation of the Second
Degree : Dr. T. Stuart.

*Invariants of the General Quadratic Form Modulo 2 : Prof. L. E.
Dickson.

On Linear Partial Differential Equations of the First Order : Mr.
J. Brill.

Prof. A. E. H. Love made an informal communication :

On the Representation of Functions by means of Series of certain
special Forms.

Thursday, April 11th, 1907.

Sir. W. D. NIVEN, Vice-President, in the Chair.

Present fourteen members.

Mr. W. M. Page was elected a member.

Mr. G. S. Le Beau was admitted into the Society.

The following papers were communicated :—

An Introduction to the Metrical Geometry of Space of n Dimensions :

Mr. H. Bateman.

Note on Perott's Theorem : Mr. H. Hilton.

Informal communications were made as follows :—

On Poisson's Integral and its relation to the proof of Fourier's Theorem : Dr. E. W. Hobson.

On the Value of the Parameters for which a Definite Integral can be Zero : Mr. H. Bateman.

Thursday, May 9th, 1907.

Prof. W. BURNSIDE, President, in the Chair.

Present seventeen members.

Dr. W. H. Young and Mr. F. J. W. Whipple were admitted into the Society.

Communications were made as follows :—

On Secular Stability : Prof. H. Lamb.

A Lemma connected with Fourier's Series : Mr. F. J. W. Whipple.

A Method for Quadratic Congruences : Dr. T. Stuart.

Thursday, June 13th, 1907.

Prof. W. BURNSIDE, President, in the Chair.

Present eighteen members and two visitors.

Mr. J. E. Littlewood and Dr. H. A. P. de Sadow Pittand were elected members.

Dr. Sadow Pittand was admitted into the Society.

The President referred to the loss sustained by the Society through the death of Dr. E. J. Routh, and the following resolution was passed :—

That this meeting of the London Mathematical Society hereby records its sense of the loss sustained by mathematical science through the death of Dr. E. J. Routh, who was a member of the Society from the time of its foundation, and whose writings and teaching contributed so greatly to the advance of mathematics in England ; and that the Secretaries be requested to convey to Mrs. Routh and the family the expression of the respectful and sympathetic condolence of the members.

The following mathematicians were elected Honorary members of the Society :—Prof. Guido Castelnuovo of Rome, George William Hill of New York, Camille Jordan of Paris, Vito Volterra of Rome.

The following papers were communicated :—

*On the Number of Representations of a Number as a Sum of $2r$ Squares, where $2r$ does not exceed 18 : Dr. J. W. L. Glaisher.

†An Extension of Eisenstein's Law of Reciprocity : Mr. A. E. Western.

†Note on a special Set of Classes of Partial Differential Equations of the Second Order : Prof. A. R. Forsyth.

†Various Extensions of Abel's Lemma : Prof. T. J. I'A. Bromwich.
The Arithmetical Nature of the Coefficients of Linear Substitutions (Third Paper) : Prof. W. Burnside.

The Invariants of the Quintic : Dr. H. F. Baker.

Informal communications were made as follows :—

On Certain Singular Points of Surfaces : Mr. A. B. Basset.

The Minimum necessary Postulates as to a Function to be defined as Analytic over a Region : Prof. E. B. Elliott.

* Printed in this volume.

† To be printed in *Proceedings*, Ser. 2, Vol. 6.

LIBRARY

Presents.

BETWEEN December, 1906, and October, 1907, the following presents were made to the Library:—

From the American Philosophical Society:—

“The Franklin Bicentennial Celebration,” Philadelphia, 1906.

From the Society of Accountants in Edinburgh:—

“A History of Accounting and Accountants,” Edinburgh, 1905.

From the respective Authors or Publishers:—

Alasia, Prof. C.—“Rassegna Matematica,” Paria, 1907.

Bachelier, Louis.—“Théorie des Probabilités Continues.”

Bashforth, Francis.—“Ballistic Experiments from 1864 to 1880,” Camb., 1907.

Dini, Prof. W.—“Lezioni di Analisi Infinitesimale,” Vol. 1, Parts 1 and 2; Pisa, 1907.

Fukuzawa, S.—“Vier Mathematische Abhandlungen,” Tokyo, 1907.

Glen, Oliver E.—“Determination of the Abstract Groups of the Order p^2qr .”

Hamburger Bibliothek.—“Katalog Math. und Physik,” Hamburg, 1906.

Huygen, F. C.—“Over de Exhaust-werking bij Locomotieven.”

Liapounoff, A.—“Sur les Figures d'Equilibre,” Pt. 1; St. Petersburg, 1906.

Mal, Devan Gandra.—“Trisection of an Angle,” 3rd edition.

Martin, Artemas.—“Table of Right-Angled Triangles,” and “Mathematical Magazine,” vol. 2, no. 12.

Ragsdale, Virginia.—“On the Arrangements of the Real Branches of Plane Algebraic Curves,” Baltimore, 1906.

Stormer, Carl.—“On the Trajectories of Electric Corpuscles.”

Walker, B. M.—“On the Resolution of Higher Singularities of Algebraic Curves.”

Coimbra: Academia Polyt. de Porto, Ann. Scientificos, vol. 1, no. 4, 1906, and vol. 2, nos. 1-3, 1907.

Glasgow: Insurance and Actuarial Soc. Transactions, ser. 5, nos. 11-18, 1903-5.

Indian Engineering, vol. 40, nos. 16-26, 1906; vol. 41, 1907; vol. 42, nos. 1-15, 1907.

Kansas: Univ. Science Bulletin, vol. 4, nos. 1-6, 1907.

London: Educational Times, vol. 60, nos. 549-559, 1907.

London: Mathematical Gazette, vol. 4, nos. 61-65, 1907.

Nautical Almanac for 1910.

Ottawa: Literary and Scientific Society, Transactions, 1906-7, no. 4.

Paris: L'Enseignement Math., ann. 9, nos. 1-5, 1907.

Tokyo: Physico-Math. Society, Proceedings, vol. 3, nos. 6-11, 1906-7; vol. 4, nos. 1-6, 1907.

Varsovie: Soc. des Naturalistes, Comptes Rendus, 1898, ann. 9; 1901-2, ann. 12, 13; 1903-4, ann. 14, 15, 1906.

Warsaw: Wiadomości Matem., tom 10, zeszyt 4-6, 1906; tom 11, zeszyt 1-4, 1907.

Exchanges.

Between December, 1906, and October, 1907, the following exchanges were received :—

- American Journal of Mathematics, vol. 29, 1907.
 American Mathematical Society, Transactions, vol. 8, nos. 1-3, 1907.
 American Mathematical Society, Bulletin, vol. 13, nos. 3-10, 1906-7; vol. 14, no. 1, 1907.
 American Philosophical Society, Proceedings, vol. 45, no. 184, 1906; vol. 46, no. 185, 1907.
 Amsterdam: Nieuw Archief, deel 7, stuk 4; deel 8, stuk 1, 1907.
 Amsterdam: Revue Semestrielle, tome 15, pts. 1, 2, 1907.
 Belgique: Académie Royale des Sciences, Bulletin, 1906, nos. 9-12; 1907, nos. 1-8.
 Berlin: Jahrbuch über die Fortschritte, bd. 35, heft 3, 1907.
 Berlin: Journal für die Mathematik, bd. 132, 1907.
 Berlin: Sitzungsberichte der K. Preuss. Akademie, 1906, nos. 39-53; 1907, nos. 1-38.
 Bordeaux: Société des Sciences, Observations Pluviométriques, 1906.
 Bordeaux: Société des Sciences, Procès-Verbaux, 1906.
 Boston: American Academy of Arts and Sciences, Proceedings, vol. 43, nos. 1-6, 1907.
 Cambridge Philosophical Society, Proceedings, vol. 14, pts. 1, 2, 1907.
 Cambridge Philosophical Society, Transactions, vol. 20, pts. 11-13, 1907.
 Cambridge, Mass.: Annals of Mathematics, vol. 8, nos. 2-4; vol. 9, no. 1, 1907.
 Catania: Accademia Gioenia, Atti, Ser. 4, vol. 19, 1906.
 Catania: Accademia Gioenia, Bollettino, fasc. 92-94, 1907.
 Dublin: Royal Irish Academy, Proceedings, vol. 25, no. 3, 1905.
 Edinburgh: Mathematical Society, Proceedings, vol. 23, 1905.
 Edinburgh: Royal Society, Proceedings, vol. 25, nos. 7-12, 1905.
 Erlangen: Physik-medicin. Societät, Sitzungsberichte, bd. 38, 1906.
 France: Société Mathématique, Bulletin, tome 34, fasc. 4, 1906; tome 35, fasc. 1, 2, 1907.
 Göttingen: Königl. Gesell. der Wissenschaften, Nachrichten, Math. Klasse, 1906, hefte 3-5; 1907, hefte 1-5.
 Göttingen: Königl. Gesell. der Wissenschaften, Mittheilungen, 1907, heft 1.
 La Haye: Archives Néerlandaises, tome 12, liv. 1-4, 1907.
 Leipzig: Beiblätter zu den Annalen der Physik, bd. 30, heft 24, 1906; bd. 31, hefte 1-20, 1906-7.
 Leipzig: K. Sächsische Gesell., Math. Klasse, Berichte, 1906, nos. 6-8; 1907, no. 1.
 Livorno: Periodico di Matematica, anno 22, fasc. 3-6, 1906-7; anno 23, fasc. 1, 1907.
 Livorno: Periodico di Matematica, Supplemento, anno 10, fasc. 2-9, 1906-7.
 London: Royal Society, Proceedings, Series A, vol. 78, nos. 525-7, 1907; vol. 79, nos. 528-532, 1907.
 London: Royal Society, Proceedings, Series B, vol. 79, nos. 528-533, 1907.
 London: Royal Society, Transactions, Series A, vol. 206, 1906.
 London: Physical Society, Proceedings, vol. 20, pts. 3-5, 1906-7.
 London: Institution of Naval Architects, Transactions, 1905.
 London: Institute of Actuaries, Journal, vol. 39, pts. 2-4, 1905; vol. 40, pt. 1, 1906.
 London: Nat. Physical Laboratory, Collected Researches, vol. 2, 1907.
 London: Nature, vol. 75, nos. 1937-1956; vol. 76, nos. 1957-1983, 1907.
 Manchester Literary and Philosophical Society, Memoirs, vol. 49, pts. 2, 3, 1905; vol. 50, pt. 1, 1906.
 Milano: Reale Istituto Lombardo, Rendiconti, vol. 39, fasc. 17-20, 1906-7; vol. 40, fasc. 1-16, 1907.
 Milano: Reale Istituto Lombardo, Memorie, vol. 20, fasc. 9, 1906.
 Modena: Regia Accademia, Memorie, vol. 6, 1906.
 Napoli: Accademia delle Scienze, Rendiconti, vol. 12, fasc. 9-12, 1906; vol. 13, fasc. 1-7, 1907.

- Palermo : Rendiconti del Circolo Matematico, tomo 22, fasc. 3, 1906 ; tomo 23, 1907 ; tomo 24, fasc. 1, 2, 1907.
- Paris : Bulletin des Sciences Mathématiques, tome 30, Oct.-Dec., 1906 ; tome 31, Jan.-Aug., 1907.
- Paris : Journal de l'Ecole Polyt., cah. 11, 1907.
- Roma : Reale Accademia dei Lincei, Rendiconti, vol. 16, sem. 2, fasc. 10-12, 1906 ; vol. 17, sem. 1, and sem. 2, fasc. 1-6, 1907.
- Roma : Reale Accademia dei Lincei, Rendiconti delle Sedute Solenni, vol. 2, pp. 269-350, 1907.
- Stockholm : Acta Mathematica, bd. 30, pt. 4, 1906.
- Torino : R. Accademia delle Scienze, Atti, vols. 42, disp. 1-11, 1907.
- Torino : R. Accademia delle Scienze, Osservazione Meteorologiche, 1906.
- Toulouse : Faculté des Sciences, Annales, tome 8, fasc. 3, 4, 1906 ; tome 9, fasc. 1, 1907.
- Upsala : Nova Acta R. Societatis Scient., ser. 4, vol. 1, fasc. 1, 1906 ; fasc. 2, 1907.
- Warsaw : Prace Matematyczno-Fizyczne, tome 18, 1907.
- Washington : United States Naval Observatory, Synopsis of Report for year ending June, 1906.
- Wien : Monatshefte für Mathematik, jahr. 18, 1907.
- Zurich : Vierteljahrsschrift, hefte 2-4, 1906 ; heft 1, 2, 1907.

International Catalogue of Scientific Literature.

In the year April, 1906, to March, 1907 (inclusive), the following exchanges were sent in the first instance to Prof. Love to be indexed for the International Catalogue of Scientific Literature :—

- "Proceedings of the Edinburgh Mathematical Society," Vol. xxiv., 1906.
- "Transactions of the Institution of Naval Architects," London, 1906.
- "Journal of the Institute of Actuaries," Vol. xl., Pts. 1-4 ; London, 1906-7.
- "Proceedings of the Manchester Literary and Philosophical Society," Vol. l., Pts. 2, 3, and Vol. lx., Pt. 1, 1906-7.
- "Transactions of the Royal Irish Academy," Vol. xxxiii., Section A, Pt. 1 ; Dublin, 1906.

The following were also sent especially for the purposes of the Catalogue :—

- "Mathematical Gazette," Nos. 56-63 ; London, 1906-7.
- "Educational Times," Nos. 540-551 ; London, 1906-7.
- "Journal of the Royal Statistical Society," Vol. xlix., Pts. 1-4 ; London, 1906.
- "Proceedings of the Royal Irish Academy," Vol. xxvi., Section A, Nos. 1, 2 ; and Vol. xxvii., Section A, Nos. 1, 2 ; Dublin, 1906-7.

OBITUARY NOTICES

AMÉDÉE MANNHEIM.

[For this notice the Council is indebted to Prof. G. Darboux.]

Avec le Colonel Mannheim disparaît un des rares géomètres qui se bornent aujourd'hui encore à faire usage des méthodes synthétiques sans faire appel en aucune manière aux ressources de l'analyse moderne. La devise qu'il avait prise était celle de Poncelet dans le *Traité des Propriétés Projectives* : "En un mot, j'ai cherché avant tout la méthode de découvrir et de démontrer en simple géométrie." Reçu à l'École Polytechnique en 1848, à l'âge de dix-sept ans, il était entré à sa sortie dans l'artillerie de terre, mais ses goûts le portaient vers la géométrie, et il ne tarda pas à se faire attacher à l'École Polytechnique où il a fait toute sa carrière, d'abord comme répétiteur, puis comme examinateur d'entrée, et enfin comme professeur de la Chaire de Géométrie Descriptive dans laquelle il succéda, en 1864, à M. de la Gournerie. Ses premiers travaux se rapportent aux méthodes de transformation en géométrie ; mais il ne tarda pas à trouver un sujet fécond de recherches prolongées dans l'étude des propriétés relatives au déplacement des figures dans l'espace et des applications de ces propriétés où il s'inspira heureusement des travaux d'Euler, de Poinso, et de Chasles. Cela lui a permis de constituer un ensemble qu'il se plaisait à désigner sous le nom de *Géométrie Cinématique*, et qu'il a développé dans son *Cours de Géométrie Descriptive*. L'étude des déplacements qui peut prendre un corps soumis à moins de cinq conditions méritait les recherches persévérantes que M. Mannheim lui a consacrées. Parmi ses travaux les plus élégants, il convient de citer une étude sur la cyclide de Dupin, des constructions géométriques des centres de courbure, des recherches sur les surfaces réglées et les systèmes des rayons rectilignes, où il a retrouvé par la géométrie les beaux théorèmes de Kummer, des travaux sur la surface des ondes de Fresnel, et sur l'optique géométrique, &c. M. Mannheim avait prit sa retraite comme professeur, il y a quelques années. Mais il ne cessait de travailler et d'envoyer aux journaux mathématiques des notes courtes contenant toujours quelque élégante propriété. Il avait réuni, il y a quelques années, ses principaux mémoires dans un volume in 4° qui porte le nom de *Développements de Géométrie Cinématique*.

EDWARD JOHN ROUTH

ONE of the most famous names in the history and the progress of mathematics in England has been transferred to the roll of record by the death of Dr. Edward John Routh on June 7th, 1907.

During the greater part of last century, nearly every able mathematical student at the University of Cambridge was prepared for his final examination by a private tutor: and among these private tutors ("coaches," to adopt the word of technical and friendly use), two names stand out, eminent beyond all others. They are Hopkins and Routh. William Hopkins was a conspicuous figure in Cambridge, teaching during the second quarter of the century: among his pupils in that period were Stokes, Cayley, Adams, Thomson, Maxwell, and Routh himself. Routh's immediate influence extended from the later fifties until 1888: and during those years, a large majority of successful students were happy in calling themselves his pupils. The University has recently decided to abolish the order of individual merit in the Mathematical Tripos, and thus, incidentally, to annihilate the significance of the title of Senior Wrangler, and of another title, as picturesque though not quite so honour-bearing. It is Routh's unique distinction to have coached twenty-four Senior Wranglers in twenty-four consecutive years: no man will ever be able to repeat the achievement.

This distinguished teacher and investigator was born in Quebec on 20th January, 1831. His father was Sir Randolph Routh, K.C.B., an old Etonian, Commissary-General in the Army, and Senior Commissariat officer at Waterloo in 1815. His boyhood was spent in London: he was educated at University College School: and later, at University College, he came under the influence of Augustus de Morgan. It may have been due to friendship with that mathematician, who was our founder, that Routh became a member of our Society at its inception in January, 1865.

He was admitted a pensioner at Peterhouse on June 1st, 1850, and commenced residence in October. Clerk Maxwell entered at the same college in that year, but he migrated to Trinity near the close of his freshman's term. Other names belonging to the same "year" are those of the present Chancellor of the University, the Duke of Devonshire; Mr. Aldis Wright,

Mr. Henry Bradshaw, Sir Leslie Stephen, and Mr. Burbury. In these days of depleted numbers of successful candidates in the Mathematical Tripos, it is not unworthy of note that the list in "Routh's year" (again to use our vernacular at Cambridge) contains over 140 names. Routh was Senior Wrangler in 1854, and Maxwell was second: the two distinguished mathematicians were bracketed for the two Smith's Prizes (for the first time on record).

Routh soon was elected a Fellow of his college and was appointed to lecture. At the time of his degree, the Peterhouse foundation included, among other famous men, William Thomson (then Professor at Glasgow, already widely known in Europe by his original work, his name now being merged in his title of Lord Kelvin); Peter Guthrie Tait (cheery in spirit, strong of brain, aggressively unconventional in social superficialities, not yet gone to Belfast nor transplanted to Edinburgh); and Steele, affectionately described to me by Tait as the greatest mathematical teacher who ever lived. But the blight of illness soon fell upon Steele: during one term, Routh took his pupils: and when Steele died, the pupils remained with Routh. Thus began his connection with the private teaching of mathematics at Cambridge: it ceased in 1888, only upon his deliberate retirement from the work.

From that time onwards, Routh's teaching occupied the greater part of his activity though not absorbing the whole of it, for he made valuable original contributions to the science he loved and served so well. But these came slowly in the earlier years of his maturity. He examined twice for the Tripos—that is, the old Tripos, in its greatest days; and, after the first occasion, he published a volume in co-operation with H. W. Watson, embodying the solutions of the questions which had been set in 1860. Already, as a young man, he had collaborated with Lord Brougham in publishing (in 1855) a volume entitled *An Analytical View of Sir Isaac Newton's Principia*; it can hardly be rash, on reading the book at this distance of time from its publication, to assume that the mathematics belonged to Routh and the opinions to Lord Brougham.

Other private tutors—Walton, Frost, Todhunter, Besant, Webb in later years, and now Herman—have been successful in taking pupils at Cambridge, and have had them in ample numbers. When I came into residence in 1877 as a freshman, the reputation of Routh as a private tutor was in a class by itself; and no undergraduate, whether confident or diffident about his powers while ambitious in his prospects, dared lightly to abstain from coaching with the maker of Senior Wranglers. Thus Routh was in large practice; the largeness of his practice, and the remarkable success of his pupils in the examination lists, maintained his

pre-eminent position. Occasionally (as in 1867 by Clifford, in 1870 by Greenhill, in 1875 by Burnside) a bold bid was made by some other teacher's pupil for the highest place in the Tripos: but, somehow, it did not succeed, and the Senior Wrangler still had "Mr. Routh for his private tutor."

No doubt can be felt that, at the epoch of his greatest influence, most of the ablest of the undergraduates went to him for their private teaching. Success certainly led to increased numbers: but it was incessant and diligent care which made Routh's teaching supremely effective. It is true that, after a time, the examiners in the Tripos were mostly old pupils of his, and so his dominance might easily be inferred as obvious: yet such an explanation is hardly even a fragment of the truth. Wherein, then, consisted the genius of his teaching?

It belonged to his time; also to the tradition which, silently transmitted to him, was maintained and strengthened. It was superbly direct for the purpose in view: and it was strong in the measured completeness with which he covered the whole ground for the Tripos. Independence on the part of a student was not encouraged; for independence would rarely, if ever, be justified by the event. Foreign books were seldom mentioned: Routh himself had summarised from them all that could be deemed useful for the examination. Above all, he was a born teacher. Regularity and steady diligence were two demands which he always made from his pupils. As a result, any one who worked carefully could feel confident that he had had the best preparation for the Tripos which was possible, and that he had been trained so as to make the most of his powers in the examination.

The method was simple in operation. His pupils went to him three times a week, for an hour on each occasion. They were arranged in classes of six to ten, selected at first mainly by their own statements of mathematical knowledge brought from school, and sifted later by their attainments and progress. Each hour's work was substantially of the nature of a lecture. A few minutes at the beginning of the hour were devoted to the inspection of written exercises, a process at which Routh was wonderfully quick. The remainder of the hour was occupied with a skilful exposition of the subject under consideration, sometimes summarised in such a way as almost to compel remembrance. The intrinsic interest of the teaching was not of uniform quality; he was at his best in subjects so diverse as solid geometry and physical optics; he seemed below his best in a subject like rigid dynamics, perhaps because (as some of us used to imagine) all he had to say on it was contained in the book which he had written.

In addition to the direct teaching, he set to all his pupils a weekly paper of a dozen problems, taken mainly from past examinations. The conditions of answering this paper were made to alternate in successive weeks. In one week, a student could devote unlimited time to the questions: in the other, he was expected to limit himself to three hours so as to feign some similarity to the pressure of the Senate House. The pupils' answers were deposited in Routh's rooms in Peterhouse on the Friday or the Saturday of any week: they were returned on the succeeding Monday, when he placed in an outer room a carefully written set of solutions. To add a stimulus of competition, every pupil's answers had been marked; and there was a list containing names and marks, so that the men even of different years were almost encouraged to regard the weekly problem-paper as a field for rivalry and comparison.

Finally, at some period before the crucial examination, there began the process of revision, in the form of bookwork papers. The answers to each such paper had to be brought two days after the paper was set, and they were examined critically in class. The pupils were brought up to the mark, so far as the teacher was concerned, partly by good humour, partly by occasional warnings that were kindly in phrase and in spirit, and always under a genial sense of his easy mastery over the whole range of subjects included in the examination.

It is obvious that the process, if not supplemented by other means, would imply a uniformity of treatment which would not be found advantageous by the eager members of even an average class. The necessary supplement was provided by Routh in the form of "manuscripts," which really were of the character of small pamphlets dealing with special portions of subjects not taught in his classes. Pupils were expected to copy these manuscripts, in default of English text-books; but diligence in taking copies was far from universal. Little or no account was paid to work done in college lectures: and attendance at a course of any University professor's lectures, save only in the case of Stokes, was almost a rare exception. Routh prepared his men thoroughly for the examination: its needs dictated and limited alike the range and the spirit of the work done with him. Out of much knowledge and experience he realised, almost as by instinct, the wants, the needs, the possibilities, of examiners and examinees. Simply and effectively, he dominated the Mathematical Tripos, almost during the whole period (1855 to 1888) of his coaching activity. In that time, he had more than six hundred pupils. On twenty-seven different occasions, he prepared the Senior Wrangler: and in my own year a class of six men, who had been taught together since their freshmen's term, occupied the first six places in the Tripos.

Exacting as was this work throughout the year (for he taught through the long vacation residence, as well as during the usual three terms), it was far from exhausting his mathematical activities. Quite early he had produced a text-book on rigid dynamics, of which no fewer than seven editions have appeared. The book, in its first form, was of modest size and occupied a single volume: but the contents were gradually increased. The seventh edition occupies a couple of closely packed volumes and is an encyclopædia of the subject, representing much original work on the part of the author as well as the results obtained by a vast number of writers. This treatise was translated into German on the initiative of Prof. Klein, of Göttingen, who contributed an introductory preface in warm commendation of its importance.

When he retired from private teaching, he remained a lecturer at Peterhouse and at Pembroke College. In the comparative leisure which then became his, he compiled a two-volume work on analytical statics, which now is in its second edition; and, later, he published a single volume on the dynamics of a particle. The most original work written by Routh—a work which secures for him an honoured place among the most distinguished writers on theoretical dynamics—was his essay, *A Treatise on the Stability of a Given State of Motion, particularly Steady Motion*, to which the Adams prize in the University of Cambridge was awarded in 1877. In that essay, he introduced the modified Lagrangian function which, as Professor Larmor says,* made a fundamental advance in the principles of the dynamical interpretation of nature.

Over thirty mathematical papers have been published under Routh's own name. Of these, perhaps the most important, and the most novel in its time, was a brief paper published in the seventh volume of the *Quarterly Journal* in 1866. There, Routh develops some properties of the curvature of twisted curves, by using the kinematical relations of a set of moving axes in three dimensions: readers of Darboux's *Théorie générale des surfaces* (of which the first volume appeared in 1887) are familiar with the frequent applications of a moving trihedron in that great treatise. It may be added that twelve of Routh's papers appeared in our own *Proceedings*.

In 1864 he married Hilda, the eldest daughter of Sir G. B. Airy, and a god-daughter of Dr. Whewell: she survives him, together with a daughter and three sons. As already stated, he was a member of this Society from its earliest days: and he was elected a Fellow of the Royal Society in 1872. Before graduating at Cambridge, he had proceeded to the degree of M.A. in London University, winning the gold medal for mathematics

* *Nature*, 27th June, 1907, p. 202.

in 1853 : and at the time of his death, he was a Fellow of that University. When the new degree of Doctor of Science was instituted at Cambridge, he was induced to apply for it together with Besant : they were admitted to the degree on 31st May, 1883, being the first two recipients. The honorary degree of Doctor of Science was conferred upon him by the University of Dublin, and that of Doctor of Laws by the University of Glasgow. His fellowship at Peterhouse had been vacated by his marriage in 1864 ; in 1883, he was elected an honorary fellow of his old college, and for many years, he has been the senior among the small and select body of honorary fellows of that ancient foundation.

The conclusion of his full activity as a private teacher was marked by an act of his old pupils in token of their gratitude. At their request, his portrait was painted by Herkomer ; and it was presented to Mrs. Routh, by Mr. Justice Stirling on their behalf, at a gathering held in the combination room at Peterhouse. Those who were present are not likely to forget the remarkable collection of distinguished men there assembled, or the appreciation then expressed for him and for his teaching, or the unaffected simplicity of his speech in his reply of thanks. A copy of the picture hangs in the dining hall of Peterhouse.

He served for one term of office, from 1888 to 1892, on the Council of the Senate of the University : but the business did not seem to interest him keenly. What did interest him, among matters academic, was the Mathematical Tripos. Broadly conservative by spirit, alike in imperial politics and in academic politics, he was willing to make small changes in the examination from time to time, especially if their tendency appeared to foster the study of applied mathematics. He was a member of the Syndicate which recommended the changes first effected in 1882, when the examination was divided : he signed the report which maintained (though without enthusiasm) the order of merit in one portion of the examination and replaced it by a classified order in the other portion. Seven years ago, an attempt was made, on the recommendation of the Special Board for Mathematics, to abolish the order of merit in that portion of the examination where it had been retained : the attempt failed. During the past academic year, a similar attempt succeeded. On both occasions, Routh was the leader of the opposition, taking an active part in the struggle.

His natural constitution must have been strong, to enable him to endure the severe and prolonged labour of teaching so many pupils : in the middle period of his work, he had over a hundred in the Michaelmas Term. His regularity was a marvel ; his days in term seemed to have a stable precision in their round, and the time on any afternoon could

almost have been told by the different stages of his walk along the Trumpington Road. Latterly, his strength began to give way: and in recent months, his health failed so completely that his death, when it came, was not unexpected. The attendance at the funeral service, in Great St. Mary's Church on the 10th of June, gave silent and significant testimony to the esteem in which the great teacher was held by his old pupils in particular and by the University at large.

July 5th, 1907.

A. R. FORSYTH.

NOTES AND CORRECTIONS.

THE result suggested in the note added to Prof. Hill's paper, pp. 340, 341 of this volume proves to be true. If in the result (VIII.), proved for $s+1$ a positive integer, s is replaced by $\delta-2$, and $\alpha+s+1$, $\beta+s+1$, $\gamma+s+1$ are replaced by α , β , γ , the result becomes

$$\frac{1}{\delta-1} \left[1 + \frac{\alpha \cdot \beta}{\gamma \cdot \delta} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)\delta(\delta+1)} + \dots \right] \\ = \frac{1}{\gamma+\delta-\alpha-\beta-1} \left[1 + \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma+\delta-\alpha-\beta)} + \frac{(\gamma-\alpha)(\gamma-\alpha+1)(\gamma-\beta)(\gamma-\beta+1)}{\gamma(\gamma+1)(\gamma+\delta-\alpha-\beta)(\gamma+\delta-\alpha-\beta+1)} + \dots \right].$$

This result has been proved to hold, when the real parts of $\delta-1$ and of $\gamma+\delta-\alpha-\beta-1$ are positive, by Dr. Barnes, using the method of contour integration developed in his paper "A New Development of the Theory of Hypergeometric Functions" to be published in Vol. 6. A more complete result, including the remainder after $s+1$ terms, has been obtained by Prof. Hill, using elementary algebraic methods only. His result is that

$$\frac{1}{\delta-1} \left[1 + \frac{\alpha \cdot \beta}{\gamma \cdot \delta} + \dots + \frac{\alpha_s \cdot \beta_s}{\gamma_s \cdot \delta_s} \right] \\ - \frac{\Pi(n+1, \gamma-1) \Pi(n+1, \gamma+\delta-\alpha-\beta-2)}{\Pi(n+1, \gamma-\alpha-1) \Pi(n+1, \gamma-\beta-1)} \frac{(n+1)^{1-\delta}}{\delta-1} \left[1 + \frac{\alpha \cdot \beta}{\delta(\gamma+n+1)} + \dots + \frac{\alpha_s \cdot \beta_s}{\delta_s(\gamma+n+1)_s} \right] \\ = \frac{1}{\gamma+\delta-\alpha-\beta-1} \left[1 + \frac{(\gamma-\alpha)(\gamma-\beta)}{\gamma(\gamma+\delta-\alpha-\beta)} + \dots + \frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{\gamma_n(\gamma+\delta-\alpha-\beta)_n} \right] \\ - \frac{\Pi(s+1, \gamma-1) \Pi(s+1, \delta-2)}{\Pi(s+1, \alpha-1) \Pi(s+1, \beta-1)} \frac{(s+1)^{\alpha+\beta-\gamma-\delta+1}}{\gamma+\delta-\alpha-\beta-1} \\ \times \left[1 + \frac{(\gamma-\alpha)(\gamma-\beta)}{(\gamma+\delta-\alpha-\beta)(\gamma+s+1)} + \dots + \frac{(\gamma-\alpha)_n(\gamma-\beta)_n}{(\gamma+\delta-\alpha-\beta)_n(\gamma+s+1)_n} \right].$$

Either side of this equation is transformed into the other by replacing

$$\alpha, \quad \beta, \quad \gamma, \quad \delta, \quad s, \quad n,$$

by

$$\gamma-\alpha, \quad \gamma-\beta, \quad \gamma, \quad \gamma+\delta-\alpha-\beta, \quad n, \quad s.$$

Dr. Barnes' result follows from the above by making n and s both infinite, provided that the real parts of $\delta-1$ and of $\gamma+\delta-\alpha-\beta-1$ are both positive, as he supposes. These results are given in papers read on November 14th, 1907.

MR. HILTON sends the following note :—

My attention has been called to the fact that Dr. G. A. Miller gave an expression for the number of sub-groups of a given type in a given Abelian group (*Annals of Math.*, October, 1904) which is equivalent to that given in my paper in *Proc. London Math. Soc.*, Ser. 2, Vol. 5, p. 3.

CORRIGENDUM.

Ser. 2, Vol. 4, p. xiv., line 1, for "University of Dublin" read "Royal Irish Academy."

PAPERS

PUBLISHED IN THE

PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY

ON SUB-GROUPS OF A FINITE ABELIAN GROUP

By HAROLD HILTON.

[Received October 7th, 1906.—Read November 8th, 1906.]

1. Let H be a sub-group of order p^r in a finite Abelian group G whose order is p^a (p prime). Let t_1 of the invariants of H be 1, t_2 be 2, ..., t_m be m . Let u_1 of the invariants of G be 1, u_2 be 2, ..., u_m be m . The quantities $t_1, t_2, t_3, \dots, u_1, u_2, u_3, \dots$ are zeros or positive integers ($u_m > 0$).

Then
$$a = u_1 + 2u_2 + 3u_3 + \dots, \quad r = t_1 + 2t_2 + 3t_3 + \dots;$$

let
$$u = u_1 + u_2 + u_3 + \dots, \quad t = t_1 + t_2 + t_3 + \dots.$$

We first find the number (X) of ways in which a base $[g_1, g_2, g_3, \dots]$ can be chosen for a sub-group of G of the same type as H .

Let $[h_1, h_2, h_3, \dots]$ be a base of G . Then G contains $(p^{u_m} - 1)p^{a - u_m}$ elements of order p^m ; namely, $h_1^{\beta_1} h_2^{\beta_2} h_3^{\beta_3} \dots$, where at least one of $\beta_1, \beta_2, \dots, \beta_{u_m}$ is prime to p .* Therefore g_1 can be chosen in $(p^{u_m} - 1)p^{a - u_m}$ ways.

When g_1 is chosen $[g_1, h_2, h_3, \dots]$ may be taken as a base of G . Then G contains $(p - 1)p^{a - u_m}$ elements of order p^m whose p^{m-1} -th powers are in $\{g_1\}$; namely, $g_1^{\beta_1} h_2^{\beta_2} h_3^{\beta_3} \dots$, where $\beta_2, \beta_3, \dots, \beta_{u_m}$ are multiples of p , β_1 is prime to p , and $\beta_{u_m+1}, \beta_{u_m+2}, \dots$ are any integers. Therefore G contains

$$[(p^{u_m} - 1) - (p - 1)]p^{a - u_m} = (p^{u_m-1} - 1)p^{1+a-u_m}$$

* See Netto's *Algebra*, Vol. II., p. 246.

elements of order p^m whose p^{m-1} -th powers are not in $\{g_1\}$; i.e., g_2 may be chosen in $(p^{u_m}-1)p^{1+\alpha-u_m}$ ways when g_1 is given.

Again, when g_1 and g_2 are chosen $[g_1, g_2, h_3, h_4, \dots]$ may be taken as a base of G . Then G contains $(p^2-1)p^{\alpha-u_m}$ elements of order p^m whose p^{m-1} -th powers are in $\{g_1, g_2\}$; namely, $g_1^{\beta_1} g_2^{\beta_2} h_3^{\beta_3} h_4^{\beta_4} \dots$, where $\beta_3, \beta_4, \dots, \beta_{u_m}$ are multiples of p , β_1 or β_2 is prime to p , and $\beta_{u_m+1}, \beta_{u_m+2}, \dots$ are any integers. Therefore G contains

$$[(p^{u_m}-1)-(p^2-1)]p^{\alpha-u_m} = (p^{u_m-2}-1)p^{2+\alpha-u_m}$$

elements of order p^m whose p^{m-1} -th powers are not in $\{g_1, g_2\}$; i.e., g_3 may be chosen in $(p^{u_m-2}-1)p^{2+\alpha-u_m}$ ways when g_1 and g_2 are given.

This reasoning may be continued to show that the first t_m generators of the base may be chosen in $\frac{f(u_m)}{f(u_m-t_m)}p^{t_m(\alpha-u_m)+\frac{1}{2}t_m(t_m-1)}$ ways; where $f(k)$ denotes $(p^k-1)(p^{k-1}-1)\dots(p^2-1)(p-1)$ and $f(0) = 1$.

Now, when $[h_1, h_2, h_3, \dots]$ is a base of G every element of order p^{m-1} in G is contained in the group whose base is

$$[h_1^p, h_2^p, \dots, h_{u_m}^p, h_{u_m+1}, h_{u_m+2}, \dots],$$

and conversely. Hence, by reasoning precisely similar to that used above, G contains $(p^{u_m+u_{m-1}}-1)p^{\alpha-u_{m-1}-2u_m}$ elements of order p^{m-1} , and the p^{m-2} -th powers of $(p^{t_m}-1)p^{\alpha-u_{m-1}-2u_m}$ of these elements are in $\{g_1, g_2, \dots, g_{t_m}\}$. Hence, as before, g_{t_m+1} can be chosen in

$$(p^{u_m+u_{m-1}-t_m}-1)p^{t_m+\alpha-u_{m-1}-2u_m}$$

ways when $g_1, g_2, \dots, g_{t_m}$ are given.

Proceeding as before, we see that g_{t_m+2} can be chosen in

$$(p^{u_m+u_{m-1}-t_m-1}-1)p^{1+t_m+\alpha-u_{m-1}-2u_m}$$

ways when $g_1, g_2, \dots, g_{t_m+1}$ are given. Thus we show that the first t_m+t_{m-1} generators of the base may be chosen in

$$\frac{f(u_m)}{f(u_m-t_m)} \frac{f(u_m+u_{m-1}-t_m)}{f(u_m+u_{m-1}-t_m-t_{m-1})} \\ \times p^{t_m(\alpha-u_m)+t_{m-1}(\alpha-u_{m-1}-2u_m)+\frac{1}{2}[t_1(t_1-1)+t_2(t_2-1)]+t_m t_{m-1}}$$

ways.

Proceeding in this way, we get

$$X = \frac{f(u_m)}{f(u_m-t_m)} \frac{f(u_m+u_{m-1}-t_m)}{f(u_m+u_{m-1}-t_m-t_{m-1})} \\ \times \frac{f(u_m+u_{m-1}+u_{m-2}-t_m-t_{m-1})}{f(u_m+u_{m-1}+u_{m-2}-t_m-t_{m-1}-t_{m-2})} \dots p^c$$

$$[c = \frac{1}{2}t(t-1) + u_1(t-t_1) + u_2(2t-t_2-2t_1) + u_3(3t-t_3-2t_2-3t_1) + \dots].$$

To find the number (Y) of distinct bases of any sub-group of G of the same type as H we put $u_1 = t_1, u_2 = t_2, u_3 = t_3, \dots$ in X . The total number (N) of sub-groups of G of the same type as H is then

$$\frac{X}{Y} = \frac{f(u_m)}{f(t_m) f(u_m - t_m)} \frac{f(u_m + u_{m-1} - t_m)}{f(t_{m-1}) f(u_m + u_{m-1} - t_m - t_{m-1})} \\ \times \frac{f(u_m + u_{m-1} + u_{m-2} - t_m - t_{m-1})}{f(t_{m-2}) f(u_m + u_{m-1} + u_{m-2} - t_m - t_{m-1} - t_{m-2})} \dots p^d \\ [d = (u_1 - t_1)(t - t_1) + (u_2 - t_2)(2t - t_2 - 2t_1) + (u_3 - t_3)(3t - t_3 - 2t_2 - 3t_1) + \dots].$$

The above reasoning shows that the necessary and sufficient conditions for the existence of sub-groups such as H are

$$u_m + u_{m-1} + \dots + u_{m-q+1} \geq t_m + t_{m-1} + \dots + t_{m-q+1} \quad (q = 1, 2, \dots, m);$$

i.e., the k -th invariant of H is not greater than the k -th invariant of G ($k = 1, 2, 3, \dots$).*

2. To find the total number (M) of sub-groups of order p^r in G , we have only to find every set of values of $t_1, t_2, t_3, \dots$ satisfying the relations

$$u_m + u_{m-1} + \dots + u_{m-q+1} \geq t_m + t_{m-1} + \dots + t_{m-q+1}$$

$$\text{and} \quad r = t_1 + 2t_2 + 3t_3 + \dots$$

Then M is the sum of the corresponding values of N . A general formula giving M for every value of r would probably be somewhat complicated.

We can, however, find the simple expression $\frac{f(u+r-1)}{f(r)f(u-1)}$ for M when $r \leq$ the smallest invariant of G . In this case

$$u_m = u, \quad u_{m-1} = u_{m-2} = u_{m-3} = \dots = 0$$

for every sub-group considered, while

$$N = \frac{f(u)}{f(u-t)f(t_1)f(t_2)\dots} p^d \\ [d = u(r-t) - rt + t_1^2 + (t_2 + 2t_1)t_2 + (t_3 + 2t_2 + 3t_1)t_3 + \dots].$$

$$\text{We have to prove} \quad \frac{f(u+r-1)}{f(r)f(u-1)} = \Sigma(N),$$

the sum being taken for all positive integral or zero values of $t_1, t_2, t_3, \dots$

* In the notation of Burnside's *Theory of Groups*, § 47, $n_i \leq m_i$. Since the above was written Prof. Burnside has informed me that this corrected form of his result was communicated to him by Prof. E. H. Moore, of Chicago, in 1899.

such that $t = t_1 + t_2 + t_3 + \dots \leq u$, $t_1 + 2t_2 + 3t_3 + \dots = r$.

This is obviously true when $u = 1$. We assume it true for all values of u less than the one considered, and use induction to prove the theorem true in general.

$$\text{Now } \frac{f(u+r-1)}{f(r)f(u-1)}$$

$$= \text{the coefficient of } x^r \text{ in } p^{-\frac{1}{2}r(r+1)}(1+px)(1+p^2x) \dots (1+p^{u+r-1}x),$$

i.e., in

$$p^{-\frac{1}{2}r(r+1)}(1+px)(1+p^2x) \dots (1+p^ux)(1+p \cdot p^ux)(1+p^2 \cdot p^ux) \dots (1+p^{r-1} \cdot p^ux) \\ = p^{-\frac{1}{2}r(r+1)} \sum_t p^{\frac{1}{2}t(t+1)} \frac{f(u)}{f(t)f(u-t)} p^{(r-t)u + \frac{1}{2}(r-t)(r-t+1)} \frac{f(r-1)}{f(r-t)f(t-1)}.$$

But, by our assumption,

$$\frac{f(r-1)}{f(r-t)f(t-1)} = \sum \frac{f(t)}{f(t-\tau)f(\tau_1)f(\tau_2) \dots} p^e$$

$$[e = t(r-t-\tau) - (r-t)\tau + \tau_1^2 + (\tau_2 + 2\tau_1)\tau_2 + (\tau_3 + 2\tau_2 + 3\tau_1)\tau_3 + \dots]$$

for all integral values of $\tau_1, \tau_2, \tau_3, \dots$ such that

$$\tau = \tau_1 + \tau_2 + \tau_3 + \dots \leq t, \quad \tau_1 + 2\tau_2 + 3\tau_3 + \dots = r-t.$$

But, if we put $t-\tau = t_1, \tau_1 = t_2, \tau_2 = t_3, \tau_3 = t_4, \dots$, we have

$$t = t_1 + t_2 + t_3 + \dots, \quad t_1 + 2t_2 + 3t_3 + \dots = r.$$

$$\text{Hence } \frac{f(u+r-1)}{f(r)f(u-1)} = \sum \frac{f(u)}{f(u-t)f(t_1)f(t_2) \dots} p^f$$

$$[f = e + (r-t)u + \frac{1}{2} \{t(t+1) + (r-t)(r-t+1) - r(r+1)\}]$$

for all values of $t_1, t_2, t_3, \dots$ such that

$$t = t_1 + t_2 + t_3 + \dots \leq u, \quad t_1 + 2t_2 + 3t_3 + \dots = r.$$

We readily verify $f = d$, which completes the proof.

3. We may illustrate the result of § 1 by finding an expression for the number of normal (self-conjugate) sub-groups of index p^2 in any group G . Let $H_1, H_2, H_3, \dots$ be these normal sub-groups, and let D be their greatest common sub-group. Since $G/H_1, G/H_2, G/H_3, \dots$ are Abelian (being of order p^2), the commutant of G is contained in $H_1, H_2, H_3, \dots$, and is therefore contained in D . Hence $\Gamma \equiv G/D$ is Abelian. Moreover, the p^2 -th power of every element of G is in $H_1, H_2, H_3, \dots$, and is therefore in D . Hence the p^2 -th power of every

element of $\Gamma = 1$. It follows that Γ is an Abelian group of the type $(2, 2, 2, \dots, 1, 1, 1, \dots)$ [y 2's and z 1's] whose order is a power of p .^{*} The number of normal sub-groups of index p^2 in G is the same as the number of sub-groups of index p^2 in Γ .

Now, by § 1, Γ contains (i.) $\frac{f(z)}{f(2)f(z-2)} p^{2y}$ sub-groups with y invariants 2 and $z-2$ invariants 1; (ii.) $\frac{f(y)}{f(2)f(y-2)}$ with $y-2$ invariants 2 and $z+2$ invariants 1; (iii.) $\frac{f(y)f(z+1)}{f(1)f(1)f(y-1)f(z)} p^{y-1}$ with $y-1$ invariants 2 and z invariants 1. The factor-group of Γ with respect to $\frac{f(y)}{f(y-1)f(1)} p^{y+z-1}$ of the sub-groups (iii.) is cyclic; the factor-group of Γ with respect to the remaining $\frac{f(y+z)}{f(2)f(y+z-2)}$ sub-groups of index p^2 is non-cyclic. This is readily proved directly or by considering the reciprocal sub-groups.[†]

Hence the factor-group of G with respect to $\frac{f(y)}{f(y-1)f(1)} p^{y+z-1}$ normal sub-groups of index p^2 is cyclic, and the factor-group with respect to the remaining $\frac{f(y+z)}{f(2)f(y+z-2)}$ normal sub-groups of index p^2 is non-cyclic.

The total number of normal sub-groups of index p^2 in G is therefore $\frac{(p^{y+z}-1)(p^{y+z-1}-1)}{(p^2-1)(p-1)} + \frac{p^y-1}{p-1} p^{y+z-1}$, where y and z are zero or positive integers. As an example we may take the group

$$a^{p^{a-1}} = b^p = 1, \quad ab = ba^{1+p^{a-2}},$$

for which $y = z = 1$.

^{*} See M. Bauer, *Nouv. Ann. Math.* [3], Vol. xix. (1900), p. 508.

[†] Weber's *Algebra*, Vol. II., p. 56.

ON BÄCKLUND'S TRANSFORMATION AND THE PARTIAL DIFFERENTIAL EQUATION $s = F(x, y, z)$

By J. E. CAMPBELL.

[Received October 11th, 1906.—Read November 8th, 1906.]

INTRODUCTORY.

If the arc of a surface is given in Gaussian coordinates by

$$ds^2 = A^2 du^2 + B^2 dv^2 + 2AB \cos \alpha \, du \, dv,$$

and the spherical image by

$$d\sigma^2 = P^2 du^2 + Q^2 dv^2 + 2PQ \cos \theta \, du \, dv,$$

where A , B , P , Q , α , and θ are known functions of u and v , the surface is itself given, save as regards its position in space, and its determination depends on the solution of Riccati's equation.

In the case of a pseudosphere (that is, a surface on which the measure of curvature is everywhere negative and equal to unity in magnitude), if we refer to lines of curvature, we may take

$$\alpha = \theta = 90^\circ$$

and $P = -A = \cos z$, $Q = -B = \sin z$,

where z is a function of u and v to be determined by

$$\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} = \sin z \cos z,$$

an equation at once transformable (with the usual notation) into

$$s = \sin z.$$

The determination of all pseudospheres depends therefore on the solution of this equation, and, if from any given solution we can deduce another, we can from the corresponding given pseudosphere deduce another pseudosphere.

Now, Bianchi* had shown how from any pseudosphere another could be

* For an account of Bianchi's transformation see Darboux, *Surfaces*, Vol. III., chapter xii., or Bianchi (German edition), *Vorlesungen über Differentialgeometrie*, Kap. xvii. My attention was first drawn to this subject by reading a thesis by Miss Coddington, of Columbia University, on "The Historical Development of Pseudospherical Surfaces."

geometrically constructed, and Bäcklund* was able to use this discovery to illustrate the transformation theory which he himself had invented.

Thus, let $z = f(x, y)$ be any integral of the equation

$$s = \sin z,$$

and, regarding $z = f(x, y)$ as the equation of a surface, let x, y, z, p, q be the coordinates of one of the surface elements. Consider now the four equations

$$x' = x, \quad y' = y, \quad p' = -p + 2a \sin \frac{1}{2}(z - z'), \quad q' = q - 2b \sin \frac{1}{2}(z + z'),$$

where a and b are two constants such that $ab = 1$.

We may regard x', y', z', p', q' as the coordinates of some other surface element; for the condition for this is

$$\frac{dp'}{dy'} = \frac{dq'}{dx'},$$

$$\text{that is, } \frac{-dp + a(dz - dz') \cos \frac{1}{2}(z - z')}{dy' (= dy)} = \frac{dq - b(dz + dz') \cos \frac{1}{2}(z + z')}{dx' (= dx)},$$

$$\text{i.e., } -s + a(q - q') \cos \frac{1}{2}(z - z') = s - b(p + p') \cos \frac{1}{2}(z + z'),$$

$$\text{which comes to } s = \sin z.$$

It is also at once verified that

$$s' = \frac{dp'}{dy'} = \sin z'.$$

We thus see that when we are given the surface $z = f(x, y)$ we can eliminate x, y, z, p, q from the four connecting equations, and obtain two equations in x', y', z', p', q' which will be in involution, that is, will be such that when we substitute from them for p' and q' in

$$dz' = p' dx' + q' dy'$$

this equation will be completely integrable; and, further, any integral of this equation will satisfy

$$s' = \sin z'.$$

In this paper I propose to discuss the more general equation

$$s = F(x, y, z),$$

in connection with Bäcklund's transformation theory, and to show that, if we exclude equations of which the complete solution is already known, the only equations of the proposed form to which Bäcklund's transformation

* *Mathematische Annalen*, t. xvii. and xix.

theory will apply and enable us from one solution to obtain others are equations reducible by change of the variables to one or other of the forms

$$s = \frac{\kappa(1-\kappa)z}{(x-y)^2} \text{ where } \kappa \text{ is a constant,}$$

$$s = z,$$

$$s = \sin z.$$

It will now be necessary* to sketch as briefly as possible Bäcklund's general theory.

Let x, y, z, p, q be the coordinates of a surface element, and u_i a function of these coordinates. Let

$$X \equiv \frac{\partial}{\partial x} + p \frac{\partial}{\partial z}, \quad Y \equiv \frac{\partial}{\partial y} + q \frac{\partial}{\partial z}, \quad P \equiv \frac{\partial}{\partial p}, \quad Q \equiv \frac{\partial}{\partial q};$$

then in operating on u_i

$$\frac{d}{dx} \equiv X + rP + sQ, \quad \frac{d}{dy} \equiv Y + sP + tQ.$$

Let

$$(i, j)_{XY} \equiv Xu_i Yu_j - Yu_i Xu_j,$$

and let $(ij)_{XP}, \dots$ be similarly defined; then, if

$$(i, j) \equiv \frac{du_i}{dx} \frac{du_j}{dy} - \frac{du_i}{dy} \frac{du_j}{dx},$$

we have

$$(ij) \equiv (ij)_{XY} + r(ij)_{PY} + s(ij)_{XP} + s(ij)_{QY} + t(ij)_{XQ} + (rt - s^2)(ij)_{PQ}.$$

Finally, denoting $(ij)_{XP} + (ij)_{YQ}$ by $[i, j]$, we can easily verify the identity

$$(23)[14] + (31)[24] + (12)[34] + (14)[23] + (24)[31] + (34)[12] \equiv 0 \quad (1)$$

for all values of x, y, z, p, q, r, s, t .

Let now $u_1 = 0, u_2 = 0, u_3 = 0, u_4 = 0$ be four equations *arbitrarily* assigned connecting two sets of variables x, y, z, p, q and x', y', z', p', q' . Can these variables be the coordinates of two surface elements?

Suppose $z' = f(x', y')$ to be one surface and x', y', z', p', q' the coordinates of one of its elements; since these coordinates are functions of x', y' , the four assigned equations may be regarded as equations connecting x, y, z, p, q and x', y' . We may therefore regard x', y' as func-

* This paper was written before the appearance of the last volume of Forsyth's *Differential Equations*, in chapter xxi. of which an account of Bäcklund's work and its recent developments is given.

tions of x, y, z, p, q and (if x, y, z, p, q are element coordinates of another surface) thus implicitly functions of x and y . We therefore must add to the four assigned equations

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \quad u_4 = 0$$

the equation which results from the elimination of $r, s, t, \frac{dx'}{dx}, \frac{dy'}{dy}, \frac{dz'}{dz}$ from the eight equations

$$\frac{du_i}{dx} + \frac{du_i}{dx'} \frac{dx'}{dx} + \frac{du_i}{dy'} \frac{dy'}{dx} = 0, \quad \frac{du_i}{dy} + \frac{du_i}{dx'} \frac{dx'}{dy} + \frac{du_i}{dy'} \frac{dy'}{dy} = 0$$

$$(i = 1, 2, 3, 4).$$

These equations give at once

$$(i, j) = (ij)' \frac{\partial (x', y')}{\partial (x, y)} \quad (i, j = 1, 2, 3, 4),$$

where $(ij)'$ has the same meaning with respect to the variables x', y' that (ij) has with respect to x, y .

From the identity (1) we therefore see that $w' = 0$ where

$$w' \equiv (23)' [14] + (31)' [24] + (12)' [34] + (14)' [23] + (24)' [31] + (34)' [12].$$

We thus have, in general, five unconnected equations to determine x, y, z, p, q in terms of $x', y', z', p', q', r', s', t'$; and the condition that

$$dz - p dx - q dy = 0$$

may be satisfied will lead therefore to two differential equations of the third order which $z' = f(x', y')$ must satisfy.

It is not difficult to prove that these two equations of the third order are compatible; and therefore we see that there are surfaces $z' = f(x', y')$, to each of which a single surface can be made to correspond by the four assigned equations.

Suppose, however, that the equation $w' = 0$ is such that from it and the four assigned equations it is possible to eliminate x, y, z, p, q : we may, by aid of these four equations, throw $w' = 0$ into an equation of Ampère's form in $x', y', z', p', q', r', s', t'$ only. If $z' = f(x', y')$ is any surface satisfying this, the equations obtained by eliminating x', y' from

$$u_1 = u_2 = u_3 = u_4 = 0$$

will be *in involution*; and any surface satisfying the corresponding Pfaffian equation

$$dz - p dx - q dy = 0$$

will also satisfy the equation $w = 0$ where

$$w \equiv (23) [14]' + (31) [24]' + (12) [34]' + (14) [23]' + (24) [31]' + (34) [12]'$$

If, now, $w = 0$ can also be thrown, by aid of the assigned four equations, into a form in which the variables x', y', z', p', q' do not occur, we have a Bäcklundian transformation enabling us to obtain, by the solution of an integrable Pfaffian equation, from any integral of one Ampèrian equation an integral of another (or it may be the same) equation of Ampère's form.

Of course, if the equations are arbitrarily assigned, they will give rise merely to a correspondence between certain surfaces and establish no Bäcklundian transformation between equations of Ampère's form.

If the equation $w' = 0$ should vanish identically, the four equations

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \quad u_4 = 0$$

will be four of the five equations required for a contact transformation, and in this case $w = 0$ will also be a mere identity.

A problem* the solution of which would be of interest would be : Given an equation of Ampère's form

$$Hr + Ks + Lt + M + N(rt - s^2) = 0, \quad (2)$$

to determine the forms of u_1, u_2, u_3, u_4 in order that the equations

$$u_1 = 0, \quad u_2 = 0, \quad u_3 = 0, \quad u_4 = 0$$

may have (2) as their w equation, and that the corresponding w' equation may be expressible in terms of x', y', z', p', q' only ; and, in particular, to find the form of the Ampèrian equation in order that the two Ampèrian equations may be the same, and thus have the property of having an infinity of integrals determined by one integral.

A particular case only of this problem will here be considered, viz., the determination of the forms u_1, u_2, u_3, u_4 , and F in order that the two Ampèrian equations may be each, in their own set of variables, of the form

$$s = F(x, y, z).$$

As the paper is rather long, I have divided it into five sections. In the first I prove that Bäcklund's transformation will not apply to any of the equations under consideration, unless at least one of the generating equations consists of a relation between the arguments x, y, z and x', y', z' only.

In the second section I use the theorem established in the first section to show that the generating equations can be thrown into the form

$$x' = f_1, \quad y' = f_2, \quad p' = f_3, \quad q' = f_4$$

* In connection with this problem see Clairin, *Ann. de l'Ecole Norm. Sup.*, 3e Sér., t. XIX. (1902), Supplement : and Goursat in a memoir, *Annales de Toulouse*, 2e Sér., t. IV (1902).

where f_1 is a function of x, y, z , and z' only, while f_2, f_3 , and f_4 are possibly functions of x, y, z, z', p , and q .

In the third section I prove that f_2 must also be independent of p and q .

In the fourth section I prove that by a simple transformation we may take the generating equations to be

$$x' = x, \quad y' = y, \quad p' = f_3(x, y, z, z', p), \quad q' = f_4(x, y, z, z', q).$$

Finally, in the fifth section I prove that, corresponding to these forms, transformations of the required type do exist, but only for equations reducible to one of the three forms

$$s = \sin z, \quad s = z,$$

and

$$s = \frac{\kappa(1-\kappa)z}{(x-y)^2}$$

where κ is a constant.

I.

1. First we shall consider the case where the four equations are such that from them no equation can be algebraically deduced connecting x, y, z and x', y', z' only.

They may therefore be thrown into the forms

$$p = f_1, \quad q = f_2, \quad p' = f_3, \quad q' = f_4$$

where f_1, f_2, f_3, f_4 are functions of the arguments x, y, z, x', y', z' only.

Let

$$X \equiv \frac{\partial}{\partial x} + f_1 \frac{\partial}{\partial z}, \quad Y \equiv \frac{\partial}{\partial y} + f_2 \frac{\partial}{\partial z}, \quad X' \equiv \frac{\partial}{\partial x'} + f_3 \frac{\partial}{\partial z'}, \quad Y' \equiv \frac{\partial}{\partial y'} + f_4 \frac{\partial}{\partial z'},$$

and let $a_{ij} \dots$ be a system of functions defined by

$$\begin{aligned} a_{11} &= Xf_1, & a_{12} &= Xf_2, & a_{13} &= Xf_3, & a_{14} &= Xf_4; \\ a_{21} &= Yf_1, & a_{22} &= Yf_2, & a_{23} &= Yf_3, & a_{24} &= Yf_4; \\ a_{31} &= X'f_1, & a_{32} &= X'f_2, & a_{33} &= X'f_3, & a_{34} &= X'f_4; \\ a_{41} &= Y'f_1, & a_{42} &= Y'f_2, & a_{43} &= Y'f_3, & a_{44} &= Y'f_4. \end{aligned}$$

It will now be found that $w = 0$ becomes

$$N(rt - s^2) + Hr + Ks + Lt + M = 0$$

and $w' = 0$ becomes

$$N'(r't' - s'^2) + H'r' + K's' + L't' + M' = 0$$

where

$$N = a_{34} - a_{43},$$

$$H = -a_{23}a_{42} - a_{22}(a_{34} - a_{43}) + a_{24}a_{32},$$

$$K = a_{23}a_{41} + a_{13}a_{42} + (a_{12} + a_{21})(a_{34} - a_{43}) - a_{24}a_{31} - a_{14}a_{32},$$

$$L = -a_{13}a_{41} - a_{11}(a_{34} - a_{43}) + a_{14}a_{31},$$

$$M = -a_{41}(a_{12}a_{23} - a_{13}a_{22}) - a_{42}(a_{13}a_{21} - a_{11}a_{23}) + (a_{34} - a_{43})(a_{11}a_{21} - a_{12}a_{21}) \\ - a_{32}(a_{11}a_{24} - a_{14}a_{21}) + a_{31}(a_{12}a_{24} - a_{14}a_{22})$$

and

$$N' = a_{12} - a_{21},$$

$$H' = -a_{41}a_{24} - a_{44}(a_{12} - a_{21}) + a_{42}a_{14},$$

$$K' = a_{41}a_{23} + a_{31}a_{24} + (a_{34} + a_{43})(a_{12} - a_{21}) - a_{42}a_{13} - a_{32}a_{14},$$

$$L' = -a_{31}a_{23} - a_{33}(a_{12} - a_{21}) + a_{32}a_{13},$$

$$M' = -a_{23}(a_{34}a_{41} - a_{31}a_{44}) - a_{24}(a_{31}a_{43} - a_{33}a_{41}) + (a_{12} - a_{21})(a_{33}a_{44} - a_{34}a_{43}) \\ - a_{14}(a_{33}a_{42} - a_{32}a_{43}) + a_{13}(a_{34}a_{42} - a_{32}a_{44}).$$

In order that N , H , L and N' , H' , L' may all vanish, we must have

$$a_{12} = a_{21}, \quad a_{34} = a_{43}, \quad a_{23}a_{42} = a_{32}a_{24}, \quad a_{13}a_{41} = a_{31}a_{14}, \quad a_{14}a_{42} = a_{41}a_{24},$$

and

$$a_{23}a_{31} = a_{13}a_{32}.$$

These lead to one of the following sets of relations:—

$$\frac{a_{23}}{a_{32}} = \frac{a_{24}}{a_{42}} = \frac{a_{13}}{a_{31}} = \frac{a_{14}}{a_{41}}, \quad a_{12} = a_{21}, \quad a_{34} = a_{43};$$

$$a_{32} = a_{23} = a_{41} = a_{14} = 0, \quad a_{12} = a_{21}, \quad a_{34} = a_{43};$$

$$a_{42} = a_{24} = a_{31} = a_{13} = 0, \quad a_{12} = a_{21}, \quad a_{34} = a_{43}.$$

Now the first set of relations leads to w and w' vanishing identically, and therefore to

$$p = f_1, \quad q = f_2, \quad p' = f_3, \quad q' = f_4,$$

being four of the equations of a contact transformation. We need not therefore consider this set at all. The third set of relations is obtained from the second by interchanging x' and y' and p' and q' ; so that we need only consider the second set, which gives, to determine f_1 , f_2 , f_3 , f_4 , the equations

$$\left(\frac{\partial}{\partial x'} + f_3 \frac{\partial}{\partial z'}\right)f_2 = 0, \quad \left(\frac{\partial}{\partial y'} + f_4 \frac{\partial}{\partial z'}\right)f_1 = 0,$$

$$\left(\frac{\partial}{\partial x} + f_1 \frac{\partial}{\partial z}\right)f_4 = 0, \quad \left(\frac{\partial}{\partial y} + f_2 \frac{\partial}{\partial z}\right)f_3 = 0,$$

$$\left(\frac{\partial}{\partial x} + f_1 \frac{\partial}{\partial z}\right)f_2 = \left(\frac{\partial}{\partial y} + f_2 \frac{\partial}{\partial z}\right)f_1, \quad \left(\frac{\partial}{\partial x'} + f_3 \frac{\partial}{\partial z'}\right)f_4 = \left(\frac{\partial}{\partial y'} + f_4 \frac{\partial}{\partial z'}\right)f_3.$$

If f_1, f_2, f_3, f_4 satisfy these equations, $w = 0$ becomes

$$s = a_{12} = \left(\frac{\partial}{\partial x} + f_1 \frac{\partial}{\partial z} \right) f_2$$

and $w' = 0$ becomes $s' = a_{24} = \left(\frac{\partial}{\partial x'} + f_3 \frac{\partial}{\partial z'} \right) f_4$,

provided that $a_{13}a_{42} - a_{24}a_{31}$ is not zero; if it were zero, w and w' would vanish identically.

We must therefore also have

$$\left(\frac{\partial}{\partial x} + f_1 \frac{\partial}{\partial z} \right) f_2 \equiv F(x, y, z) \quad \text{and} \quad \left(\frac{\partial}{\partial x'} + f_3 \frac{\partial}{\partial z'} \right) f_4 \equiv F(x', y', z'),$$

and, if all of these equations can be satisfied,

$$s = F(x, y, z)$$

will have the required property.

2. We now write $u = f_1$, $v = f_2$, and then we have, to define f_3 and f_4 ,

$$f_3 \frac{\partial v}{\partial z'} + \frac{\partial v}{\partial x'} = 0, \quad f_4 \frac{\partial u}{\partial z'} + \frac{\partial u}{\partial y'} = 0;$$

and from

$$\left(\frac{\partial}{\partial x} + f_1 \frac{\partial}{\partial z} \right) f_4 = 0$$

we see that

$$\frac{\partial \left(u, \frac{\partial u}{\partial x} \right)}{\partial (y', z')} + u \frac{\partial \left(u, \frac{\partial u}{\partial z} \right)}{\partial (y', z')} = 0.$$

This may be written in the form

$$\frac{\partial \left(u, \frac{\partial}{\partial x} \log u + \frac{\partial u}{\partial z} \right)}{\partial (y', z')} = 0;$$

and therefore $\frac{\partial}{\partial x} \log u + \frac{\partial u}{\partial z}$ is a function of u, x', x, y, z only; that is,

Xu is such a function.

Let

$$Xu = f(u, x', x, y, z).$$

From the equations

$$a_{12} = a_{21} = F(x, y, z)$$

we have

$$Yu = F(xyz) = Xv,$$

and we see that X and Y are commutative operators. It follows that

$$XF = XYu = YXu = Yf = F \frac{\partial f}{\partial u} + \frac{\partial f}{\partial y} + v \frac{\partial f}{\partial z}.$$

We shall now prove that f cannot contain z explicitly, though it may implicitly, as being a function of u .

If it did contain z explicitly, we should conclude from the above equations that v must be a function of x', u, x, y, z ; say

$$v = \phi(u, x', x, y, z).$$

Now $Y'u$ and $Y'x'$ are both zero, and therefore w is $Y'\phi$; that is, $Y'v$ is zero; that is $a_{42} = 0$. But $Y'F = 0$, and therefore $Y'Yu$ and $YY'u$ are both zero; so that

$$YY' - Y'Y = a_{24} \frac{\partial}{\partial z'} - a_{42} \frac{\partial}{\partial z}$$

annihilates u .

Now a_{42} is zero, and, as we saw that $a_{13}a_{42} - a_{24}a_{31}$ could not be zero, we cannot take a_{24} to be zero; we conclude that $\partial u / \partial z'$ must be zero.

Now, if $\partial u / \partial z'$ were zero, so would be $\partial u / \partial y'$, and therefore u depend only on x', x, y, z ; then from

$$\left(\frac{\partial}{\partial x} + v \frac{\partial}{\partial z} \right) u = Yu = F(x, y, z)$$

we should conclude either that v did not contain y' or z' , or that $\partial u / \partial z$ was zero.

If u did not contain z , we should then have

$$\frac{\partial u}{\partial y} = F(x, y, z),$$

and, as u does not contain z , neither can F . But, if F does not contain z , the equation $s = F$ is integrable, and as such has been excluded from consideration.

If v did not contain y' or z' , then we should have

$$p = f_1(x', x, y, z), \quad q = f_2(x', x, y, z),$$

and, eliminating x' , we should have a relation between x, y, z, p, q which could not possibly lead to a Bäcklundian transformation.

We conclude that f does not contain z explicitly, and therefore

$$Xu = f(u, x', x, y)$$

and

$$\frac{\partial F}{\partial x} + u \frac{\partial F}{\partial z} = F \frac{\partial f}{\partial u} + \frac{\partial f}{\partial y}. \quad (1)$$

It will now be shown that this equation must be, not a mere equation connecting u, x', x, y, z , but an identity in these arguments.

First we notice that by our assumption as to the form of the four equations

$$p = f_1, \quad q = f_2, \quad p' = f_3, \quad q' = f_4$$

defining the Bäcklundian transformation there can be no equation connecting x, y, z, x', y', z' ; for, by the definition of such a transformation, four, and only four, unconnected equations can exist between the coordinates of the two sets of elements.

The equation (1) then, if it is not a mere identity, must give u in terms of x', x, y, z only; but we have shown that such an hypothesis is inconsistent with the limitation placed on the proposed problem. We conclude that (1) is an identity in the arguments u, x', x, y, z .

3. Differentiating (1) with respect to u and z successively, we have

$$\frac{\partial^2 F}{\partial z^2} = \frac{\partial F}{\partial z} \frac{\partial^2 f}{\partial u^2},$$

and therefore, since by our limitations F must contain z , $\partial^2 f / \partial u^2$ does not contain u .

We therefore have $f = A + Bu + \frac{1}{2}Cu^2$

where, since f does not explicitly contain z , A , B , and C are functions of x, y , and x' only. Substituting this value of f in the identity (1),

$$\frac{\partial F}{\partial x} + u \frac{\partial F}{\partial z} \equiv (B + Cu)F + \frac{\partial A}{\partial y} + u \frac{\partial B}{\partial y} + \frac{1}{2}u^2 \frac{\partial C}{\partial y},$$

and therefore

$$\frac{\partial C}{\partial y} = 0, \quad \frac{\partial F}{\partial z} = CF + \frac{\partial B}{\partial y}, \quad \frac{\partial F}{\partial x} = BF + \frac{\partial A}{\partial y}. \quad (2)$$

From the second two of these equations we have, by the condition of compatibility,

$$F \frac{\partial C}{\partial x} + \frac{\partial^2 B}{\partial x \partial y} - B \frac{\partial B}{\partial y} + C \frac{\partial A}{\partial y} \equiv 0;$$

and, since F must contain z , we therefore conclude that

$$\frac{\partial C}{\partial x} = 0 \quad \text{and} \quad C \frac{\partial A}{\partial y} + \frac{\partial^2 B}{\partial x \partial y} - B \frac{\partial B}{\partial y} = 0.$$

Differentiating the identities (2) with respect to x' , we see that, since F contains z while A , B , and C do not, we must have

$$\frac{\partial C}{\partial x'} = 0, \quad \frac{\partial B}{\partial x'} = 0, \quad \frac{\partial^2 A}{\partial y \partial x'} = 0;$$

C is therefore a mere constant c , while B and $\partial A / \partial y$ are functions of x and y only.

4. First we take the case where c is not zero. From

$$\frac{\partial F}{\partial z} = cF + \frac{\partial B}{\partial y}$$

we have

$$F + \frac{1}{c} \frac{\partial B}{\partial y} = e^{cz} \phi(x, y)$$

where ϕ is some function of x and y only. From this, combined with

$$\frac{\partial F}{\partial x} = BF + \frac{\partial A}{\partial y} \quad \text{and} \quad c \frac{\partial A}{\partial y} + \frac{\partial^2 B}{\partial x \partial y} - B \frac{\partial B}{\partial y} = 0,$$

we see that

$$B\phi = \frac{\partial \phi}{\partial x}.$$

and therefore

$$F + \frac{1}{c} \frac{\partial^2}{\partial x \partial y} \log \phi = e^{c(z + 1/c \log \phi)};$$

so that the Ampèrian equation is

$$\frac{d^2}{dx dy} \left(z + \frac{1}{c} \log \phi \right) = e^{c(z + 1/c \log \phi)}.$$

The equation is therefore reducible by a point transformation to the form

$$s = e^{cs}$$

where c is a constant.

This is Liouville's equation, of which the solution is known (Forsyth, *Differential Equations*, 1885 edition, p. 421), and therefore is not to be considered.

5. Next we take the case where c is zero, and we have

$$\frac{\partial F}{\partial z} = \frac{\partial B}{\partial y}, \quad \frac{\partial^2 B}{\partial x \partial y} = B \frac{\partial B}{\partial y}.$$

It is easily seen from the second of these equations that

$$B = \frac{a''}{a'} + \frac{2a'}{\beta - a}$$

where a is a function of x only and β of y only, and the dashes denote the function differentiated with respect to its argument.

$$\text{We therefore have} \quad \frac{\partial F}{\partial z} = -\frac{2a'\beta'}{(\beta - a)^2};$$

$$\text{so that the equation is} \quad s = -\frac{2a'\beta'}{(\beta - a)^2} z + \phi(x, y).$$

If $z = z_1$, where z_1 is a function of x and y , is a particular integral of this equation, by a transformation $z' = z - z'$ we can deprive the equation of the function ϕ ; and then, by taking

$$y' = \beta, \quad x' = \alpha,$$

we can bring it to the form

$$s = - \frac{2z}{(y-x)^2}.$$

Now this equation has (Darboux, *Surfaces*, Vol. II., p. 143) the integral

$$z = 2(\alpha - \beta) / (x - y) - \alpha' - \beta'$$

where α is an arbitrary function of x and β of y ; and therefore may also be excluded from further consideration.

The general conclusion is therefore that there is no Bäcklundian transformation of the type required, and, if a transformation exists, it must include as one of its "generating" equations a relation between x, y, z, x', y', z' alone.

II.

6. We now know that the four generating equations of the Bäcklundian transformation which leaves $s = F(x, y, z)$ unaltered must involve an equation connecting x, y, z and x', y', z' only.

Let them be thrown into the forms

$$x' = f_1, \quad y' = f_2, \quad p' = f_3, \quad q' = f_4,$$

where f_1, f_2, f_3 , and f_4 are functions of the arguments x, y, z, p, q , and z' . It is always possible to throw the equations into these forms unless they involve a relation between those arguments. That such a relation cannot exist in the Bäcklundian transformation we are considering may be proved formally; I do not, however, give the proof here, lest it should add unnecessarily to the length of a paper already long, in that its nature is not essentially different from the methods of proof now to be employed.

The generating equations being now in the assumed form, the first Ampèrian equation is found to be

$$f_4 \dot{f}_1 (23) + (f_4 \dot{f}_2 - 1) (31) + (f_4 \dot{f}_3 - f_3 \dot{f}_4) (12) + f_3 \dot{f}_2 (14) + (1' - f_3 \dot{f}_1) (24) = 0$$

where $\dot{f}$ denotes the differential coefficient of f with respect to z' .

The second Ampèrian equation is

$$\begin{aligned}
 (r't' - s'^2)[12] + r' \{ (1 - f_4 \dot{f}_2)[41] + f_4 \dot{f}_1[42] - f_4 \dot{f}_4[12] \} \\
 + t' \{ (1 - f_3 \dot{f}_1)[23] + f_3 \dot{f}_2[13] - f_3 \dot{f}_3[12] \} \\
 + s' \{ f_3 \dot{f}_2[41] + (f_3 \dot{f}_1 - 1)[24] + f_4 \dot{f}_1[23] + (f_4 \dot{f}_2 - 1)[31] \\
 + (f_3 \dot{f}_4 + f_4 \dot{f}_3)[12] \} \\
 + f_3 \dot{f}_3[14] + f_4 \dot{f}_3[24] + (1 - f_3 \dot{f}_1 - f_4 \dot{f}_2)[34] \\
 + f_4 \dot{f}_4[32] + f_3 \dot{f}_4[31] = 0.
 \end{aligned}$$

Since the coefficients of $r't' - s'^2$, r' and t' , must be zero,

$$[12] = 0, \quad (1 - f_4 \dot{f}_2)[41] + f_4 \dot{f}_1[42] = 0, \quad (1 - f_3 \dot{f}_1)[23] + f_3 \dot{f}_2[13] = 0.$$

By aid of these and the identity

$$(23)[14] + (31)[24] + (12)[34] + (14)[23] + (24)[31] + (34)[12] \equiv 0,$$

the first Ampèrian equation gives

$$\begin{aligned}
 \{ [23] f_4 \dot{f}_1 + [41] f_3 \dot{f}_2 \} \{ (14)[23] + (24)[31] \} \\
 + [23] \{ [41] (f_4 \dot{f}_3 - f_3 \dot{f}_4) + [34] f_4 \dot{f}_1 \} (12) = 0. \quad (3)
 \end{aligned}$$

Conversely, if this equation holds, so also does the first Ampèrian equation, unless at least one of the following sets of conditions holds,

$$\begin{aligned}
 [23] = 0, \quad \dot{f}_2 = 0; \quad [41] = 0, \quad \dot{f}_1 = 0; \\
 [23] = 0, \quad [31] = 0; \quad [41] = 0, \quad [42] = 0, \quad (4)
 \end{aligned}$$

when (3) becomes a mere identity.

7. As the generating equations must involve a relation between x, y, z and x', y', z' only, $(12)_{PQ}$ is zero. Our object will now be to prove that neither f_1 nor f_2 can involve p or q , and first we assume that both do involve one or other of these arguments.

Suppose f_1 contains p , say: then, since $(12)_{PQ}$ is zero, if Pf_2 were zero, Qf_2 would be zero; and, as this is contrary to our hypothesis, we must suppose that f_1 and f_2 both contain p .

We shall now prove that an Ampèrian equation of the kind we are concerned with cannot be of the form

$$a(12) + b(13) = 0 \quad (5)$$

where a and b are any functions of x, y, z, p, q , and z' .

Since $(12)_{PQ} = 0$, we have $b(13)_{PQ} = 0$. We cannot take b to be zero; for the equation would then reduce to $(12) = 0$, an equation which, as being soluble by Monge's method, falls without the scope of this investigation.

We then have $(13)_{PQ} = 0$, and therefore, since $(12)_{PQ} = 0$, and f_1 contains p , $(23)_{PQ} = 0$; so that, μ being some function of x, y, z, p, q , and z' , $Q - \mu P$ annihilates f_1, f_2 , and f_3 .

Since the coefficients of r and t are to be zero in (5), and

$$(12)_{QY} = \mu(12)_{PY}, \quad (13)_{QY} = \mu(13)_{PY},$$

$$\text{we have} \quad a(12)_{QY} + b(13)_{QY} = 0, \quad a(12)_{QX} + b(13)_{QX} = 0;$$

and therefore, unless μ is zero, we also have

$$a(12)_{PX} + b(13)_{PX} = 0.$$

But this would make the coefficient of s zero; so that we must take μ to be zero.

Since f_1, f_2, f_3 do not now contain q , the equation takes the form

$$s \{a(12)_{XP} + b(13)_{XP}\} + a(12)_{XY} + b(13)_{XY} = 0$$

where

$$a(12)_{PY} + b(13)_{PY} = 0;$$

and $[12] = 0$ gives $X - \lambda P$ as an annihilator of f_1 and f_2 where λ is some function of x, y, z, p , and z' .

Noting that

$$(12)_{XY} = \lambda(12)_{PY},$$

we have

$$(sPf_1 + Yf_1)(X - \lambda P)f_3 = 0.$$

We cannot take $(X - \lambda P)f_3 = 0$; for that would reduce the equation to a mere identity. If, then, the Ampèrian equation is

$$s = F(x, y, z),$$

$FP + Y$ and Q would both be annihilators of f_1 ; and therefore also $\partial/\partial z$, and so $\partial F/\partial z \cdot P$. But P does not annihilate f_1 , and $\partial F/\partial z = 0$ would mean that the Ampèrian equation was integrable, and therefore outside our investigation.

Similarly we see that there can be no equation of the form

$$a(12) + b(23) = 0.$$

8. Suppose now that we have an equation of the form.

$$a(23) + b(31) + c(12) = 0, \quad (6)$$

and that $(13)_{PQ}$ is not zero. Since the coefficient of $rt - s^2$ is to be zero,

and $Q - \lambda P$ annihilates f_1 and f_2 but not f_3 , we must have

$$aPf_2 = bPf_1.$$

Since the coefficient of r must also be zero, the equation (6) takes the form

$$Pf_1(23) + Pf_2(31) + Pf_3(12) = 0, \quad (7)$$

and the vanishing of the coefficient of t in this gives

$$\begin{vmatrix} Pf_1 & Qf_1 & Xf_1 \\ Pf_2 & Qf_2 & Xf_2 \\ Pf_3 & Qf_3 & Xf_3 \end{vmatrix} = 0.$$

This, combined with the fact that $(12)_{PQ}$ is zero but not $(13)_{PQ}$, gives $(12)_{PX}$ and $(12)_{QX}$ both equal to zero. Since $[12] = 0$, we also have $(12)_{QY} = 0$, and thus, unless $Qf_1 = Qf_2 = 0$, we should have $(12) = 0$. As we cannot have this, we must take $Qf_1 = Qf_2 = 0$. Since $[12] = 0$, we may say that $X - \lambda P$ annihilates f_1 and f_2 where λ is some function of x, y, z, p , and z' .

The equation (6) now takes the form

$$(sQf_3 + Xf_3 - \lambda Pf_3)(12)_{YP} = 0.$$

Now, we cannot ever have $(12)_{YP} = 0$; for it gives $(12)_{YQ} = 0$; and therefore, since $(12)_{XP} + (12)_{YQ} = 0$, we have $(12)_{XP} = 0$; but

$$(12)_{YP} = (12)_{XP} = 0$$

gives, since Pf_1 is not zero, $(12)_{XY} = 0$ and thus $(12) = 0$.

If, therefore, the Ampèrian equation is

$$s = F(x, y, z),$$

we see that

$$FQ + X - \lambda P$$

annihilates f_3 .

9. Returning now to the equation (3), assume for the present that none of the conditions (4) hold. We cannot then have

$$[23]f_4\dot{f}_1 + [41]f_3\dot{f}_2 = 0;$$

for that would reduce the first Ampèrian equation to the form $(12) = 0$.

We therefore have $[23](14)_{PQ} + [31](24)_{PQ} = 0$.

Let $Q - \mu P$ annihilate f_1 and f_2 ; then

$$\{[23]Pf_1 + [31]Pf_2\} (Q - \mu P)f_4 = 0$$

$$\text{and } [23]Pf_1 + [31]Pf_2 = Pf_3(12)_{PX} + Qf_3(12)_{PY} = (12)_{PY}(Q - \mu P)f_3.$$

It follows, since $(12)_{PY}$ cannot be zero, that $Q - \mu P$ must annihilate f_3 or f_4 .

We therefore take f_3 to be so annihilated—if we had chosen to take f_4 instead of f_3 , we should have used, instead of (3), the corresponding equation obtained by interchanging the suffixes 3 and 4 and also 1 and 2.

We now have $(12)_{PQ} = (13)_{PQ} = (23)_{PQ} = 0$, and cannot therefore have $(14)_{PQ} = 0$; in fact, if we had, we should be able to obtain from the generating equations three equations not containing p or q and thus be able to express z in terms of $x', y', z', p',$ and q' only; that we cannot so do was the assumption made at the beginning of this section.

The equation (3) is now of the form

$$(24)[31] + (41)[32] + c(12) = 0$$

$$\text{where } c = [23]\{(f_4\dot{f}_3 - f_3\dot{f}_4)[41] + f_4\dot{f}_1[34]\} / \{f_4\dot{f}_1[23] + f_3\dot{f}_2[41]\},$$

and $(14)_{PQ}$ is not zero. It follows that Q annihilates $f_1, f_2,$ and f_3 , and that

$$FQ + X - \lambda P$$

annihilates $f_1, f_2,$ and f_4 .

We now prove that neither $\dot{f}_1$ nor $\dot{f}_2$ can be zero. If $\dot{f}_1$ were zero, we could not have $[41]$ zero; for, by hypothesis, the conditions (4) do not hold; we must therefore have $f_4\dot{f}_2 - 1 = 0$; f_4 does not therefore contain q . But this we have just seen is impossible, since $(14)_{PQ}$ is not zero.

If $\dot{f}_2 = 0$, then $f_3\dot{f}_1 = 1$; since f_2 has the two annihilators $\partial/\partial z'$ and $FQ + X - \lambda P$, and not P , we see by taking the alternant that $\lambda = 0$. It follows that $FQ + X - \lambda P$ annihilates $\dot{f}_1$, and therefore also f_3 . Since f_3 does not contain q , this would make $[23] = 0$, and thus, contrary to hypothesis, one of the conditions (4) holds.

Now, if neither $\dot{f}_1$ nor $\dot{f}_2$ is zero, the second Ampèrian equation takes the form

$$s' \{[41]f_3\dot{f}_2 + [23]f_4\dot{f}_1\} - f_3f_4 \{[23]\dot{f}_4\dot{f}_1 + [34]\dot{f}_1\dot{f}_2 + [41]\dot{f}_2\dot{f}_3\} = 0$$

where $1 - f_3\dot{f}_1 - f_4\dot{f}_2$ must not be zero; that is,

$$s' - f_3\dot{f}_4 = cf_3\dot{f}_2/[23]. \quad (7')$$

By (7) we have

$$cPf_2 = [32]Pf_4; \quad (7'')$$

and therefore, the second Ampèrian equation being

$$s' = F(f_1, f_2, z') = F', \text{ say,}$$

$$F'PF_2 = f_3(\dot{f}_4Pf_2 - \dot{f}_2Pf_4).$$

Also, since

$$(1 - f_3\dot{f}_1)[23] + f_3\dot{f}_2[13] = 0,$$

we have

$$Pf_2 = f_3(\dot{f}_1Pf_2 - \dot{f}_2Pf_1);$$

and therefore

$$F'(12)_{z'P} + (24)_{z'P} = 0, \quad (8)$$

where we write, for $(12)_{\partial/\partial z', P}$, simply $(12)_{z'P}$.

Let $F'Q + X - \lambda P$ be T ; then

$$T \frac{\partial}{\partial z'} - \frac{\partial}{\partial z'} T = \dot{\lambda} P, \quad T \frac{\partial}{\partial z} - \frac{\partial}{\partial z} T = \frac{\partial \lambda}{\partial z} P - \frac{\partial F}{\partial z} Q,$$

$$TP - PT = P\lambda \cdot P - \frac{\partial}{\partial z};$$

and, noting that, as T destroys f_1 and f_2 , it must destroy F' , and that

$$T(12)_{z'P} = P\lambda(12)_{z'P} - (12)_{z'z}, \quad T(24)_{z'P} = P\lambda(24)_{z'P} - (24)_{z'z},$$

we have

$$F'(12)_{z'z} + (24)_{z'z} = 0. \quad (9)$$

It follows that

$$F'(12)_{Pz} + (24)_{Pz} = 0. \quad (10)$$

Operate again with T on (9), and, since

$$T(12)_{z'z} = \dot{\lambda}(12)_{Pz} + \frac{\partial \lambda}{\partial z}(12)_{z'P},$$

$$T(24)_{z'z} = \dot{\lambda}(24)_{Pz} + \frac{\partial \lambda}{\partial z}(24)_{z'P} - \frac{\partial F}{\partial z}\dot{f}_2Qf_4,$$

we get

$$\frac{\partial F}{\partial z}\dot{f}_2Qf_4 = 0.$$

But this is impossible, for $\partial F/\partial z = 0$ would mean that the Ampèrian equation was integrable, and we have proved that $\dot{f}_2$ is not zero, and f_4 not annihilated by Q .

We conclude that, if none of the conditions (4) hold, either f_1 or f_2 is independent of p and q .

10. We must now examine the cases where some of the sets of conditions (4) hold. Let these conditions be respectively referred to as A , B , C , and D . If A holds, we see at once that, unless $[42]$ is zero, the

first Ampèrian equation would be of the form

$$a(12) + b(24) = 0,$$

which we have seen is impossible.

On the other hand, if $[42] = 0$, we have the condition D .

We thus need only consider the condition A in conjunction with the condition D . Similarly B in conjunction with C . We cannot have C and D together; for that would make the coefficient of s' zero.

We shall now show that C cannot hold without B . Assume that C holds, but not B ; then A cannot hold, for it leads to D , and we have, from

$$[23]Pf_1 + [31]Pf_2 = (12)_{PY}(Q - \mu P)f_3,$$

that

$$(13)_{PQ} = 0,$$

and therefore $(14)_{PQ}$ is not zero. The Ampèrian equation is now

$$\{f_4\dot{f}_1[34] + (f_4\dot{f}_3 - f_3\dot{f}_4)[41]\}(12) + (24)[41](1 - f_3\dot{f}_1) + (41)[14]f_3\dot{f}_2 = 0.$$

We cannot have $\dot{f}_1$ zero; for, since B does not hold, that would mean

$$1 - f_4\dot{f}_2 = 0;$$

and, from the form of the Ampèrian equation and the fact that $(14)_{PQ}$ is not zero, f_2 does not contain q ; so that neither would f_4 , which is absurd.

Nor can we have $\dot{f}_2$ zero; for that would reduce the equation to the form

$$a(12) + b(24) = 0.$$

Since neither $\dot{f}_1$ nor $\dot{f}_2$ is zero, the second Ampèrian equation is, by (7)' and (7)'',

$$s' = f_3\dot{f}_4 - f_3\dot{f}_2Pf_4/Pf_2,$$

and we thus have

$$F'Pf_2 = f_3(24)_{Pq}. \quad (11)$$

Now, f_1 , f_2 , and f_4 are annihilated by T , as is also f_3 , since

$$[23] = [31] = 0.$$

Operate therefore with T , and we shall obtain

$$F'\frac{\partial f_2}{\partial z} = f_3(24)_{zz}; \quad (12)$$

and therefore, since $\dot{f}_2$ is not zero,

$$(24)_{Pz} = 0. \quad (13)$$

Operating on (13) again with T , we obtain

$$\dot{f}_2\frac{\partial F}{\partial z}Qf_4 = 0.$$

We thus see that C and B must go together, and similarly we see that D and A must. Taking B and C together, the identity becomes

$$[24](31) + [34](12) = 0,$$

and, as we cannot have $[24] = 0$, we must have $(13)_{PQ} = 0$; and the first Ampèrian equation becomes

$$\{[24](f_4\dot{f}_3 - f_3\dot{f}_4) + [34](1 - f_4\dot{f}_2)\}(12) + [24](24) + [42]f_3\dot{f}_2(41) = 0,$$

whilst the second is

$$s'[24] = [24]f_4\dot{f}_3 + (1 - f_4\dot{f}_2)[34].$$

As before, we thus obtain the equation

$$F'Pf_2 = f_3(24)_{Pz},$$

and by operating with T see that this would lead to the inadmissible equation $\dot{f}_2 = 0$. Similarly for the cases A and D in conjunction. We thus conclude that either f_1 is independent of p and q , or else f_2 is.

We may take f_1 therefore to be a function of x, y, z , and z' only.

III.

11. Our next object is to prove that f_2 cannot contain p or q : we therefore assume that it contains p ; a similar method would be applied if we were to assume that it contains q .

Our first step will be to prove that one of the conditions A, B, C, D must now hold. Suppose that none of them hold; then, since the equation (3) is now equivalent with the first Ampèrian one, we must have

$$(14)_{PQ}[23] + (24)_{PQ}[31] = 0;$$

for we cannot have $[23]f_4\dot{f}_1 + [41]f_3\dot{f}_2 = 0$.

We then have $(24)_{PQ}[31] = 0$. If $(24)_{PQ}$ were zero, $Q - \mu P$ would annihilate f_2 and f_4 (since f_2 is not independent of p), and therefore $[14] = 0$ would be a consequence of $[12] = 0$. But $[14] = 0$ must lead to condition B or D . We thus have $[31] = 0$.

Since $[31] = 0$ and the conditions (4) do not hold, by equating the coefficients of r and t to zero, we see that

$$\begin{vmatrix} (14)_{PY}, & (12)_{PY} \\ (14)_{QX}, & (12)_{QX} \end{vmatrix} = 0;$$

and therefore, since $(24)_{PQ}$ is not zero, Xf_1 is zero or Yf_1 is. If Xf_1 were

zero, f_1 would depend on y and z' only, and, as $[13] = 0$ must now lead to $1 - f_3 \dot{f}_1 = 0$, the same is true for f_3 . This would mean a relation between x' , p' , and z' which is, of course, impossible. Similarly for $Yf_1 = 0$.

12. We must therefore assume one of the conditions A , B , C , D .

Assume that B does not hold.

If C holds, D cannot, as it would make the coefficient of s' zero. From $[31] = [12] = 0$, we see that $(23)_{PQ} = 0$, and therefore, from the first Ampèrian equation,

$$(1 - f_3 \dot{f}_1)(24)_{PQ} = 0.$$

To take $(24)_{PQ} = 0$ would lead to $[14] = 0$, and thus to B or D ; so that we must take $1 - f_3 \dot{f}_1 = 0$. From $[23] = 0$, this gives us

$$Xf_3 Pf_2 + Yf_3 Qf_2 = 0;$$

and therefore $(13)_{XY} = 0$. As f_1 and f_3 contain neither p nor q , this would give $(13) = 0$, which is impossible.

Similarly, we see that, if D holds, we must have, from $[14] = 0$ and $[12] = 0$, that $[24]_{PQ} = 0$; and therefore, from the Ampèrian equation,

$$f_3 \dot{f}_1 (23)_{PQ} = 0.$$

As B does not hold, we have $(23)_{PQ} = 0$; and therefore, as Pf_2 is not zero, $(34)_{PQ} = 0$.

But this would mean that three equations could be obtained independent of p and q , and this we have proved is impossible.

Suppose, now, that A holds: then, from the Ampèrian equation and the identity, we have

$$f_4 \dot{f}_1 (23)_{PQ} + (1 - f_3 \dot{f}_1)(24)_{PQ} = 0, \quad [14](23)_{PQ} + [31](24)_{PQ} = 0;$$

and, as we cannot have $(23)_{PQ} = (24)_{PQ} = 0$, we must have

$$f_4 \dot{f}_1 [31] + (1 - f_3 \dot{f}_1)[41] = 0.$$

But this combined with A and

$$f_4 \dot{f}_1 [42] + (1 - f_4 \dot{f}_2)[41] = 0$$

would lead to $(12) = 0$ or a mere identity, instead of an equation of the kind we require.

We must therefore take the condition B . With this we cannot have $[31] = 0$; for it would lead to $(23)_{PQ} = 0$, and thus, through the Ampèrian equation, to $(24)_{PQ} = 0$, which we have seen is impossible.

13. The Ampèrian equation now takes the form

$$a(13) - b(12) = 0,$$

where $a = (1 - f_4 \dot{f}_2)[31] + [24]$, $b = (f_3 \dot{f}_4 - f_4 \dot{f}_3)[31] + [34]$,

and $[14] = \dot{f}_1 = 0$, $[23] + f_3 \dot{f}_2[13] = 0$,

but $[13]$ is not zero.

Since the coefficients of r and t must vanish, we have

$$Xf_1(aQf_3 - bQf_2) = Yf_1(aPf_3 - bPf_2) = 0.$$

If neither Xf_1 nor Yf_1 were zero, we should have $(23)_{PQ} = 0$ and so $[13] = 0$. We must therefore have $Xf_1 = 0$ or $Yf_1 = 0$. If $Yf_1 = 0$, we have, from $[12] = 0$, that $Pf_2 Xf_1 = 0$; but, by hypothesis, Pf_2 is not zero, and we cannot have $Xf_1 = Yf_1 = 0$; for then would f_1 be a mere constant.

We must therefore take $Xf_1 = 0$ and so $Qf_2 = 0$. From $[14] = 0$, we now see that f_2 and f_4 are independent of q , and we also have

$$aPf_3 = bPf_2;$$

and, since $[13]$ is not zero, f_3 must contain q .

We may take $f_1 \equiv y$, and we find for the Ampèrian equation

$$sPf_2Qf_3 + Pf_2Xf_3 - Pf_3Xf_2 = 0,$$

which, by aid of $[23] + f_3 \dot{f}_2[13] = 0$, becomes

$$sPf_2 + f_3 \dot{f}_2 + Yf_2 = 0.$$

If this is to coincide with $s = F(x, y, z)$,

we see that $FP + Y + f_3 \frac{\partial}{\partial z'}$

annihilates f_2 .

We cannot have $\dot{f}_2 = 0$; for then would both $FP + Y$ and Q be annihilators of f_2 and we should thus be led to $\partial F / \partial z = 0$. It follows that $f_3 = f + q\phi$, where f and ϕ are functions of the arguments x, y, z, p , and z' only, and that f_2 has the two annihilators

$$F \frac{\partial}{\partial p} + \frac{\partial}{\partial y} + f \frac{\partial}{\partial z'} \quad \text{and} \quad \frac{\partial}{\partial z} + \phi \frac{\partial}{\partial z'}.$$

Equating to zero the coefficient of q in $[23] + f_3 \dot{f}_2[13] = 0$, we get

$$Xf_2 P\phi = X\phi Pf_2.$$

The second Ampèrian equation is

$$s' \{ [42] + (f_4 \dot{f}_2 - 1) [31] \}$$

$$+ f_3 \dot{f}_4 [31] + f_4 \dot{f}_3 [24] + (1 - f_4 \dot{f}_2) [34] + f_4 \dot{f}_4 [32] = 0.$$

Expressing [34] and [24] in terms of a , b , and [31], this becomes

$$a(s' - f_4 \dot{f}_3) = b(1 - f_4 \dot{f}_2).$$

The relation $aPf_3 = bPf_2$ gives, on substituting for a and b their values and making use of $[23] + f_3 \dot{f}_2 [13] = 0$,

$$Yf_4Pf_2 = Pf_4(Yf_2 + f_3\dot{f}_2) + (f_4\dot{f}_3 - f_3\dot{f}_4)Pf_2 + (1 - f_4\dot{f}_2)Pf_3. \quad (14)$$

If the second Ampèrian equation is therefore to coincide with

$$s' = F(y, f_2, z') = F'' \quad \text{say,}$$

we must have

$$F'' = \left(FP + Y + f_3 \frac{\partial}{\partial z'} \right) f_4.$$

Since F'' and f_4 do not contain q , we must have f_4 also annihilated by

$$\frac{\partial}{\partial z} + \phi \frac{\partial}{\partial z'}.$$

It follows that f_4 is a function of f_2 , x , y , and p only: let

$$f_4 = \Phi(f_2, x, y, p).$$

14. Since Φ is to be regarded as the function of the arguments f_2 , x , y , and p , we shall understand by $\partial\Phi/\partial p$, for instance, the partial coefficient with respect to p when x , y , and f_2 are kept constant; so that

$$Pf_4 = P\Phi = \frac{\partial\Phi}{\partial f_2} Pf_2 + \frac{\partial\Phi}{\partial p}.$$

We therefore have

$$F'' = F \frac{\partial\Phi}{\partial p} + \frac{\partial\Phi}{\partial y}, \quad (15)$$

since $FP + Y + f_3 \frac{\partial}{\partial z'}$ annihilates f_2 .

Differentiating this equation with respect to z' and z , we get

$$\frac{\partial F''}{\partial z'} = \left(F \frac{\partial^2\Phi}{\partial p \partial f_2} + \frac{\partial^2\Phi}{\partial y \partial f_2} - \frac{\partial F''}{\partial f_2} \right) \frac{\partial f_2}{\partial z'}, \quad (16)$$

$$- \frac{\partial F}{\partial z} \frac{\partial\Phi}{\partial p} = \left(F \frac{\partial^2\Phi}{\partial p \partial f_2} + \frac{\partial^2\Phi}{\partial y \partial f_2} - \frac{\partial F''}{\partial f_2} \right) \frac{\partial f_2}{\partial z}; \quad (17)$$

so that

$$\phi \frac{\partial F''}{\partial z'} = \frac{\partial F}{\partial z} \frac{\partial\Phi}{\partial p}.$$

It follows that

$$\begin{aligned} X \left(\frac{\partial F}{\partial z} \frac{\partial \Phi}{\partial p} \right) / P \left(\frac{\partial F}{\partial z} \frac{\partial \Phi}{\partial p} \right) \\ = \left(\frac{\partial^2 F'}{\partial f_2 \partial z'} \phi X f_2 + \frac{\partial F'}{\partial z'} X \phi \right) / \left(\frac{\partial^2 F'}{\partial f_2 \partial z'} \phi P f_2 + \frac{\partial F'}{\partial z'} P \phi \right) = \frac{X f_2}{P f_2}, \end{aligned}$$

since

$$X f_2 P \phi = X \phi P f_2;$$

that is,

$$\left(\frac{\partial F}{\partial z} \left(\frac{\partial^2 \Phi}{\partial p \partial x} + \frac{\partial^2 \Phi}{\partial p \partial f_2} X f_2 \right) + \frac{\partial \Phi}{\partial p} X \frac{\partial F}{\partial z} \right) / \frac{\partial F}{\partial z} \left(\frac{\partial^2 \Phi}{\partial p^2} + \frac{\partial^2 \Phi}{\partial p \partial f_2} P f_2 \right) = \frac{X f_2}{P f_2};$$

so that

$$\left(\frac{\partial F}{\partial z} \frac{\partial^2 \Phi}{\partial p \partial x} + \frac{\partial \Phi}{\partial p} \frac{\partial}{\partial z} X F \right) / \frac{\partial F}{\partial z} \frac{\partial^2 \Phi}{\partial p^2} = \frac{X f_2}{P f_2}.$$

Operating with X and P on (15), we have

$$\frac{\partial^2 \Phi}{\partial x \partial y} + F \frac{\partial^2 \Phi}{\partial p \partial x} + \frac{\partial \Phi}{\partial p} X F + \left(F \frac{\partial^2 \Phi}{\partial p \partial f_2} + \frac{\partial^2 \Phi}{\partial y \partial f_2} - \frac{\partial F'}{\partial f_2} \right) X f_2 = 0, \quad (18)$$

$$\frac{\partial^2 \Phi}{\partial p \partial y} + F \frac{\partial^2 \Phi}{\partial p^2} + \left(F \frac{\partial^2 \Phi}{\partial p \partial f_2} + \frac{\partial^2 \Phi}{\partial y \partial f_2} - \frac{\partial F'}{\partial f_2} \right) P f_2 = 0, \quad (19)$$

and we thus obtain

$$\begin{aligned} \left(\frac{\partial^2 \Phi}{\partial p \partial y} + F \frac{\partial^2 \Phi}{\partial p^2} \right) \left(\frac{\partial F}{\partial z} \frac{\partial^2 \Phi}{\partial p \partial x} + \frac{\partial \Phi}{\partial p} \frac{\partial}{\partial z} X F \right) \\ = \frac{\partial F}{\partial z} \frac{\partial^2 \Phi}{\partial p^2} \left(\frac{\partial^2 \Phi}{\partial x \partial y} + F \frac{\partial^2 \Phi}{\partial p \partial x} + \frac{\partial \Phi}{\partial p} X F \right). \quad (20) \end{aligned}$$

Now this equation involves only the arguments p , x , y , f_2 , and z' , and, unless it is an identity, we could by means of it express f_2 in terms of x , y , z , and p ; but, as we have proved that f_2 is not zero, this is impossible, and therefore it is an identity.

We shall now obtain an identity in the arguments x , y , p , f_2 , and z which will also be required. Equating to zero the coefficient of q in (14) we get

$$\left\{ f_4 P f_2 \frac{\partial}{\partial z'} + (1 - f_4 f_2) P \right\} \log \phi = 0,$$

since $\frac{\partial}{\partial z} + \phi \frac{\partial}{\partial z'}$ annihilates both f_2 and f_4 .

Eliminating $P f_2 / (1 - f_4 f_2)$ by means of (16) and (19), we have

$$P f_2 / (1 - f_4 f_2) = \left(\frac{\partial^2 \Phi}{\partial p \partial y} + F \frac{\partial^2 \Phi}{\partial p^2} \right) / \left(\frac{\partial F'}{\partial f_2} - F \frac{\partial^2 \Phi}{\partial p \partial f_2} - \frac{\partial^2 \Phi}{\partial y \partial f_2} + \Phi \frac{\partial F'}{\partial z'} \right);$$

and, since $\phi = \frac{\partial F}{\partial z} \frac{\partial \Phi}{\partial p} / \frac{\partial F'}{\partial z'}$ and $F' = F \frac{\partial \Phi}{\partial p} + \frac{\partial \Phi}{\partial y}$,

we thus obtain the required identity

$$\begin{aligned} & \left(\frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial y \partial p} + F' \frac{\partial^2 \Phi}{\partial p^2} - \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial p^2} \right) \left(\Phi \frac{\partial \Phi}{\partial p} \frac{\partial^2 F'}{\partial z'^2} + \frac{\partial \Phi}{\partial p} \frac{\partial^2 F'}{\partial z' \partial f_2} - \frac{\partial F'}{\partial z'} \frac{\partial^2 \Phi}{\partial p \partial f_2} \right) \\ &= \frac{\partial F'}{\partial z'} \frac{\partial^2 \Phi}{\partial p^2} \left(\frac{\partial F'}{\partial f_2} \frac{\partial \Phi}{\partial p} + \Phi \frac{\partial \Phi}{\partial p} \frac{\partial F'}{\partial z'} + \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial p \partial f_2} - \frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial y \partial f_2} - F' \frac{\partial^2 \Phi}{\partial p \partial f_2} \right). \end{aligned} \quad (21)$$

Also, from $[23] + f_3 \dot{f}_2 [13] = 0$, we have, since the terms containing q vanish,

$$Pf Xf_2 = Pf_2(Xf + \phi F).$$

15. The investigation now divides into two parts corresponding to the cases $\partial^2 \Phi / \partial p^2 = 0$ and $\partial^2 \Phi / \partial p^2 \neq 0$.

Taking the first, we have

$$\frac{\partial F}{\partial z} \frac{\partial^2 \Phi}{\partial p \partial x} + \frac{\partial \Phi}{\partial p} X \frac{\partial F}{\partial z} = 0, \quad \frac{\partial^2 \Phi}{\partial p^2} = 0.$$

It is easily seen that $\partial \Phi / \partial p = 0$ is inadmissible—in fact, it would at once give $\partial F' / \partial z' \cdot \phi = 0$. Since, then, $\partial \Phi / \partial p$ is independent of p but not zero, $\partial^2 F' / \partial z'^2$ must be zero.

Since an equation of the form

$$s = z\theta(x, y) + \phi(x, y)$$

can by a simple transformation be deprived of the term ϕ on the right, we may take ϕ to be zero, and Φ is then given by

$$\frac{\partial}{\partial x} \left(\theta \frac{\partial \Phi}{\partial p} \right) = 0;$$

so that $\theta \partial \Phi / \partial p =$ a function of y and f_2 only, say $= \psi(y, f_2)$.

From (19) we see that we cannot have $\partial^2 \Phi / \partial p \partial y$ and $\partial^2 \Phi / \partial p^2$ both zero: for that would involve $\partial F' / \partial z'$ being zero.

Since $F' = z'\theta(y, f_2) = z'\theta'$, say, the second identity (21) gives

$$\frac{\partial \Phi}{\partial p} \frac{\partial \theta'}{\partial f_2} = \theta' \frac{\partial^2 \Phi}{\partial p \partial f_2}.$$

It follows that $\theta' / (\partial \Phi / \partial p)$ is independent of f_2 , and therefore $\theta' \theta / \psi$ is: but θ depends only on x and y , and therefore θ' / ψ is independent of f_2 . It follows that θ' / ψ depends only on y .

Now

$$\phi = \frac{\partial F}{\partial z} \frac{\partial \Phi}{\partial p} \Big/ \frac{\partial F'}{\partial z'} = \frac{\psi}{\theta'};$$

and therefore, as ϕ depends only on y and is not zero, and

$$f_3 = q\phi + f,$$

we may, without essentially altering the proposed problem, take ϕ to be -1 .

The equations

$$\theta \frac{\partial \Phi}{\partial p} = \psi, \quad F' = F \frac{\partial \Phi}{\partial p} + \frac{\partial \Phi}{\partial y}, \quad PfXf_2 = Pf_2(Xf + \phi F)$$

now become, since $\psi = -\theta'$,

$$\theta \frac{\partial \Phi}{\partial p} + \theta' = 0, \quad \frac{\partial \Phi}{\partial y} = \theta'(z + z'), \quad (Xf - \theta z) Pf_2 = Xf_2 Pf.$$

Operate with $FP + \partial/\partial y + f \partial/\partial z'$, which annihilates f_2 , on the second of these equations, and we get

$$f = -z \frac{\partial}{\partial y} \log \frac{\theta'}{\theta} + \frac{\partial}{\partial y} \left(\frac{1}{\theta'} \frac{\partial \Phi}{\partial y} \right)$$

where, in operating with $\partial/\partial y$, f_2 is to be regarded as constant.

From this we deduce

$$\begin{aligned} Xf &= \frac{\partial^2}{\partial x \partial y} \left(\frac{1}{\theta'} \frac{\partial \Phi}{\partial y} \right) + z \frac{\partial^2}{\partial x \partial y} \log \theta - p \frac{\partial}{\partial y} \log \frac{\theta'}{\theta} \\ &\quad + \left\{ \frac{\partial^2}{\partial y \partial f_2} \left(\frac{1}{\theta'} \frac{\partial \Phi}{\partial y} \right) - z \frac{\partial^2}{\partial y \partial f_2} \log \frac{\theta'}{\theta} \right\} Xf_2, \\ Pf &= \left\{ \frac{\partial^2}{\partial y \partial f_2} \left(\frac{1}{\theta'} \frac{\partial \Phi}{\partial y} \right) - z \frac{\partial^2}{\partial y \partial f_2} \log \frac{\theta'}{\theta} \right\} Pf_2; \end{aligned}$$

and therefore

$$\frac{\partial^2}{\partial x \partial y} \left(\frac{1}{\theta'} \frac{\partial \Phi}{\partial y} \right) + z \frac{\partial^2}{\partial x \partial y} \log \theta - z\theta = p \frac{\partial}{\partial y} \log \frac{\theta'}{\theta}.$$

This is an identity in the arguments x, y, z, p , and f_2 , and z does not occur in θ, θ' , or ϕ . We therefore have

$$\theta = \frac{\partial^2}{\partial x \partial y} \log \theta.$$

The equation

$$s = F = \theta z$$

is therefore a linear one soluble by Laplace's method, and as such falls outside the scope of the present investigation.

16. We now come to the case where $\partial^2 \Phi / \partial p^2$ is not zero. The identity (20) may be written

$$\frac{\frac{\partial^2 \Phi}{\partial p \partial y} \frac{\partial^2 \Phi}{\partial p \partial x} - \frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial^2 \Phi}{\partial p^2}}{\frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial p^2}} + \frac{\frac{\partial p \partial y}{\partial^2 \Phi}}{\frac{\partial^2 \Phi}{\partial p^2}} X \log \frac{\partial F}{\partial z} + \frac{F^2}{\frac{\partial F}{\partial z}} \frac{\partial}{\partial z} X \log F = 0. \quad (22)$$

Since Φ does not contain z ,

$$\frac{\partial}{\partial z} \left\{ \frac{F^2}{\frac{\partial F}{\partial z}} \frac{\partial}{\partial z} X \log F \right\} + \frac{\frac{\partial^2 \Phi}{\partial p \partial y}}{\frac{\partial^2 \Phi}{\partial p^2}} \frac{\partial}{\partial z} \left(X \log \frac{\partial F}{\partial z} \right) = 0;$$

and therefore, since

$$\frac{\partial}{\partial z} \frac{F^2}{\frac{\partial F}{\partial z}} \frac{\partial^2}{\partial z^2} \log F = F \frac{\partial^2}{\partial z^2} \log \frac{\partial F}{\partial z}$$

and

$$\frac{\partial}{\partial z} \frac{F^2}{\frac{\partial F}{\partial z}} \frac{\partial^2}{\partial x \partial z} \log F = F \frac{\partial^2}{\partial x \partial z} \log \frac{\partial F}{\partial z},$$

and so

$$\frac{\partial}{\partial z} \left\{ \frac{F^2}{\frac{\partial F}{\partial z}} \frac{\partial}{\partial z} X \log F \right\} = F \frac{\partial}{\partial z} \left(X \log \frac{\partial F}{\partial z} \right),$$

we either have

$$F + \frac{\partial^2 \Phi}{\partial p \partial y} \bigg/ \frac{\partial^2 \Phi}{\partial p^2} = 0$$

or

$$\frac{\partial}{\partial z} \left(X \log \frac{\partial F}{\partial z} \right) = 0.$$

The first, making F independent of z , is inadmissible, and we thus see, from the second, that F must be of the form $ae^{\beta z} + \gamma$ where β is a function of y only and a and γ are functions of x and y .

Without loss of generality it is easily seen that we may take γ to be zero, and we may say

$$F = \theta(x, y) e^{z\psi(y)},$$

and therefore

$$F' = \theta(y, f_2) e^{z'\psi(f_2)} = \theta' e^{z'\psi'} \quad \text{say.}$$

The second identity (21) may now be written

$$\begin{aligned} & \left(\frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial p \partial y} + F' \frac{\partial^2 \Phi}{\partial p^2} - \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial p^2} \right) \left(\Phi \frac{\partial \Phi}{\partial p} \psi' + \frac{\partial \Phi}{\partial p} \frac{\partial}{\partial f_2} [\psi' z' + \log (\theta' \psi')] - \frac{\partial^2 \Phi}{\partial p \partial f_2} \right) \\ &= \frac{\partial^2 \Phi}{\partial p^2} \left(F' \frac{\partial \Phi}{\partial p} \frac{\partial}{\partial f_2} (\psi' z' + \log \theta') + F' \psi' \Phi \frac{\partial \Phi}{\partial p} \right. \\ & \quad \left. + \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial p \partial f_2} - \frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial y \partial f_2} - F' \frac{\partial^2 \Phi}{\partial p \partial f_2} \right). \end{aligned}$$

From the known form of F' we see that this must vanish identically, as though F' and z' were independent. We thus have

$$\frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial y \partial p} - \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial p^2} = 0; \quad \frac{\partial \Phi}{\partial y} \frac{\partial^2 \Phi}{\partial p \partial f_2} - \frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial y \partial f_2} = 0.$$

It follows that $\frac{\partial \Phi}{\partial y} / \frac{\partial \Phi}{\partial p}$ is independent of p and f_2 . Let it be written λ , where λ depends therefore on x and y only.

The identity (22) now gives

$$\frac{\partial^2 \Phi}{\partial x \partial y} \frac{\partial^2 \Phi}{\partial p^2} - \frac{\partial^2 \Phi}{\partial p \partial y} \frac{\partial^2 \Phi}{\partial p \partial x} = \frac{\partial \Phi}{\partial p} \frac{\partial^2 \Phi}{\partial p \partial y} \left(\frac{\partial}{\partial x} \log \theta + p \psi \right);$$

so that
$$\frac{\partial \lambda}{\partial x} = \lambda \left(\frac{\partial}{\partial x} \log \theta + p \psi \right).$$

Since λ and θ are independent of p , and ψ is not zero, λ must be zero; that is, $\partial \Phi / \partial y = 0$. We therefore have, from

$$F' = F \frac{\partial \Phi}{\partial p} + \frac{\partial \Phi}{\partial y} \quad \text{and} \quad \phi \frac{\partial F'}{\partial z'} = \frac{\partial F}{\partial z} \frac{\partial \Phi}{\partial p},$$

that
$$\phi = \psi / \psi'.$$

Now ϕ is annihilated by

$$f_4 P f_2 \frac{\partial}{\partial z'} + (1 - f_4 f_2) P,$$

and therefore
$$\frac{\partial \psi'}{\partial f_2} P f_2 = 0.$$

It follows that $\psi(y)$ must be a mere constant which we may take to be unity.

We now have

$$F = \theta e^z, \quad F' = \theta' e^{z'}, \quad \text{and} \quad F'' = F \frac{\partial \Phi}{\partial p}.$$

Operating on
$$z' = z + \log \frac{\partial \Phi}{\partial p} + \log \frac{\theta}{\theta'}$$

with
$$FP + \frac{\partial}{\partial y} + f \frac{\partial}{\partial z'},$$

which annihilates f_2 , we get

$$f = \theta e^z \frac{\partial}{\partial p} \log \frac{\partial \Phi}{\partial p} + \frac{\partial}{\partial y} \log \frac{\theta}{\theta'};$$

and thus

$$\begin{aligned} Xf = & \left(\frac{\partial \theta}{\partial x} + p \theta \right) e^z \frac{\partial}{\partial p} \log \frac{\partial \Phi}{\partial p} + \theta e^z \frac{\partial^2}{\partial p \partial x} \log \frac{\partial \Phi}{\partial p} + \frac{\partial^2}{\partial x \partial y} \log \theta \\ & + \left(\theta e^z \frac{\partial^2}{\partial p \partial f_2} \log \frac{\partial \Phi}{\partial p} - \frac{\partial^2}{\partial y \partial f_2} \log \theta \right) Xf_2, \end{aligned}$$

$$Pf = \theta e^z \frac{\partial^2}{\partial p^2} \log \frac{\partial \Phi}{\partial p} + \left(\theta e^z \frac{\partial^2}{\partial p \partial f_2} \log \frac{\partial \Phi}{\partial p} - \frac{\partial^2}{\partial y \partial f_2} \log \theta \right) Pf_2.$$

Eliminating f by means of

$$Xf_2Pf = (Xf + F\phi)Pf_2 = (Xf + \theta e^z)Pf_2,$$

we get

$$\begin{aligned} Pf_2 \left[\frac{\partial^2}{\partial x \partial y} \log \theta + \theta e^z \left\{ \left(\frac{\partial}{\partial x} \log \theta + p \right) \frac{\partial}{\partial p} \log \frac{\partial \Phi}{\partial p} + \frac{\partial^2}{\partial p \partial x} \log \frac{\partial \Phi}{\partial p} + 1 \right\} \right] \\ = Xf_2 \left(\frac{\partial^2}{\partial p^2} \log \frac{\partial \Phi}{\partial p} \right) \theta e^z. \end{aligned}$$

It follows that

$$\frac{\partial^2}{\partial x \partial y} \log \theta = 0;$$

and therefore the equation can, by a simple transformation, be reduced to the integrable form

$$s = e^z.$$

We have thus finally proved that neither f_1 nor f_2 can contain p or q .

IV.

17. We have now to consider the possible Bäcklundian transformations when two of the generating equations connect x, y, z and x', y', z' only.

The generating equations may be taken in the form

$$x' = f_1, \quad y' = f_2, \quad p' = f_3, \quad q' = f_4,$$

where f_1 and f_2 are functions of the arguments x, y, z , and z' only, and f_3 and f_4 functions of these arguments and p and q .

The first Ampèrian equation is

$$f_4 \dot{f}_1 (23) + (f_4 \dot{f}_2 - 1) (31) + (f_4 \dot{f}_3 - f_3 \dot{f}_1) (12) + f_3 \dot{f}_2 (14) + (1 - f_3 \dot{f}_1) (24) = 0,$$

where $\dot{f}$ denotes the differential coefficient of f with respect to z' .

Let

$$\begin{aligned} A &= f_4 \dot{f}_1 Yf_2 + (1 - f_4 \dot{f}_2) Yf_1, & B &= f_3 \dot{f}_2 Yf_1 + (1 - f_3 \dot{f}_1) Yf_2, \\ C &= f_4 \dot{f}_1 Xf_2 + (1 - f_4 \dot{f}_2) Xf_1, & D &= f_3 \dot{f}_2 Xf_1 + (1 - f_3 \dot{f}_1) Xf_2. \end{aligned}$$

From the fact that f_1 and f_2 do not contain p or q , we see that the coefficient of $rt - s^2$ in the equation is zero: the vanishing of the coefficients r and t gives

$$APf_3 + BPf_4 = 0, \quad CQf_3 + DQf_4 = 0.$$

The vanishing of the coefficients of r' and t' in the second Ampèrian equation gives

$$(1 - f_4 \dot{f}_2) [41] + f_4 \dot{f}_1 [42] = 0, \quad (1 - f_3 \dot{f}_1) [23] + f_3 \dot{f}_2 [13] = 0,$$

since $[12] = 0$: that is

$$CPf_4 + AQf_4 = 0, \quad BQf_3 + DPf_3 = 0.$$

It is easily seen that the coefficient of s in the first equation is

$$CPf_3 + DPf_4 - AQf_3 - BQf_4;$$

so that A, B, C, D cannot all be zero.

If none of them are zero, then

$$\frac{Pf_3}{B} = \frac{Pf_4}{-A} = \frac{Qf_3}{-D} = \frac{Qf_4}{C};$$

but, as this would make the coefficient of s to be zero, at least one of them must vanish.

It is easily proved that we must have

$$B = C = Pf_3 = Qf_4 = 0$$

or

$$A = D = Pf_4 = Qf_3 = 0.$$

We take the first case only, the other being derivable by interchanging x and y .

18. We shall now prove that both Xf_1 and Yf_2 are zero, and to do so begin by assuming that neither is zero. The Ampèrian equation is seen to be

$$s(AQf_3 - DPf_4) = (f_3\dot{f}_2Yf_4/Yf_2 - f_4\dot{f}_1Xf_3/Xf_1 - f_3\dot{f}_4 + f_4\dot{f}_3)(12)_{XY}.$$

To simplify this we have

$$A = f_4\dot{f}_1(12)_{XY}/Xf_1, \quad D = f_3\dot{f}_2(12)_{XY}/Yf_2,$$

$$f_3 = Yf_2/(\dot{f}_1Yf_2 - \dot{f}_2Yf_1), \quad f_4 = Xf_1/(\dot{f}_2Xf_1 - \dot{f}_1Xf_2),$$

$$\frac{Xf_3}{f_3} = \frac{XYf_2}{Yf_2} - \frac{f_3}{Yf_2}(\dot{f}_1XYf_2 - \dot{f}_2XYf_1 + Yf_2X\dot{f}_1 - Yf_1X\dot{f}_2),$$

$$\frac{Yf_4}{f_4} = \frac{XYf_1}{Xf_1} - \frac{f_4}{Xf_1}(\dot{f}_2XYf_1 - \dot{f}_1XYf_2 + Xf_1Y\dot{f}_2 - Xf_2Y\dot{f}_1),$$

$$\frac{\dot{f}_3}{f_3} = \frac{Y\dot{f}_2}{Yf_2} - \frac{f_3}{Yf_2}(\dot{f}_1Y\dot{f}_2 - \dot{f}_2Y\dot{f}_1 + \ddot{f}_1Yf_2 - \ddot{f}_2Yf_1),$$

$$\frac{\dot{f}_4}{f_4} = \frac{X\dot{f}_1}{Xf_1} - \frac{f_4}{Xf_1}(\dot{f}_2X\dot{f}_1 - \dot{f}_1X\dot{f}_2 + \ddot{f}_2Xf_1 - \ddot{f}_1Xf_2),$$

$$Pf_4 = f_4^2\dot{f}_1(f_1f_2)_{xx}/(Xf_1)^2, \quad Qf_3 = f_3^2\dot{f}_2(f_1f_2)_{yy}/(Yf_2)^2,$$

where we denote by $(f_1 f_2)_{zz}$

$$\frac{\partial f_1}{\partial x} \frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z} \frac{\partial f_2}{\partial x},$$

and where $(f_1 f_2)_{zy}$ has a similar meaning.

By aid of these equations, combined with the fact that $(12)_{xy}$ cannot be zero without reducing the equation to a mere identity, we see that it becomes

$$\begin{aligned} s(f_1 f_2)_{zz} = & \frac{\dot{f}_1}{\dot{f}_2} Y f_2 X \dot{f}_2 - \frac{\dot{f}_2}{\dot{f}_1} X f_1 Y \dot{f}_1 + \left(\frac{\ddot{f}_1}{\dot{f}_1} - \frac{\ddot{f}_2}{\dot{f}_2} \right) X f_1 Y f_2 \\ & + X f_1 Y \dot{f}_2 - X \dot{f}_1 Y f_2 - \dot{f}_1 X Y f_2 + \dot{f}_2 X Y f_1. \end{aligned}$$

If we take $\bar{X} \equiv X f_1 \frac{\partial}{\partial z'} - \dot{f}_1 X, \quad \bar{Y} \equiv Y f_2 \frac{\partial}{\partial z'} - \dot{f}_2 Y,$

this may be written in either of the forms

$$s(f_1 f_2)_{zz} = \frac{\dot{f}_2}{\dot{f}_1} \bar{X} (\bar{Y} f_1 / \dot{f}_2),$$

$$s(f_2 f_1)_{zz} = \frac{\dot{f}_1}{\dot{f}_2} \bar{Y} (\bar{X} f_2 / \dot{f}_1).$$

It should be noticed that our assumption that neither $X f_1$ nor $Y f_2$ is zero involves the fact that neither can $\dot{f}_1$ nor $\dot{f}_2$ be zero; to suppose either zero causing the coefficient of s to vanish.

Since the coefficients of pq , of p , and of q must all vanish on the right, we have from the first of these facts that

$$\left(\frac{\partial f_1}{\partial z} \frac{\partial}{\partial z'} - \dot{f}_1 \frac{\partial}{\partial z} \right) \{ (f_1 f_2)_{zz'} / \dot{f}_2 \} = 0;$$

and therefore $(f_1 f_2)_{zz'} / \dot{f}_2$ must be a function of the arguments f_1 , x , and y only.

Let then

$$(f_1 f_2)_{zz'} / \dot{f}_2 = 1 / \frac{\partial F}{\partial f_1}$$

when F is some function of the arguments f_1 , x , y . It now follows that

$$f_2 = \Phi(z - F, x, y),$$

and the equation takes the form

$$s \dot{f}_1 / \frac{\partial F}{\partial f_1} = \bar{X} \left\{ \left(q - \frac{\partial F}{\partial y} \right) / \frac{\partial F}{\partial f_1} + \frac{\partial \Phi}{\partial y} / \frac{\partial \Phi}{\partial z} \frac{\partial F}{\partial f_1} \right\}.$$

As the coefficient of q must be zero, $\bar{X}$ must annihilate $\partial F/\partial f_1$, and, as it also annihilates f_1 , we see that $\partial F/\partial f_1$ must contain x , only as involved in f_1 , if at all.

Since the coefficient of p is also zero,

$$\frac{\partial f_1}{\partial z} \frac{\partial}{\partial z'} - \dot{f}_1 \frac{\partial}{\partial z}$$

must annihilate $\left(\frac{\partial \Phi}{\partial y} - \frac{\partial F}{\partial y} \frac{\partial \Phi}{\partial z} \right) / \frac{\partial F}{\partial f_1} \frac{\partial \Phi}{\partial z}$;

and therefore, since it annihilates F and its derivatives, it must annihilate $\frac{\partial \Phi}{\partial y} / \frac{\partial \Phi}{\partial z}$. It follows that $\frac{\partial \Phi}{\partial y} / \frac{\partial \Phi}{\partial z}$ is independent of z .

From what we have now proved we see that

$$f_2 = \Phi \{ z - \phi(f_1, y) - \psi(x, y), x \}$$

where ϕ and f are functional symbols, and the equation is

$$s \dot{f}_1 = \frac{\partial \phi}{\partial f_1} \left(\frac{\partial f_1}{\partial x} \frac{\partial}{\partial z'} - \dot{f}_1 \frac{\partial}{\partial x} \right) \left(q - \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial y} \right) / \frac{\partial \phi}{\partial f_1} = \dot{f}_1 \frac{\partial^2 \psi}{\partial x \partial y};$$

that is $s = \frac{\partial^2 \psi}{\partial x \partial y}$;

so that the equation is integrable.

It follows that our assumption that neither Xf_1 nor Yf_2 was zero is not tenable.

19. Take Xf_1 to be zero, the other case being deducible by interchange of x and y and of x' and y' .

If $Xf_1 = 0$, we cannot have $Xf_2 = 0$, as that would involve a relation between x' , y' , and z' ; we therefore have $\dot{f}_1 = 0$.

We may now take $f_1 \equiv y$, when the relations become

$$Pf_3 = Qf_4 = 0, \quad A = 1 - f_4 \dot{f}_2, \quad D = Xf_2, \quad f_3 \dot{f}_2 + Yf_2 = 0.$$

Assume that Yf_2 is not zero, and the equation is

$$s \{ (1 - f_4 \dot{f}_2) Qf_3 - Xf_2 Pf_4 \} = (f_3 \dot{f}_4 - f_4 \dot{f}_3 + Yf_4) Xf_2 + (f_4 \dot{f}_2 - 1) Xf_3. \quad (23)$$

We can now say that

$$f_3 = f + q\phi,$$

where f and ϕ are functions of the arguments x , y , z , and z' : let the Ampèrian equation be

$$s = F(x, y, z).$$

It will be convenient to introduce the operators $\bar{f}$ and $\bar{\phi}$ where

$$\bar{f} \equiv f \frac{\partial}{\partial z'} + \frac{\partial}{\partial y}, \quad \bar{\phi} \equiv \phi \frac{\partial}{\partial z'} + \frac{\partial}{\partial z},$$

and the symbols u and v where

$$u \equiv (F\phi + Xf) / Xf_2, \quad v \equiv X\phi / Xf_2.$$

From the Ampèrian equation (23) we obtain, by equating to zero, the coefficient of q , and the term independent of q ,

$$(\bar{f} + FP)f_4 = u + (\dot{f} - \dot{f}_2 u)f_4, \quad \bar{\phi}f_4 = v + (\dot{\phi} - \dot{f}_2 v)f_4. \quad (24)$$

Also, since $\bar{f}$ and $\bar{\phi}$ both annihilate f_2 ,

$$(\bar{f}\phi - \bar{\phi}f)\dot{f}_2 = 0;$$

and therefore, since $\dot{f}_2$ is not zero (Yf_2 being assumed different from zero), the operators $\bar{f}$ and $\bar{\phi}$ are commutative.

The next step is to prove that either $\partial F / \partial z$ is zero, when the equation would be soluble, or that $Pf_4 = (1 - f_4 \dot{f}_2)\phi / Xf_2$. As the latter conclusion would involve the vanishing of the coefficient of s , we should thus be driven to the assumption that Yf_2 is also zero.

We easily obtain, by operating with $\bar{\phi}$ and $\bar{f}$ on (24),

$$\begin{aligned} \bar{\phi}(\bar{f} + FP)f_4 &= (1 - f_4 \dot{f}_2)\bar{\phi}u + (\dot{f} - \dot{f}_2 u)v + f_4(\dot{f} - \dot{f}_2 u)(\dot{\phi} - \dot{f}_2 v) + f_4\bar{\phi}\dot{f} + f_4 u \dot{\phi} \dot{f}_2, \\ \bar{f}\bar{\phi}f_4 &= (1 - f_4 \dot{f}_2)\bar{f}v + (\dot{\phi} - \dot{f}_2 v)u + f_4(\dot{\phi} - \dot{f}_2 v)(\dot{f} - \dot{f}_2 u) + f_4\bar{f}\dot{\phi} + f_4 v \dot{f} \dot{f}_2 \\ &\quad - (\dot{\phi} - \dot{f}_2 v)FPf_4; \end{aligned}$$

and therefore, since $\bar{f}\bar{\phi} = \bar{\phi}\bar{f}$ and $\bar{\phi}P = P\bar{\phi}$,

$$\begin{aligned} \frac{\partial F}{\partial z} Pf_4 + FP\{v + f_4(\dot{\phi} - \dot{f}_2 v)\} - (\dot{\phi} - \dot{f}_2 v)FPf_4 \\ = (1 - f_4 \dot{f}_2)(\bar{\phi}u - \bar{f}v) + (\dot{f} - \dot{f}_2 u)v + (\dot{f}_2 v - \dot{\phi})u + f_4(\bar{\phi}\dot{f} - \bar{f}\dot{\phi}) + f_4 \dot{f}_2 (u\dot{\phi} - v\dot{f}). \end{aligned}$$

This simplifies to

$$\frac{\partial F}{\partial z} Pf_4 + (1 - f_4 \dot{f}_2)(FPv + \bar{f}v - \bar{\phi}u - \dot{f}v + \dot{\phi}u) = f_4(\bar{\phi}\dot{f} - \bar{f}\dot{\phi}).$$

$$\text{Now} \quad \frac{\partial}{\partial z'} \bar{f} = \bar{f} \frac{\partial}{\partial z'} + \dot{f} \frac{\partial}{\partial z'}, \quad \frac{\partial}{\partial z'} \bar{\phi} = \bar{\phi} \frac{\partial}{\partial z'} + \dot{\phi} \frac{\partial}{\partial z'};$$

$$\text{and therefore, since} \quad \frac{\partial}{\partial z'} \bar{f}\bar{\phi} = \frac{\partial}{\partial z'} \bar{\phi}\bar{f},$$

$$\text{we have} \quad \bar{f}\dot{\phi} = \dot{\phi}\bar{f}.$$

$$\text{Also } Pv = \left(\frac{\partial \phi}{\partial z} - v \frac{\partial f_2}{\partial z} \right) / Xf_2, \quad \bar{f}v = \{ (X\bar{f}\phi - Xf(\phi - vf_2)) \} / Xf_2;$$

$$\bar{\phi}u = \left(\frac{\partial F}{\partial z} \phi + F\bar{\phi}\phi + X\bar{\phi}f - fX\phi + uf_2X\phi \right) / Xf_2;$$

so that we have, finally,

$$\frac{\partial F}{\partial z} Pf_4 - (1 - f_4\dot{f}_2) \frac{\partial F}{\partial z} \phi / Xf_2;$$

that is, $\partial F / \partial z$ is zero or $Pf_4 = (1 - f_4\dot{f}_2) \phi / Xf_2$.

We thus see Yf_2 must be zero as well as Xf_1 , and consequently $\dot{f}_2 = 0$ also.

We may therefore take

$$f_1 \equiv y, \quad f_2 \equiv x, \quad Pf_3 = Qf_4 = 0;$$

but it will now be more convenient to interchange x and y , and therefore take

$$f_1 \equiv x, \quad f_2 \equiv y, \quad Pf_4 = Qf_3 = 0.$$

V.

20. We have now come to the case where the four generating equations are of the form

$$x' = x, \quad y' = y, \quad p' = f_3(x, y, z, z', p), \quad q' = f_4(x, y, z, z', q),$$

and the Ampèrian equations are

$$s = F(x, y, z), \quad s' = F(x', y', z') = F(x, y, z') = F' \quad (\text{say}).$$

We assume that F must contain z , and we have

$$F' = Yf_3 + f_4\dot{f}_3 + FPf_3 = Xf_4 + f_3\dot{f}_4 + FQf_4. \quad (25)$$

It will be noticed that these are identities in the arguments x, y, z, p, q , and z' ; for, otherwise, z' would be expressible in terms of x, y, z, p , and q , which it cannot be from the generating equations. For a similar reason we cannot have $\dot{f}_3 = 0$ or $\dot{f}_4 = 0$.

From the forms of the identities, we also see that f_3 must be linear in p and f_4 in q .

$$\text{Let } f_3 \equiv Ap + C, \quad f_4 \equiv Bq + D,$$

$$\bar{A} = A \frac{\partial}{\partial z'} + \frac{\partial}{\partial z}, \quad \bar{B} = B \frac{\partial}{\partial z'} + \frac{\partial}{\partial z}, \quad \bar{C} = C \frac{\partial}{\partial z'} + \frac{\partial}{\partial x}, \quad \bar{D} = D \frac{\partial}{\partial z'} + \frac{\partial}{\partial y},$$

when, from (25), we have

$$\left. \begin{aligned} \bar{B}A = \bar{A}B = \bar{D}A = \bar{C}B = \bar{B}C = \bar{A}D = 0, \\ F' - AF = \bar{D}C, \quad F' - BF = \bar{C}D \end{aligned} \right\} \quad (26)$$

If A and B both contain z' , then B having the annihilators $\bar{A}$ and $\bar{C}$, also has $\bar{A}\bar{C} - \bar{C}\bar{A}$; but this is impossible (since B contains z') unless $\bar{A}$ and $\bar{C}$ are commutative. But $\bar{A}$ and $\bar{D}$ are commutative, and therefore $\bar{A}$ is commutative with $\bar{D}\bar{C} - \bar{C}\bar{D}$; that is, $\bar{A}$ is commutative with $\partial/\partial z'$, or $\bar{C}$ and $\bar{D}$ are commutative. The first alternative is inconsistent with the hypothesis that A contains z' ; the second, since

$$F' - AF = \bar{D}C \quad \text{and} \quad F' - BF = \bar{C}D,$$

gives $A = B$, which is impossible, as it would cause the coefficient of s to vanish.

It follows that A and B cannot both contain z' . Suppose that A does not: then, since $\bar{B}A = 0$, A does not contain z ; nor does it contain y , since $\bar{D}$ annihilates it. It is thus a function of x only.

If B contains z' , then we have seen that $\bar{A}$ and $\bar{C}$ are commutative: operating with $\bar{A}$ on

$$F' - AF = \bar{D}C, \quad F' - BF = \bar{C}D,$$

we easily have, since A is a function of x only,

$$\bar{A}F' - A\bar{A}F = \bar{A}\bar{D}C = \bar{D}\bar{A}C = \bar{D}\bar{C}A = 0,$$

$$\bar{A}F' - B\bar{A}F = \bar{A}\bar{C}D = \bar{C}\bar{A}D = 0,$$

and therefore $(A - B)\bar{A}F = 0$.

Since F must contain z , we cannot have $\bar{A}F = 0$; and therefore $B = A$, which we have seen is impossible. We conclude that B does not contain z' .

Since B does not contain z' and is annihilated by $\bar{A}$, it does not contain z either; and, since $\bar{C}B = 0$, it must be a function of y only.

21. We now have, from $\bar{A}D = \bar{B}C = 0$,

$$C = f(z' - Bz, x, y), \quad D = \phi(z' - Az, x, y);$$

and therefore, from (26),

$$F' - AF = D\dot{C} + C_y, \quad F' - BF = C\dot{D} + D_x$$

where $\dot{C}$ denotes $\partial C/\partial z'$ and C_y , $\partial C/\partial y$ and $\dot{D}$, D_x have similar meanings.

We now have

$$(A - B)F = C\dot{D} + D_x - D\dot{C} - C_y, \quad (27)$$

$$(A - B)F' = A\dot{C}\dot{D} + AD_x - B\dot{D}\dot{C} - BC_y. \quad (28)$$

$$\begin{aligned}\text{Now} \quad \frac{\partial}{\partial x} \left(\frac{\partial}{\partial z} + A \frac{\partial}{\partial z'} \right) - \left(\frac{\partial}{\partial z} + A \frac{\partial}{\partial z'} \right) \frac{\partial}{\partial x} &= \frac{dA}{dx} \frac{\partial}{\partial z'}, \\ \frac{\partial}{\partial y} \left(\frac{\partial}{\partial z} + B \frac{\partial}{\partial z'} \right) - \left(\frac{\partial}{\partial z} + B \frac{\partial}{\partial z'} \right) \frac{\partial}{\partial y} &= \frac{dB}{dy} \frac{\partial}{\partial z'};\end{aligned}$$

and therefore

$$\frac{\partial}{\partial z} D_x = -A\dot{D}_x - \frac{dA}{dx} \dot{D}, \quad \frac{\partial}{\partial z} C_y = -B\dot{C}_y - \frac{dB}{dy} \dot{C}. \quad (29)$$

Since F does not contain z' , we get, by differentiating (27),

$$C\ddot{D} + \dot{D}_x = D\ddot{C} + \dot{C}_y;$$

and, differentiating (28) with respect to z , we have, by (29) and

$$\frac{\partial}{\partial z} D = -A\dot{D}, \quad \frac{\partial}{\partial z} C = -B\dot{C},$$

$$\text{that} \quad A^2(C\ddot{D} + \dot{D}_x) + A \frac{dA}{dx} \dot{D} = B^2(D\ddot{C} + \dot{C}_y) + B \frac{dB}{dy} \dot{C}.$$

$$\text{Let now} \quad z' - Az = u, \quad z' - Bz = v;$$

then we may regard C as a function of v , x , y , and D as a function of u , x , y .

So regarded

$$\dot{C} = \frac{\partial f}{\partial v}, \quad \ddot{C} = \frac{\partial^2 f}{\partial v^2}, \quad C_y = -z \frac{\partial f}{\partial v} \frac{dB}{dy} + \frac{\partial f}{\partial y},$$

$$\dot{D} = \frac{\partial \phi}{\partial u}, \quad \ddot{D} = \frac{\partial^2 \phi}{\partial u^2}, \quad D_x = -z \frac{\partial \phi}{\partial u} \frac{dA}{dx} + \frac{\partial \phi}{\partial x},$$

and

$$C\ddot{D} + \dot{D}_x = f \frac{\partial^2 \phi}{\partial u^2} + \frac{u-v}{A-B} \frac{\partial^2 \phi}{\partial u^2} \frac{dA}{dx} + \frac{\partial^2 \phi}{\partial u \partial x},$$

$$D\ddot{C} + \dot{C}_y = \phi \frac{\partial^2 f}{\partial v^2} + \frac{u-v}{A-B} \frac{\partial^2 f}{\partial v^2} \frac{dB}{dy} + \frac{\partial^2 f}{\partial x \partial y}.$$

It follows that

$$(A^2 - B^2) \left(f \frac{\partial^2 \phi}{\partial u^2} + \frac{u-v}{A-B} \frac{\partial^2 \phi}{\partial u^2} \frac{dA}{dx} + \frac{\partial^2 \phi}{\partial u \partial x} \right) + A \frac{dA}{dx} \frac{\partial \phi}{\partial u} - B \frac{dB}{dy} \frac{\partial f}{\partial v} = 0,$$

$$(B^2 - A^2) \left(\phi \frac{\partial^2 f}{\partial v^2} + \frac{v-u}{B-A} \frac{\partial^2 f}{\partial v^2} \frac{dB}{dy} + \frac{\partial^2 f}{\partial v \partial y} \right) + B \frac{dB}{dy} \frac{\partial f}{\partial v} - A \frac{dA}{dx} \frac{\partial \phi}{\partial u} = 0.$$

Differentiating the first of these equations with respect to v ,

$$(A^2 - B^2) \left(\frac{\partial f}{\partial v} \frac{\partial^2 \phi}{\partial u^2} - \frac{1}{A-B} \frac{\partial^2 \phi}{\partial u^2} \frac{dA}{dx} \right) = B \frac{dB}{dy} \frac{\partial^2 f}{\partial v^2}.$$

Similarly from the second we deduce that

$$(B^2 - A^2) \left(\frac{\partial \phi}{\partial u} \frac{\partial^2 f}{\partial v^2} - \frac{1}{B-A} \frac{\partial^2 f}{\partial v^2} \frac{dB}{dy} \right) = A \frac{dA}{dx} \frac{\partial^2 \phi}{\partial u^2}.$$

Now, unless $A^2 = B^2$, we must have

$$\frac{\partial^2 f}{\partial v^2} = \frac{\partial^2 \phi}{\partial u^2} = 0;$$

for, if they are not both zero, we have

$$(A^2 - B^2)^2 \left\{ \frac{\partial f}{\partial v} - \frac{\frac{dA}{dx}}{A-B} \right\} \left\{ \frac{\partial \phi}{\partial u} - \frac{\frac{dB}{dy}}{B-A} \right\} + AB \frac{dA}{dx} \frac{dB}{dy} = 0,$$

which at once gives
$$\frac{\partial^2 f}{\partial v^2} = \frac{\partial^2 \phi}{\partial u^2} = 0,$$

unless
$$A^2 - B^2 = 0.$$

22. Assume now that $A^2 \neq B^2$, and we may take

$$C = \alpha(z' - Bz) + \beta, \quad D = \gamma(z' - Az) + \delta.$$

Where α , β , γ , and δ are functions of x and y only

$$C_y = \alpha_y(z' - Bz) + \beta_y - \alpha \frac{dB}{dy} z, \quad D_x = \gamma_x(z' - Az) + \delta_x - \gamma \frac{dA}{dx} z.$$

We thus have

$$F' - AF' = \alpha_y(z' - Bz) + \beta_y - \alpha \frac{dB}{dy} z + \alpha \gamma(z' - Az) + \alpha \delta,$$

$$F' - BF' = \gamma_x(z' - Az) + \delta_x - \gamma \frac{dA}{dx} z + \gamma \alpha(z' - Bz) + \gamma \beta,$$

and it follows that we must have

$$F = \lambda z + \mu, \quad F' = \lambda z' + \mu,$$

where λ and μ are functions of x and y only.

Without loss of generality we may take $\mu = 0$, and we have, to determine λ ,

$$\lambda u = \alpha_y v + \beta_y - \alpha \frac{dB}{dy} \frac{v-u}{A-B} + \alpha \gamma u + \alpha \delta,$$

$$\lambda v = \gamma_x u + \delta_x - \gamma \frac{dA}{dx} \frac{v-u}{A-B} + \gamma \alpha v + \gamma \beta.$$

It follows that

$$\alpha_y = \frac{a \frac{dB}{dy}}{A-B}, \quad \gamma_x = \frac{\gamma \frac{dA}{dx}}{B-A}, \quad \alpha_y = \gamma_x, \quad \delta\alpha + \beta_y = \beta\gamma + \delta_x = 0;$$

and therefore
$$\alpha = \frac{\theta(x)}{A-B}, \quad \gamma = \frac{\phi(y)}{B-A},$$

and so
$$\theta(x) \frac{dB}{dy} = \phi(y) \frac{dA}{dx}.$$

We may thus put (if neither A nor B is constant)

$$\alpha = \kappa \frac{\partial}{\partial x} \log(A-B), \quad \gamma = \kappa \frac{\partial}{\partial y} \log(A-B),$$

where κ is a constant, and we get

$$\lambda = \alpha\gamma \left(1 - \frac{1}{\kappa}\right) = \kappa(1-\kappa) \frac{dA}{dx} \frac{dB}{dy} / (A-B)^2.$$

The equation is thus

$$s = \kappa(1-\kappa) \frac{dA}{dx} \frac{dB}{dy} z / (A-B)^2.$$

If either A or B is a constant, the equation is integrable, and therefore without the scope of this enquiry. Take therefore A and B to be the independent variables, and the equation becomes

$$s = \kappa(1-\kappa) z / (x-y)^2, \quad (30)$$

and the corresponding transformation formulæ are

$$p' = xp + \frac{\kappa(z'-yz)}{x-y} + f(x, y), \quad q' = yq + \frac{\kappa(z'-xz)}{y-x} + \phi(x, y),$$

$$x' = x, \quad y' = y,$$

where f and ϕ are functions which satisfy the equations

$$\frac{\partial f}{\partial y} + \frac{\kappa\phi}{x-y} = 0, \quad \frac{\partial \phi}{\partial x} + \frac{\kappa f}{y-x} = 0.$$

If we denote the equation (Darboux, II., p. 54)

$$s - \frac{ap}{x-y} - \frac{bq}{y-x} - \frac{cz}{(x-y)^2} = 0 \quad (31)$$

where a , b , and c are constants by (a, b, c) , then we know that for any transformation

$$z' = (x-y)^a z$$

the form is unaltered, and

$$c+a(1-b), \quad c+b(1-a)$$

are invariantive.

The invariants of the equation (31) regarded as a particular case of the more general linear one are

$$\frac{c+a(1-b)}{(x-y)^2} \quad \text{and} \quad \frac{c+b(1-a)}{(x-y)^2}.$$

It is easily seen that f satisfies the equation

$$(0, 1, -\kappa^2) = 0,$$

and ϕ the equation

$$(1, 0, -\kappa^2) = 0,$$

while the equation (30) is $\{0, 0, \kappa(1-\kappa)\} = 0$.

All of these equations can be solved if κ is an integer (Forsyth, *Theory of Differential Equations*, Vol. VI., p. 53, Ex. 1).

23. We now pass to the case when A is constant, and therefore B also.

We have, since A and B are constant,

$$\alpha = \theta(x), \quad \gamma = \phi(y),$$

and the equation $s = F$ becomes

$$s = \alpha\gamma z,$$

and, by a simple change of the independent variables, we may take

$$\alpha = \gamma = 1.$$

The equation is now

$$s = z, \tag{32}$$

and the Bäcklundian transformation which corresponds is

$$x' = x, \quad y' = y, \quad p' = Ap + z' - Bz + f(x, y), \quad q' = Bq + z' - Az + \phi(x, y),$$

where f and ϕ satisfy the equations

$$\frac{\partial f}{\partial y} + \phi = 0, \quad \frac{\partial \phi}{\partial x} + f = 0;$$

so that f and ϕ both satisfy the equation (32).

24. Finally, we have to consider the case where $A^2 = B^2$; that is, since we cannot have $A = B$, $A + B = 0$. Now, as A is a function of x only, and B of y only, we cannot have $A + B = 0$ unless A is a constant which we may take to be unity, and we then take $B = -1$.

C is a function of $z' + z$, x and y , and D of $z' - z$, x and y , and we have

$$C\ddot{D} + \dot{D}_x = D\ddot{C} + \dot{C}_y \tag{33}$$

where D and D_x are annihilated by $\partial/\partial z' + \partial/\partial z$ and C and C_y by $\partial/\partial z' - \partial/\partial z$.

Operating with these two annihilators successively on (33), we get

$$\dot{\ddot{C}}\dot{\ddot{D}} = \dot{\ddot{D}}\dot{\ddot{C}}.$$

As we cannot have $\dot{\ddot{C}}$ or $\dot{\ddot{D}}$ equal to zero, for neither $\dot{f}_3$ nor $\dot{f}_4$ can be, we must have

$$\dot{\ddot{C}}/\dot{\ddot{C}} = \dot{\ddot{D}}/\dot{\ddot{D}};$$

and therefore each of these quantities must be a function of x and y only.

We thus have

$$C = f_1 \sin \{(z' + z)f + \phi_1\} + \alpha, \quad D = f_2 \sin \{(z' - z)f + \phi_2\} + \beta$$

where $f, f_1, f_2, \phi_1, \phi_2, \alpha$, and β are functions of x and y only.

$$\text{Since } 2F' = D\dot{C} + C_y + C\dot{D} + D_x, \quad 2F = C\dot{D} + D_x - D\dot{C} - C_y,$$

we get from the fact that F does not contain z' that f_1 is a function of x only, f a mere constant, and

$$\frac{\partial \phi_1}{\partial y} + \beta f = \frac{\partial \phi_2}{\partial x} + \alpha f = 0.$$

We now have

$$2F' = ff_1f_2 \sin (2zf + \phi_1 + \phi_2) - \frac{1}{f} \left(\frac{\partial^2 \phi_1}{\partial x \partial y} + \frac{\partial^2 \phi_2}{\partial x \partial y} \right),$$

$$2F = ff_1f_2 \sin (2zf + \phi_1 - \phi_2) - \frac{1}{f} \left(\frac{\partial^2 \phi_1}{\partial x \partial y} - \frac{\partial^2 \phi_2}{\partial x \partial y} \right);$$

and therefore, by a simple transformation of the variables, we have as our equation

$$s = \sin z,$$

and as our transformation formulæ

$$x' = x, \quad y' = y, \quad p' = p + 2 \sin \frac{z' + z}{2}, \quad q' = q + 2 \sin \frac{z' - z}{2}.$$

THE DIOPHANTINE EQUATION $x^n - Ny^n = z$

By Major P. A. MACMAHON, F.R.S.

[Received December 7th, 1906.—Read December 13th, 1906.]

1. The Diophantine equality

$$\lambda x = \mu y + z$$

is equivalent to the Diophantine inequality

$$\lambda x \geq \mu y.$$

It has been shewn* that the arithmetically independent solutions of this inequality are obtained by forming the descending intermediate series of convergents to the continued fraction

$$\frac{\mu}{\lambda}.$$

If this finite series be

$$\frac{1}{0}, \frac{a_0}{\beta_0}, \frac{a_1}{\beta_1}, \dots, \frac{a_t}{\beta_t}, \frac{a_{t+1}}{\beta_{t+1}}, \dots, \frac{a_{n'}}{\beta_{n'}},$$

the general solution is

$$x = (A+1)\alpha_t + B\alpha_{t+1},$$

$$y = (A+1)\beta_t + B\beta_{t+1},$$

where A, B are arbitrary positive (including zero) integers.

The fundamental solutions are given by

$$x = \alpha_t,$$

$$y = \beta_t,$$

and it was shewn long ago by Sylvester that, if

$$\frac{p_0}{q_0}, \frac{p_1}{q_1}, \dots, \frac{p_n}{q_n}$$

* "The Diophantine Inequality $\lambda x \geq \mu y$," *Camb. Phil. Trans.*, Vol. XIX., Part I., 1900.

be the principal convergents to $\frac{\mu}{\lambda}$,

and $\frac{p'_0}{q'_0}, \frac{p'_1}{q'_1}, \dots, \frac{p'_n}{q'_n}$

the principal convergents to the fraction whose partial quotients are the same as those of

$$\frac{\mu}{\lambda},$$

but taken in reverse order,

$$\begin{aligned} p_n q_s - p_s q_n &= (-)^{s+1} p_{n-s-1}, \\ p_{n-1} q_s - p_s q_{n-1} &= (-)^{s+1} q'_{n-s-1}. \end{aligned}$$

These important results involve the complete solution of the problem of finding the arithmetically independent solutions of the Diophantine equality

$$\lambda x = \mu y + z;$$

for, taking the descending intermediate series of convergents to

$$\frac{p_n}{q_n} \quad \text{or} \quad \frac{\mu}{\lambda},$$

$$\begin{aligned} p_n \beta_t - \alpha_t q_n &= p_n (q_{2m} + s q_{2m+1}) - (p_{2m} + s p_{2m+1}) q_n \\ &= -p'_{n-2m-1} + s p'_{n-2m-2}; \end{aligned}$$

hence $\lambda \alpha_t - \mu \beta_t = p'_{n-2m-3} + (a_{2m+1} - s) p'_{n-2m-2}$,

where a_{2m+2} is the partial quotient which is the maximum value of s .

Now $p'_{n-2m-3} + (a_{2m+1} - s) p'_{n-2m-2}$

is the numerator of one of the descending intermediate series of convergents to the fraction which is the reverse of

$$\frac{\mu}{\lambda}.$$

Writing this series $\frac{\alpha'_0}{\beta'_0}, \frac{\alpha'_1}{\beta'_1}, \dots, \frac{\alpha'_{n'}}{\beta'_{n'}}$,

$$\lambda \alpha_t - \mu \beta_t = \alpha'_{n'-t-1}$$

where, when $t = n'$, we take α'_{-1} to be zero.

We have therefore the following theorem:—

“The fundamental arithmetically independent solutions of the Diophantine equality

$$\lambda x = \mu y + z$$

are formed by constructing

“(i.) the descending intermediate series of convergents to the fraction $\frac{\mu}{\lambda}$, viz.,

$$\frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1}, \dots, \frac{\alpha_{n'}}{\beta_{n'}}.$$

“(ii.) The descending intermediate series to the fraction which is the reverse of $\frac{\mu}{\lambda}$, viz.,

$$\frac{\alpha'_0}{\beta'_0}, \frac{\alpha'_1}{\beta'_1}, \dots, \frac{\alpha'_{n'}}{\beta'_{n'}}.$$

“The solutions are then $n' + 1$ in number, viz.,

$$x = \alpha_t, \quad y = \beta_t, \quad z = \alpha'_{n'-t-1},$$

where α'_{-1} is to be taken equal to zero.”

As an example, take $31x = 222y + z$,

where $\frac{222}{31} = 7 + \frac{1}{6} + \frac{1}{5}, \quad \frac{222}{43} = 5 + \frac{1}{6} + \frac{1}{7}:$

the descending series to $\frac{222}{31}$

is $\frac{1}{0}, \frac{8}{1}, \frac{15}{2}, \frac{22}{3}, \frac{29}{4}, \frac{36}{5}, \frac{43}{6}, \frac{222}{31}.$

and to $\frac{222}{43}$

$$\frac{1}{0}, \frac{6}{1}, \frac{11}{2}, \frac{16}{3}, \frac{21}{4}, \frac{26}{5}, \frac{31}{6}, \frac{222}{43},$$

and we derive the fundamental solutions

x	y	z
1	0	31
8	1	26
15	2	21
22	3	16
29	4	11
36	5	6
43	6	1
222	31	0

and every other solution is a linear function of these eight.

2. It is a corollary from the foregoing theorem that the fundamental (that is arithmetically independent) solutions of the Diophantine inequality

$$x \geq N^{1/m} y,$$

N not being a perfect m -th power, are obtained by forming the infinite descending intermediate series of convergents to

$$N^{1/m}.$$

If this series be $\frac{1}{0}, \frac{a_1}{\beta_1}, \dots, \frac{a_t}{\beta_t}, \frac{a_{t+1}}{\beta_{t+1}}, \dots$, ad inf.,

the general solution is $x = (A+1)\alpha_t + B\alpha_{t+1}$,

$$y = (A+1)\beta_t + B\beta_{t+1}.$$

Hence this is also the solution of the Diophantine inequality

$$x^m \geq Ny^m;$$

and thence we say, in regard to the Diophantine equality

$$x^m = Ny^m + z,$$

that the fundamental values of x and y are as above, and it remains to determine the properties of z .

My first aim is to connect z with the continued fraction which denotes

$$N^{1/m}.$$

Putting, in a usual notation,

$$\begin{aligned} N^{1/m} &= x_1 = a_1 + \frac{1}{x_2}, \\ x_2 &= a_2 + \frac{1}{x_3}, \\ &\dots \dots \dots, \\ x_k &= a_k + \frac{1}{x_{k+1}}, \end{aligned}$$

let the principal convergents be

$$\frac{p_{-1}}{q_{-1}}, \frac{p_0}{q_0}, \frac{p_1}{q_1}, \dots$$

or

$$\frac{0}{1}, \frac{1}{0}, \frac{a_1}{1}, \dots;$$

we have

$$\begin{aligned} x_{n+2} &= \frac{q_n N^{1/m} - p_n}{p_{n+1} - q_{n+1} N^{1/m}} \\ &= \frac{(-)^m (p_n p_{n+1}^{m-1} - N q_n q_{n+1}^{m-1}) + p_{n+1}^{m-2} N^{1/m} + p_{n+1}^{m-3} q_{n+1} N^{2/m} + \dots + q_{n+1}^{m-2} N^{(m-1)/m}}{(-)^{n+1} (p_{n+1}^m - N q_{n+1}^m)}. \end{aligned}$$

The expressions $(-)^n (p_n p_{n+1}^{m-1} - N q_n q_{n+1}^{m-1}) = L_{n+1}$,

$$(-)^{n+1} (p_{n+1}^m - N q_{n+1}^m) = M_{n+1}$$

are essentially positive; L_{n+1} has been termed a "rational dividend," and M_{n+1} a "divisor."

Substituting this value of x_{n+2} in the relation

$$x_{n+1} x_{n+2} = a_{n+1} x_{n+2} + 1$$

and comparing coefficients, we find, putting

$$(-)^n (p_n^{m-1} p_{n+1} - N q_n^{m-1} q_{n+1}) = K_{n+1},$$

the two fundamental relations

$$L_n L_{n+1} + (-)^{n+1} N (p_{n+1}^{m-1} q_n^{m-1} - p_n^{m-1} q_{n+1}^{m-1}) = a_{n+1} L_{n+1} M_n + M_n M_{n+1}, \quad (1)$$

$$L_n + K_{n+1} = a_{n+1} M_n, \quad (2)$$

where, comparing the coefficients of $N^{s/m}$, the first relation corresponds to $s = 0$, and the second relation is given by all values of s from 1 to $m-1$, and no other relation is obtainable.

Now, the intermediate convergent of the descending series appertaining to

$$N^{1/m}$$

is, in general,

$$\frac{\alpha_t}{\beta_t},$$

where

$$\alpha_t = p_{2n} + s p_{2n+1},$$

$$\beta_t = q_{2n} + s q_{2n+1};$$

so that the general expression of z is

$$\begin{aligned} z &= [(A+1)(p_{2n} + s p_{2n+1}) + B \{ p_{2n} + (s+1)p_{2n+1} \}]^m \\ &\quad - N [(A+1)(q_{2n} + s q_{2n+1}) + B \{ q_{2n} + (s+1)q_{2n+1} \}]^m \\ &= [(A+B+1)p_{2n} + \{s(A+B+1) + B\} p_{2n+1}]^m \\ &\quad - N [(A+B+1)q_{2n} + \{s(A+B+1) + B\} q_{2n+1}]^m; \end{aligned}$$

or, writing

$$A+B+1 = u,$$

$$s(A+B+1) + B = u_s,$$

we have

$$z = (p_{2n}^m - N q_{2n}^m, p_{2n}^{m-1} p_{2n+1} - N q_{2n}^{m-1} q_{2n+1}, \dots, p_{2n+1}^m - N q_{2n+1}^m) \chi u, u_s)^m.$$

There are $m+1$ coefficients of which

$$\text{the first } p_{2n}^m - Nq_{2n}^m = M_{2n},$$

$$\text{the second } p_{2n}^{m-1} p_{2n+1} - Nq_{2n}^{m-1} q_{2n+1} = K_{2n+1} = a_{2n+1} M_{2n} - L_{2n},$$

$$\text{the } m\text{-th } p_{2n} p_{2n+1}^{m-1} - Nq_{2n} q_{2n+1}^{m-1} = L_{2n+1},$$

$$\text{the } m+1\text{-th } p_{2n+1}^m - Nq_{2n+1}^m = -M_{2n+1}.$$

We have to express the remaining coefficients from the 3rd to the $(m-1)$ -th inclusive in terms of the numbers L , M and the partial quotients a . Postponing the general consideration of this matter, I observe that the expression for z , viz.,

$$(up_{2n} + u_s p_{2n+1})^m - N(uq_{2n} + u_s q_{2n+1})^m,$$

is a linear transformation of $x^m - Ny^m$

to new variables u, u_s ;

the modulus of transformation being

$$\begin{vmatrix} p_{2n} & p_{2n+1} \\ q_{2n} & q_{2n+1} \end{vmatrix} = 1.$$

Hence we may employ known results in the theory of invariants of binary forms in the investigation.

It is convenient to consider in particular and in detail the cases of $m = 2, 3, 4, \dots$ in order to lead up to the theory of the general case.

$$3. \text{ The case } m = 2. \quad x^2 = Ny^2 + z.$$

The general value of z is

$$(M_{2n}, L_{2n+1}, -M_{2n+1} \chi(u, u_s)^2,$$

where

$$u = A + B + 1,$$

$$u_s = su + B,$$

A, B being arbitrary positive integers, and s a positive integer not exceeding a_{2n+2} .

Since here $K_{2n+1} = L_{2n+1}$, we have the relations

$$L_{2n} L_{2n+1} + N = a_{2n+1} L_{2n+1} M_{2n} + M_{2n} M_{2n+1},$$

$$L_{2n} + L_{2n+1} = a_{2n+1} M_{2n},$$

and thence

$$M_{2n} M_{2n+1} + L_{2n+1}^2 = N,$$

where $M_{2n} M_{2n+1}$ is an essentially positive number. This relation at once establishes the periodicity of the numbers M , the numbers L , and the numbers α . Also, for given values of A , B , it is clear that z has a finite number of numerical values.

The relation

$$M_{2n} M_{2n+1} + L_{2n+1}^2 = N$$

is obtained at once by equating the invariants of

$$x^2 - Ny^2$$

and

$$(M_{2n}, L_{2n+1}, -M_{2n+1} \chi(u, u_s)).$$

It leads to a relation between three values of z , if such values depend upon the same three numbers

$$M_{2m}, L_{2m+1}, L_{2m+1};$$

for write $z_{A, B, s} = (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(u_1, u_2))^2 = \alpha_u^2 = \beta_u^2 = \dots$,

$$z_{C, D, t} = (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(v_1, v_2))^2 = \alpha_v^2 = \beta_v^2 = \dots,$$

$$z_{E, F, r} = (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(w_1, w_2))^2 = \alpha_w^2 = \beta_w^2 = \dots$$

We may clearly eliminate

$$M_{2m}, L_{2m+1}, -M_{2m+1}$$

between these three equations and the invariant relation

$$M_{2n} M_{2n+1} + L_{2n+1}^2 = N;$$

for, squaring the symbolic identity

$$\alpha_u \beta_v - \alpha_v \beta_u = (\alpha \beta)(uv),$$

we obtain

$$\alpha_u^2 \beta_v^2 + \alpha_v^2 \beta_u^2 - (\alpha \beta)^2 (uv)^2 = 2 \alpha_u \alpha_v \beta_u \beta_v$$

and two similar identities which lead to the relations

$$z_{C, D, t} z_{E, F, r} + N (vw)^2 = \{ (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(v_1, v_2) \chi(w_1, w_2) \}^2,$$

$$z_{E, F, r} z_{A, B, s} + N (wu)^2 = \{ (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(w_1, w_2) \chi(u_1, u_2) \}^2,$$

$$z_{A, B, s} z_{C, D, t} + N (uv)^2 = \{ (M_{2m}, L_{2m+1}, -M_{2m+1} \chi(u_1, u_2) \chi(v_1, v_2) \}^2,$$

and, since $(\alpha\beta)^2 = -2N$,

$$\alpha_v \alpha_w = (M_{2m}, L_{2m+1}, -M_{2m+1}) \chi(v_1, v_2) \chi(w_1, w_2).$$

Moreover $(vw) \alpha_u + (wu) \alpha_v + (uv) \alpha_w = 0$;

whence $(vw)^2 \alpha_u^2 = (wu)^2 \alpha_v^2 + (uv)^2 \alpha_w^2 + 2(wu)(uv) \alpha_v \alpha_w$,

or

$$(vw)^2 z_{A, B, s} = (wu)^2 z_{C, D, t} + (uv)^2 z_{E, F, r} + 2(wu)(uv) \sqrt{\{z_{C, D, t} z_{E, F, r} + N(vw)^2\}}$$

and rationalising,

$$\begin{aligned} (vw)^4 z_{A, B, s}^2 + (wu)^4 z_{C, D, t}^2 + (uv)^4 z_{E, F, r}^2 - 2(wu)^2 (uv)^2 z_{C, D, t} z_{E, F, r} \\ - 2(uv)^2 (vw)^2 z_{E, F, r} z_{A, B, s} - 2(vw)^2 (wu)^2 z_{A, B, s} z_{C, D, t} \\ = 4(uv)^2 (vw)^2 (wu)^2 N. \end{aligned}$$

This relation can also be obtained as follows:—

The four quadratics $\alpha_u^2, \alpha_v^2, \alpha_w^2, \alpha_x^2$

are connected by the relation

$$\alpha_x^2 = \frac{(xv)(xw)}{(uv)(uw)} \alpha_u^2 + \frac{(xw)(xu)}{(vw)(vu)} \alpha_v^2 + \frac{(xu)(xv)}{(wu)(wv)} \alpha_w^2,$$

since the right-hand side is a homogeneous quadratic function of x_1, x_2 , and has the values

$$\alpha_u^2, \alpha_v^2, \alpha_w^2$$

for $x = u, v, w$ respectively; if we compare the coefficients of

$$x_1^2, x_1 x_2, x_2^2$$

on the two sides of the identity, we obtain the expressions of

$$M_{2m}, L_{2m+1}, M_{2m+1}$$

as linear functions of

$$\alpha_u^2, \alpha_v^2, \alpha_w^2.$$

Moreover, equating the discriminants of the two quadratic functions of x_1, x_2 , we obtain the sought quadratic relation between

$$\alpha_u^2, \alpha_v^2, \alpha_w^2.$$

The expressions

$$(vw), (wu), (uv)$$

involve arbitrary positive integers

$$A, B, C, D, E, F,$$

which can be determined so that

$$(vw)^2 = (wu)^2 = (uv)^2 = 1;$$

and then

$$z_{A, B, s}^2 + z_{C, D, t}^2 + z_{E, F, r}^2 - 2z_{C, D, t} z_{E, F, r} - 2z_{E, F, r} z_{A, B, s} - 2z_{A, B, s} z_{C, D, t} = 4N,$$

a representation of $4N$ as a ternary quadratic form.

Also

$$z_{C, D, t} z_{E, F, r} + N,$$

$$z_{E, F, r} z_{A, B, s} + N,$$

$$z_{A, B, s} z_{C, D, t} + N$$

are, each of them, perfect squares.

Now, it is clear that (vw) , (wu) , (uv)

cannot all be of the same sign.

If we choose the constants so that

$$(vw) = -(wu) = -(uv) = +1,$$

$$z_{A, B, s} = z_{C, D, t} + z_{E, F, r} + 2\sqrt{(z_{C, D, t} z_{E, F, r} + N)},$$

$$z_{C, D, t} = z_{E, F, r} + z_{A, B, s} - 2\sqrt{(z_{E, F, r} z_{A, B, s} + N)},$$

$$z_{E, F, r} = z_{A, B, s} + z_{C, D, t} - 2\sqrt{(z_{A, B, s} z_{C, D, t} + N)}.$$

In every case two of the signs attached to the radical will be negative and one positive.

Mr. G. B. Mathews has pointed out to me that, if

$$a = (vw)^2 \alpha_u^2, \quad b = (wu)^2 \alpha_v^2, \quad c = (uv)^2 \alpha_w^2,$$

$$f = (wu)(uv) \alpha_v \alpha_w, \quad g = (uv)(vw) \alpha_w \alpha_u, \quad h = (vw)(wu) \alpha_u \alpha_v,$$

the ternary quadratic form

$$(a, b, c, f, g, h | x, y, z)^2 = \{(vw) \alpha_u x + (wu) \alpha_v y + (uv) \alpha_w z\}^2,$$

$$bc - f^2 = ca - g^2 = ab - h^2 = -(vw)^2 (wu)^2 (uv)^2 N,$$

$$gh - af = hf - bg = fg - ch = -(vw)^2 (wu)^2 (uv)^2 N,$$

$$a - b = b - g = c - h,$$

$$a - b - c - 2f = b - c - a - 2g = c - a - b - 2h = 0,$$

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0,$$

and

$$af + bg + ch = + 2N(vw)^2(wu)^2(uv)^2,$$

$$gh + hf + fg = - N(vw)^2(wu)^2(uv)^2.$$

4. *The case* $m = 3$.— $x^3 = Ny^3 + z$.

The general value of z is

$$(M_{2n}, a_{2n+1} M_{2n} - L_{2n}, L_{2n+1}^{2n+1}, -M_{2n+1} \chi(u, u_s)^3;$$

and we have the relation

$$L_{2n} L_{2n+1} + N(p_{2n+1} q_{2n} + p_{2n} q_{2n+1}) = a_{2n+1} L_{2n+1} M_{2n} + M_{2n} M_{2n+1}.$$

Since $(p_{2n+1} q_{2n} + p_{2n} q_{2n+1})^2 = 1 + 4p_{2n} q_{2n} p_{2n+1} q_{2n+1}$

and $(p_{2n+1} q_{2n} + p_{2n} q_{2n+1}) - (p_{2n} q_{2n-1} + p_{2n-1} q_{2n}) = 2a_{2n+1} p_{2n} q_{2n},$

there is no difficulty in eliminating p and q and obtaining a relation between the quantities L , M , a , and N .

The following is a better method of procedure.

Equating the invariants of $x^3 - Ny^3$

and $(M_{2n}, a_{2n+1} M_{2n} - L_{2n}, L_{2n+1}, -M_{2n+1} \chi(u, u_s)^3,$

we obtain at once

$$M_{2n}^2 M_{2n+1}^2 + 6M_{2n} M_{2n+1} L_{2n+1} (a_{2n+1} M_{2n} - L_{2n}) - 4(a_{2n+1} M_{2n} - L_{2n})^3 M_{2n+1} \\ + 4M_{2n} L_{2n+1}^3 - 3(a_{2n+1} M_{2n} - L_{2n})^2 L_{2n+1}^2 = N^2.$$

This invariant relation between

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, a_{2n+1}, \text{ and } N$$

is the fundamental property of the continued fraction to

$$N^{\frac{1}{3}}.$$

It may be written

$$M_{2n}^2 M_{2n+1}^2 - 6M_{2n} M_{2n+1} L_{2n} L_{2n+1} + 4M_{2n+1} L_{2n}^3 + 4M_{2n} L_{2n+1}^3 - 3L_{2n}^2 L_{2n+1} \\ + 6(M_{2n} M_{2n+1} L_{2n+1} - 2M_{2n+1} L_{2n}^2 + L_{2n} L_{2n+1}^2) M_{2n} a_{2n+1} \\ + 3(4M_{2n+1} L_{2n} - L_{2n+1}^2) M_{2n}^2 a_{2n+1}^2 \\ - 4M_{2n+1} M_{2n}^3 a_{2n+1}^3 \\ = N^2.$$

From the Hessian of z , we obtain

$$\begin{aligned} & [M_{2n} L_{2n+1} - (a_{2n+1} M_{2n} - L_{2n})^2, \frac{1}{2} \{ -M_{2n} M_{2n+1} - L_{2n+1} (a_{2n+1} M_{2n} - L_{2n}) \}, \\ & \quad -M_{2n+1} (a_{2n+1} M_{2n} - L_{2n}) - L_{2n+1}^2] (u + u_s)^2 \\ & = -Nxy = -N(up_{2n} + u_s p_{2n+1})(uq_{2n} + u_s q_{2n+1}), \end{aligned}$$

yielding by comparison of coefficients

$$\begin{aligned} & M_{2n} L_{2n+1} - (a_{2n+1} M_{2n} - L_{2n})^2 = -Np_{2n} q_{2n}, \\ & -M_{2n} M_{2n+1} - L_{2n+1} (a_{2n+1} M_{2n} - L_{2n}) = -N(p_{2n} q_{2n+1} + p_{2n+1} q_{2n}), \\ & -M_{2n+1} (a_{2n+1} M_{2n} - L_{2n}) - L_{2n+1}^2 = -Np_{2n+1} q_{2n+1}. \end{aligned}$$

Of these the second relation has been previously found as a fundamental equation derivable from the continued fraction. Similarly from the cubic covariant

$$\begin{aligned} & -M_{2n}^2 M_{2n+1} - 3M_{2n} (a_{2n+1} M_{2n} - L_{2n}) L_{2n+1} + 2(a_{2n+1} M_{2n} - L_{2n})^3 \\ & = -N(p_{2n}^3 + Nq_{2n}^3), \end{aligned}$$

and three other relations.

By the solution of a cubic equation it is obvious that z can be expressed in terms of

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, N.$$

Let a value of z be

$$(M_{2n}, K_{2n+1}, L_{2n+1}, -M_{2n+1} \chi u_1, u_2)^3 = \alpha_u^3,$$

where

$$K_{2n+1} = a_{2n+1} L_{2n} - M_{2n},$$

and let other values of z , depending upon the same consecutive pair of principal convergents

$$\frac{p_{2n}}{q_{2n}}, \frac{p_{2n+1}}{q_{2n+1}},$$

be

$$\alpha_v^3, \alpha_w^3, \alpha_x^3, \alpha_y^3;$$

then we have the obvious identity

$$\alpha_y^3 = \frac{(yv)(yw)(yx)}{(uv)(uw)(ux)} \alpha_u^3 + \frac{(yw)(yx)(yu)}{(vw)(vx)(vu)} \alpha_v^3 + \frac{(yx)(yu)(yv)}{(wx)(wu)(wv)} \alpha_w^3 + \frac{(yu)(yv)(yw)}{(xu)(xv)(xw)} \alpha_x^3,$$

which is symmetrical in u, v, w, x, y , in that an identity remains whatever substitution be impressed upon these letters. Each side of this identity is a homogeneous cubic function of

$$y_1, y_2,$$

and, equating the discriminants, we obtain a homogeneous quartic function of

$$\alpha_u^3, \alpha_v^3, \alpha_w^3, \alpha_x^3$$

equal to a constant. This constant is a function of

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, a_{2n+1},$$

homogeneous and of degree 4, which, as shewn above, is equal to

$$N^2.$$

Hence, by giving special values to

$$u_1, u_2, v_1, v_2, w_1, w_2, x_1, x_2,$$

we get a representation of N^2 by means of a quaternary quartic form.

5. *The case $m = 4$.*— $x^4 = Ny^4 + z$.

We require an expression for

$$p_{2n}^2 p_{2n+1}^2 - N q_{2n}^2 q_{2n+1}^2 = X_{2n+1}.$$

Since

$$p_{2n+2} = a_{2n+2} p_{2n+1} + p_{2n},$$

we have

$$\begin{aligned} M_{2n+2} &= M_{2n} + 4(a_{2n+1} M_{2n} - L_{2n}) a_{2n+2} + 6X_{2n+1} a_{2n+2}^2 + 4L_{2n+1} a_{2n+2}^3 \\ &\quad - M_{2n+1} a_{2n+2}^4, \\ -L_{2n+2} &= a_{2n+1} M_{2n} - L_{2n} + 3X_{2n+1} a_{2n+2} + 3L_{2n+1} a_{2n+2}^2 - M_{2n+1} a_{2n+2}^3, \\ X_{2n+2} &= X_{2n+1} + 2L_{2n+1} a_{2n+2} - M_{2n+1} a_{2n+2}^2. \end{aligned}$$

The second of these relations enables the required expression.

From the first two relations, eliminating X_{2n+1} ,

$$\begin{aligned} &M_{2n+2} + 2L_{2n+2} a_{2n+2} \\ &= M_{2n} + 2(a_{2n+1} M_{2n} - L_{2n}) a_{2n+2} - 2L_{2n+1} a_{2n+2}^3 + M_{2n+1} a_{2n+2}^4. \end{aligned}$$

We now investigate the relations obtained from invariant theory. The expressions for z are

$$(up_{2n} + u_s p_{2n+1})^4 - N(uq_{2n} + u_s q_{2n+1})^4$$

and $(M_{2n}, a_{2n+1} M_{2n} - L_{2n}, X_{2n+1}, L_{2n+1}, -M_{2n+1} \chi(u, u_s))^4$

Equating the two invariants of the two forms

$$\begin{aligned} -M_{2n}M_{2n+1} - 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} + 3X_{2n+1}^2 &= -N, \\ -M_{2n}X_{2n+1}M_{2n+1} + 2(a_{2n+1}M_{2n} - L_{2n})X_{2n+1}L_{2n+1} - M_{2n}L_{2n+1}^2 \\ &\quad + (a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} - X_{2n+1}^3 = 0, \end{aligned}$$

whence

$$\begin{aligned} \{ -M_{2n}M_{2n+1} - 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} \} X_{2n+1} - 3M_{2n}M_{2n+1}X_{2n+1} \\ + 6(a_{2n+1}M_{2n} - L_{2n})L_{2n+1}X_{2n+1} - 3M_{2n}L_{2n+1}^2 \\ + 3(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} + NX_{2n+1} = 0; \end{aligned}$$

therefore

$$\begin{aligned} X_{2n+1} &= \frac{3M_{2n}L_{2n+1}^2 - 3(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1}}{-M_{2n}M_{2n+1} - 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - 3M_{2n}M_{2n+1} \\ &\quad + 6(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} + N} \\ &= 3 \frac{(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} - M_{2n}L_{2n+1}^2}{4M_{2n}M_{2n+1} - 2(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - N}, \end{aligned}$$

the required expression of X_{2n+1} .

Thence, from the invariant relations,

$$\begin{aligned} M_{2n}M_{2n+1} + 4(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - N \\ = 27 \left\{ \frac{(a_{2n+1}M_{2n} - L_{2n})^2M_{2n+1} - M_{2n}L_{2n+1}^2}{4M_{2n}M_{2n+1} - 2(a_{2n+1}M_{2n} - L_{2n})L_{2n+1} - N} \right\}^2, \end{aligned}$$

the invariant relation between

$$M_{2n}, M_{2n+1}, L_{2n}, L_{2n+1}, a_{2n+1} \text{ and } N,$$

which is the fundamental property of the continued fraction to

$$N^{\frac{1}{4}}.$$

Let the value of z

$$(u_1p_{2n} + u_2p_{2n+1})^4 - N(u_1q_{2n} + u_2q_{2n+1})^4$$

be denoted by

$$\alpha_u^4;$$

then we have the identity

$$\begin{aligned} \alpha_z^4 &= \frac{(zv)(zw)(zx)(zy)}{(uv)(uw)(ux)(uy)} \alpha_u^4 + \frac{(zw)(zx)(zy)(zu)}{(vw)(vx)(vy)(vu)} \alpha_v^4 + \frac{(zx)(zy)(zu)(zv)}{(wx)(wy)(wu)(wv)} \alpha_w^4 \\ &\quad + \frac{(zy)(zu)(zv)(zw)}{(xy)(xu)(xv)(xw)} \alpha_x^4 + \frac{(zu)(zv)(zw)(zx)}{(yu)(yv)(yw)(yx)} \alpha_y^4; \end{aligned}$$

and, equating the invariants of degrees 2 and 3, we find

(i.) a homogeneous quadratic function of

$$\alpha_u^4, \alpha_v^4, \alpha_w^4, \alpha_x^4, \alpha_y^4$$

equal to N ,

(ii.) a homogeneous cubic function of the same values of z equal to zero.

Results in integers are obtained by giving special values to

$$u_1, u_2, v_1, v_2, w_1, w_2, x_1, x_2, y_1, y_2$$

consistent with the definition of these quantities.

We obtain representation of N

by means of a quinary quadratic form.

In general the special properties of the continued fraction to

$$N^{1/m}$$

are obtainable from the invariant theory of the binary m -ic.

THE ASYMPTOTIC EXPANSION OF INTEGRAL FUNCTIONS DEFINED BY GENERALISED HYPERGEOMETRIC SERIES

By E. W. BARNES, Sc.D.

[Received December 3rd, 1906.—Read December 13th, 1906.]

1. The present paper is one of a series* in which the author has endeavoured to make a contribution to the theory of integral functions defined by Taylor's series.

Generalised hypergeometric functions form a wide class of integral functions whose asymptotic expansions are closely connected with the theory of linear differential equations. They appear to originate with Clausen†; two important papers are due to Thomae,‡ and the investigations of Goursat§ should also be consulted. But the detailed

* Barnes, (α) "The Asymptotic Expansion of $\sum_{n=0}^{\infty} \frac{x^n}{n! (x+\theta)}$ and the Singularities of

$g(x, \theta) = \sum_{n=0}^{\infty} \frac{x^n}{n+\theta}$," *Quarterly Journal of Mathematics*, Vol. xxxvii., pp. 289-313.

——— (β) "The Asymptotic Expansion of Integral Functions defined by Taylor's Series," *Philosophical Transactions of the Royal Society (A)*, Vol. 206, pp. 249-297.

——— (γ) "On certain Functions defined by Taylor's Series of Finite Radius of Convergence," *Proceedings of the London Mathematical Society*, Ser. 2, Vol. 4, pp. 284-316.

——— (δ) "On the Asymptotic Expansion of the Integral Functions

$$\sum_{n=0}^{\infty} \frac{x^n \Gamma(1+an)}{\Gamma(1+n)} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{x^n \Gamma(1+n\theta)}{\Gamma(1+n+n\theta)},$$

Cambridge Philosophical Transactions, Vol. xx., pp. 215-232.

——— (ε) "On the Use of Factorial Series in an Asymptotic Expansion," *Quarterly Journal of Mathematics*, Vol. xxxviii., pp. 116-140.

——— (η) "On Functions defined by Simple Types of Hypergeometric Series," *Cambridge Philosophical Transactions*, Vol. xx., pp. 253-279.

† Clausen, (1828) *Crelle*, Bd. iii., pp. 89-92.

‡ Thomae, (1869) *Mathematische Annalen*, Bd. ii., pp. 427-444.

——— (1879) *Crelle*, Bd. lxxxvii., pp. 222-349.

§ Goursat, (1883) *Annales de l'Ecole Normale Supérieure*, Sér. 2, T. xii., pp. 261-395.

——— (1884) *Acta Mathematica*, Bd. v., pp. 97-120.

development of the theory is largely due to Pochhammer,* whose voluminous writings form an interesting study.

In the present paper I take substantially the notation of Pochhammer. The general type of series considered is

$$1 + \frac{a_1 \dots a_p}{1 \cdot \rho_1 \dots \rho_q} x + \frac{a_1(a_1+1) \dots a_p(a_p+1)}{1 \cdot 2 \cdot \rho_1(\rho_1+1) \dots \rho_q(\rho_q+1)} x^2 + \dots$$

$$= \frac{\Gamma(\rho_1) \dots \Gamma(\rho_q)}{\Gamma(a_1) \dots \Gamma(a_p)} \sum_{n=0}^{\infty} \frac{\Gamma(a_1+n) \dots \Gamma(a_p+n)}{n! \Gamma(\rho_1+n) \dots \Gamma(\rho_q+n)} x^n$$

wherein $p \leq q$. This series we shall denote by

$${}_pF_q \{ a_1, \dots, a_p; \rho_1, \dots, \rho_q; x \}$$

or, briefly, by ${}_pF_q(x)$.

The series satisfies the differential equation

$$\left[(\mathfrak{S} + a_1) \dots (\mathfrak{S} + a_p) - \frac{d}{dx} (\mathfrak{S} + \rho_1 - 1) \dots (\mathfrak{S} + \rho_q - 1) \right] y = 0 \quad (1)$$

wherein $\mathfrak{S} = x(d/dx)$. The equation is of order $(q+1)$, and q other linearly independent solutions are given by

$$x^{1-\rho_m} {}_pF_q \{ a_1 - \rho_m + 1, \dots, a_p - \rho_m + 1; 2 - \rho_m, \rho_1 - \rho_m + 1, \dots, \rho_q - \rho_m + 1; x \}$$

where $m = 1, 2, \dots, q$. Among the quantities $\rho_r - \rho_m + 1$, that corresponding to $m = r$ is to be omitted.

It subsequently proves convenient to make a change in our notation; so that, for brevity, we write

$$Q_0(x) = \frac{\prod_{r=1}^q \Gamma(1 - \rho_r)}{\prod_{r=1}^p \Gamma(1 - a_r)} {}_pF_q \{ a_1, \dots, a_p; \rho_1, \dots, \rho_q; (-)^{\mu} x \},$$

* Pochhammer, (1886) "Ueber die Differentialgleichung der allgemeineren hypergeometrische Reihe mit zwei endlichen singulären Punkten," *Crelle*, Bd. cii., pp. 76-159.

— (1888) "Ueber gewisse partielle Differentialgleichungen denen hypergeometrische Integrale genügen," *Mathematische Annalen*, Bd. xxxiii., pp. 353-371.

— (1891) "Ueber die Differentialgleichung der allgemeineren F -Reihe," *Mathematische Annalen*, Bd. xxxviii., pp. 586-597.

— (1891) "Ueber die Differentialgleichungen der Reihen $F(\rho, \sigma; x)$ und $F(\rho, \sigma, \tau; x)$," *Mathematische Annalen*, Bd. xli., pp. 197-218.

— (1893) "Ueber die Reduction der Differentialgleichung der allgemeineren F -Reihe," *Crelle*, Bd. cxii., pp. 58-86.

— (1895) "Ueber die Differentialgleichungen der F -Reihen 3-ter Ordnung," *Mathematische Annalen*, Bd. xli., pp. 584-605.

— (1898) "Ueber die Differentialgleichungen der F -Reihen 4-ter Ordnung," *Mathematische Annalen*, Bd. l., pp. 285-302.

The reader of Pochhammer's papers will note that an earlier paper (1870, *Crelle*, Bd. lxxiii., pp. 135-157) dealt with series of a different type, to which the name "hypergeometric" was also given.

$$Q_m(x) = x^{1-\rho_m} \frac{\Gamma(\rho_m-1) \prod_{t=1}^q \Gamma(\rho_m-\rho_t)}{\prod_{t=1}^p \Gamma(\rho_m-\alpha_t)} \\ \times {}_pF_q \{ 1+\alpha_1-\rho_m, \dots, 1+\alpha_p-\rho_m; 2-\rho_m, \dots, \rho_q-\rho_m+1; (-)^m x \} \\ (m = 1, 2, \dots, q).$$

Then, evidently, the $(q+1)$ linearly independent solutions of the differential equation (1) are

$$Q_m \{ (-)^m x \} \quad (m = 0, 1, 2, \dots, q).$$

We shall put $\mu = q+1-p$, so that μ is an integer ≥ 1 .

2. Various types of integral functions were considered by the author in the memoir "On the Asymptotic Expansion of Integral Functions defined by Taylor's Series." Parts IX. and X. of this paper contained a *résumé* of results relating to the most simple hypergeometric integral functions ${}_1F_1\{a; \rho; x\}$ and ${}_0F_1\{\rho; x\}$. The proofs of such results were given in detail in a subsequent paper.* By means of contour integrals involving gamma functions of the variable in the subject of integration, it was shewn to be possible to develop the theory of ${}_1F_1\{a; \rho; x\}$ with considerable simplicity; and from it the theory of ${}_0F_1\{\rho; x\}$ was deduced by Kummer's formula

$${}_0F_1\{\rho; x\} = e^{2x^{\frac{1}{2}}} {}_1F_1\{\rho-\frac{1}{2}; 2\rho-1; -4x^{\frac{1}{2}}\}.$$

These two functions are the most simple examples of the two classes into which higher hypergeometric integral transcendents can be divided, the division corresponding to $\mu = 1$ or $\mu > 1$. References to the history and literature of the asymptotic theory of these two elementary transcendents will be found in the paper just cited.

The corresponding theory of the higher transcendents which forms the subject of the present investigation is by no means an obvious extension of more elementary results. The asymptotic expansions which arise for the two simple functions ${}_1F_1\{a; \rho; x\}$ and ${}_0F_1\{\rho; x\}$ have themselves the form of hypergeometric series. In the more complex cases this not true. Moreover, there does not appear to be any analogue to Kummer's formula which we can use to deduce the cases when $\mu > 1$ from cases when $\mu = 1$. Nor has any analogue to Gauss's expression of ${}_2F_1\{a, \beta; \gamma; 1\}$ in terms of gamma functions been discovered, and Mellin denies the possibility of its existence.

* *Loc. cit.*, § 1, Paper (η).

3. The results of the present theory were adumbrated by Stokes,* who shewed that, *if x be real and positive,*

$$\frac{\prod_{r=1}^p \Gamma(a_r)}{\prod_{r=1}^q \Gamma(\rho_r)} {}_pF_q \{x\} = (2\pi)^{\frac{1}{2}(1-\mu)} \mu^{-\frac{1}{2}} x^{[\sum a - \sum \rho + \frac{1}{2}(\mu-1)]/\mu} \exp\{\mu x^{1/\mu}\} J$$

where J tends to unity as x tends to infinity.

The general theory when x can take any complex value is due to Orr.† In his first paper Orr shewed that, if

$$|\arg x| < \pi \{1 + \frac{1}{2}\mu\} \quad \text{and} \quad m = 1, 2, \dots, p,$$

$$\begin{aligned} & \frac{1}{\sin \pi a_m} Q_0(-x) + \sum_{r=1}^q \frac{Q_r(-x)}{\sin \pi(\rho_r - a_m)} \\ &= \frac{\Gamma(a_m) \prod_{r=1}^q \Gamma(a_m - \rho_r + 1)}{\pi \prod_{r=1}^p \Gamma(1 + a_m - a_r)} (x)^{-a_m} {}_{q+1}F_{p-1} \{a_m, a_m + 1 - \rho_1, \dots, a_m + 1 - \rho_q; \\ & \quad a_m - a_1 + 1, \dots, a_m - a_p + 1; -1/x\}. \end{aligned} \quad (\text{A})$$

This result, which is obtained almost intuitively in Part I. of the present paper, was, with Orr, the main outcome of some thirty pages of laborious analysis in which it was derived by an elaborate process of induction: the notable simplification obtained by the present methods will be at once apparent to the reader.

In his second paper Orr obtained the dominant term of the asymptotic equality which is given in Parts II. and III. of the present memoir. In Part III. I shew that, if $|\arg x| < \pi$ and $m = 1, 2, \dots, \mu$,

$$\begin{aligned} & Q_0(x) + \sum_{r=1}^q e^{(\mu-2m)(\rho_r-1)\pi i} Q_r(x) \\ &= \exp\{-\mu e^{(\mu-2m)\pi i/\mu} x^{1/\mu}\} (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} x^{\theta/\mu} e^{(\mu-2m)\theta\pi i} \sum_{n=0}^{\infty} \frac{\lambda_n e^{n(2m-\mu)\pi i/\mu}}{x^{n/\mu}} \end{aligned} \quad (\text{B})$$

where

$$\theta = [\sum a - \sum \rho + \frac{1}{2}(\mu-1)]/\mu.$$

Orr's second paper bears traces of great compression: the argument is most difficult to follow, and at times the methods seem open to serious objection. Moreover, the method only gives the dominant term of the asymptotic expansion in question, and it is obviously desirable to shew how subsequent terms arise in the general case, even though their complexity is such that they are not actually obtained.

* Stokes, *Proceedings of the Cambridge Philosophical Society*, Vol. vi., pp. 362-366 (1839).

† Orr, *Cambridge Philosophical Transactions*, Vol. xvii., pp. 171-199 (1898); pp. 283-290 (1899).

Further, Orr only adumbrates the converse theory of the expression of the hypergeometric integral functions separately in terms of groups of asymptotic expansions. He suggests that such should be obtained by elimination between the results of the equations (A) and (B). In this paper I shew that the contour integrals which I employ give such expressions even more easily than they give the converse equalities. And the nature of certain apparent redundancies which the latter equations seem to contain is at once apparent from the present theory, which appears to solve every problem which the asymptotic theory of the series presents. The importance of the theory and its generality must be my excuse for the length of this paper.

4. The simple method used in Part I. to obtain the formula (A) of § 3 is an obvious application of ideas which I have employed for many classes of functions. The idea of employing contour integrals involving gamma functions of the variable in the subject of integration appears to be due to Pincherle,* whose suggestive paper was the starting point of the investigations of Mellin,† though the type of contour and its use can be traced back to Riemann [*Œuvres Mathématiques* (1898), pp. 166–167]. The formula (B) is, however, only obtained by the use of factorial series of a somewhat complicated type, and the reduction of these series to more simple forms and the development of their properties are the most novel features of the present investigation. This use of factorial series to obtain the asymptotic expansion of integral functions for the region at infinity for which they are exponentially infinite is due to the author, and was employed in a recent paper‡ in which his asymptotic expansion of $G_\beta(x, \theta)$ when $R(x) > 0$ § was obtained anew. Each type of integral function discussed gives rise to a new type of factorial series; but wide classes of such series possess similar properties, and the investigation in Part III. of the present paper is, to some extent, based on the properties of the more elementary series

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{n! (n+\theta)^\beta}$$

previously discussed.

5. When $\mu = 1$ the results for generalised hypergeometric integral functions differ in character from, and can be obtained by more simple analysis than, those which hold good for other integral values of μ .

* Pincherle, "Sulle funzioni ipergeometriche generalizzate" (1888), *Atti d. R. Accademia dei Lincei*, Ser. 4, *Rendiconti*, Vol. iv., pp. 694–700, and pp. 792–799.

† Mellin, *Acta Societatis Scientiarum Fennicae* (1895), T. xx., No. 12.

‡ *Loc. cit.*, § 1, Paper (ε).

§ *Loc. cit.*, § 1, Paper (β), Part III.

Consequently in Part II. of the present paper this special case is considered separately.

I construct the function ${}_pS_p(s)$ defined when $R(s) > R(\Sigma\alpha - \Sigma\rho)$ by the factorial series

$$\sum_{t=0}^{\infty} \frac{\Gamma(t-s)}{\Gamma(t+1)} \prod_{r=1}^p \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-\alpha_r+t-s)}$$

and shew by transformations depending on contour integrals

(1) that ${}_pS_p(s)$ admits of analytic continuation over the whole of the finite portion of the s -plane;

(2) that it has simple poles at the points

$$s = \Sigma\alpha - \Sigma\rho - r \quad (r = 0, 1, 2, \dots, \infty)$$

at which the residues can be calculated with sufficient labour, and no other finite singularities, except the obvious ones which belong to $\Gamma(-s)$ and $\Gamma(1-\rho_r-s)$;

(3) that, if $s = u + iv$, when u and v are real, and if u be finite and $\epsilon > 0$, $|{}_pS_p(s)| \exp\{(\frac{3}{2}\pi - \epsilon)|v|\}$ can, by taking $|v| > a$ sufficiently large positive quantity V , be made less than any arbitrarily assigned positive quantity η ;*

(4) that, if both u and $|v|$ be large and positive and $\epsilon > 0$,

$$|{}_pS_p(s)| < \eta_u \exp\{-\frac{3}{2}\pi + \epsilon\}|v|$$

where, for any finite positive value of k , $\eta_u \exp\{ku\}$ can be made as small as we please by taking u sufficiently large, if s be not in the immediate vicinity of one of the poles of ${}_pS_p(s)$.

From these results the asymptotic expansion of that combination of generalised hypergeometric integral functions ($\mu = 1$) which is exponentially infinite at infinity is deduced.

6. *The case when μ is an integer > 1* is discussed in Part III. There I construct the function ${}_pS_q(s)$ defined when

$$R(s) > R[\Sigma\alpha - \Sigma\rho + \frac{1}{2}(\mu-1)]/\mu$$

by the series

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu) \prod_{r=1}^q \Gamma(1-\rho_r+t/\mu-s)}{\prod_{r=1}^{\mu} \Gamma(t/\mu+r/\mu) \prod_{r=1}^p \Gamma(1-\alpha_r+t/\mu-s)},$$

and by more elaborate methods shew that it possesses similar properties:—

(1) ${}_pS_q(s)$ is an analytic function over the whole s -plane.

(2) Apart from the obvious poles which arise from $\Gamma(-\mu s)$ and $\Gamma(\mu - \mu\rho_r - \mu s)$, it has for its sole finite singularities simple poles at

* In this case I say that $|{}_pS_p(s)| \exp\{(\frac{3}{2}\pi - \epsilon)|v|\}$ tends uniformly to zero as $|v|$ tends to infinity.

the points $[\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu - 1) - r]/\mu \quad (r = 0, 1, 2, \dots, \infty)$
at which the residues can be calculated with sufficient labour.

(3) The expression $|{}_pS_q(s)| \exp\{[(\mu + 1)\pi - \epsilon]|v|\}$ tends uniformly to zero as $|v|$ tends to infinity, if $\epsilon > 0$.

(4) $|{}_pS_q(s)| < \eta_u \exp\{[-(\mu + 1)\pi - \epsilon]|v|\}$ when u and $|v|$ are large and positive, where $\eta_u \exp\{ku\}$ can be made as small as we please by taking u sufficiently large and independent of $|v|$.

From these results we deduce the complete series of asymptotic expansions (B) given in § 3.

I do not in the present paper attempt to apply the methods used to more general types of factorial series (*e.g.*, when μ is not an integer), nor do I consider the associated problem of the nature of the finite singularities of generalised hypergeometric series of finite radius of convergence. These questions I hope to consider on a future occasion.

PART I.

The Asymptotic Expansion of Linear Combinations of Generalised Hypergeometric Integral Functions which are not exponentially Infinite at Infinity.

7. Let $p \leq q$ and $\mu = q + 1 - p$. We proceed to shew that there are p asymptotic equalities of the type

$$\begin{aligned} & \frac{\Gamma(\alpha_1) \Gamma(1 - \rho_1) \dots \Gamma(1 - \rho_q)}{\Gamma(1 - \alpha_2) \dots \Gamma(1 - \alpha_p)} {}_pF_q\{ \alpha_1, \dots, \alpha_p; \rho_1, \dots, \rho_q; (-)^{p-q}x \} \\ & + \sum_{r=1}^q \frac{\Gamma(\alpha_1 - \rho_r + 1) \Gamma(\rho_r - 1) \prod_{t=1}^q \Gamma(\rho_r - \rho_t)}{\prod_{t=2}^p \Gamma(\rho_r - \alpha_t)} \\ & \times x^{1-\rho_r} {}_pF_q\{ \alpha_1 - \rho_r + 1, \dots, \alpha_p - \rho_r + 1; \\ & \quad 2 - \rho_r, 1 - \rho_r + \rho_1, \dots, 1 - \rho_r + \rho_q; (-)^{p-q}x \}, \\ & = \frac{\Gamma(\alpha_1) \prod_{t=1}^q \Gamma(\alpha_1 - \rho_t + 1)}{\prod_{t=2}^p \Gamma(1 + \alpha_1 - \alpha_t)} x^{-\alpha_1} {}_{q+1}F_{p-1}\{ \alpha_1, \alpha_1 - \rho_1 + 1, \dots, \alpha_1 - \rho_q + 1; \\ & \quad \alpha_1 - \alpha_2 + 1, \dots, \alpha_1 - \alpha_p + 1; -1/x \}, \quad (A) \end{aligned}$$

each valid when $|\arg x| < \pi(1 + \frac{1}{2}\mu)$. In the asymptotic equality the error which results from stopping at the $(k + 1)$ -th term of the series for ${}_{q+1}F_{p-1}\{-1/x\}$ is a quantity J_k where $|J_k x^{\alpha_1 + k}|$ can be made as small as we please by taking $|x|$ sufficiently large. The other $(p - 1)$ asymptotic equalities are obtained by interchanging α_1 with $\alpha_2, \alpha_3, \dots, \alpha_p$ respectively.

Let us denote the expression on the left-hand side of the asymptotic equality (A) by Π .

We will first shew that

$$\Pi = -\frac{1}{2\pi i} \int \frac{\Gamma(-s) \Gamma(a_1+s) \prod_{t=1}^q \Gamma(1-\rho_t-s)}{\prod_{t=2}^p \Gamma(1-\alpha_t-s)} x^s ds, \quad (B)$$

the integral being taken round a contour which embraces the positive half of the real axis and encloses all the poles of the subject of integration except those of $\Gamma(a_1+s)$.

In the first place, since $q+1 > p$, the integral is absolutely convergent. By Cauchy's theory of residues it is therefore equal to the sum of the residues of the subject of integration inside the contour.

Now the residue of $\Gamma(n-s)$ at its pole $s = n+m$ is $(-)^{m-1}/m!$. Hence the integral is equal to

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \frac{\Gamma(a_1+n) \prod_{t=1}^q \Gamma(1-\rho_t-n)}{\prod_{t=2}^p \Gamma(1-\alpha_t-n)} \\ & + \sum_{r=1}^q x^{1-\rho_r} \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \frac{\Gamma(\rho_r-1-n) \prod_{t=1}^q \Gamma(\rho_r-\rho_t-n) \Gamma(a_1+1-\rho_r+n)}{\prod_{t=2}^p \Gamma(\rho_r-\alpha_t-n)} = \Pi. \end{aligned}$$

8. We may next alter the contour of integration so as to give us the asymptotic expansion (A).

By the asymptotic expansion of the gamma function we know that, if $|\arg s| < \pi$ and $|s|$ be large,

$$\Gamma(s+a) = s^{s-\frac{1}{2}+a} e^{-s} \sqrt{2\pi J}$$

where $|J|$ tends uniformly to unity as $|s|$ tends to infinity.

Hence, if $s = u+iv$ where u is finite and $|v|$ is large,

$$|\Gamma(s+a)| = \exp \left\{ -\frac{1}{2}\pi |v| \right\} |v^{a+u-\frac{1}{2}} e^{\pm \frac{1}{2}i\pi a}| \sqrt{2\pi J},$$

the $+$ or $-$ sign being taken as v positive or negative.

Hence the contour integral (B) will vanish when taken round that part of a great circle at infinity for which $R(s) > -k$ where k is any finite positive quantity, provided

$$|\arg x| < (q+3-p)\frac{1}{2}\pi,$$

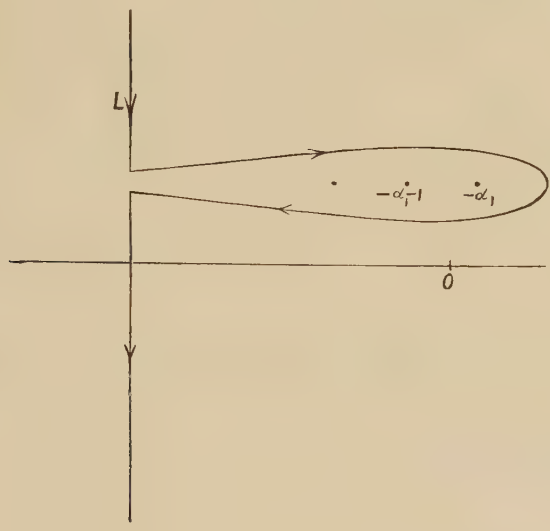
i.e.,

$$|\arg x| < \pi(1+\frac{1}{2}\mu).$$

If, then, L is a line parallel to the imaginary axis which passes between the points $-a_1-k$ and $-a_1-k-1$, we shall have $\Pi = -\frac{1}{2\pi i} \int_L$ together with the sum of the residues of

$$\frac{\Gamma(-s) \Gamma(a_1+s) \prod_{t=1}^q \Gamma(1-\rho_t-s) x^s}{\prod_{t=2}^p \Gamma(1-\alpha_t-s)}$$

at its poles $-a_1, -a_1-1, \dots, -a_1-k$.



Thus
$$\Pi = x^{-a_1} \sum_{n=0}^k \frac{(-x)^{-n}}{n!} \frac{\Gamma(a_1+n) \prod_{t=1}^q \Gamma(1-\rho_t+a_1+n)}{\prod_{t=2}^p \Gamma(1-\alpha_t+a_1+n)} + J_k,$$

and it is evident that $|J_k x^{a_1+k}|$ for any finite value of k can be made as small as we please by taking $|x|$ greater than an assignable positive large quantity.

We thus have the given asymptotic expansion.

9. Let us consider the set of expansions of the type (A) in detail.

When $p = q = 1$, so that $\mu = 1$, we have one asymptotic expansion of the sum of two hypergeometric integral functions, and this asymptotic equality is valid when $|\arg x| < \frac{3}{2}\pi$.

The single expansion is therefore equal to two separate expansions

when $R(x) < 0$.[†] From these two we may eliminate one of the hypergeometric integral functions and obtain an asymptotic expansion for the other when $R(x) < 0$ whose dominant term is only algebraically infinite (or zero). The work has been carried out in my previous paper,[‡] where it has been shewn that we get the asymptotic equality

$${}_1F_1\{a; \rho; x\} = \frac{\Gamma(\rho)}{\Gamma(\rho-a)} (-x)^{-a} {}_2F_0\{a, 1+a-\rho; -1/x\}.$$

More generally, when $p = q > 1$, so that $\mu = 1$, we have p asymptotic expansions of the sum of $(p+1)$ hypergeometric integral functions, and these asymptotic equalities are valid when $|\arg x| < \frac{3}{2}\pi$. There are thus in all $2p$ relations when $R(x) < 0$. At first sight we might be tempted to say that by eliminating p of the hypergeometric integral functions we can obtain p different asymptotic expansions, when $R(x) < 0$, for the remaining hypergeometric function. This, however, is evidently absurd, and, in point of fact, of the $2p$ asymptotic equalities only $p+1$ are independent, as may be shewn by the somewhat laborious algebra which results from writing down the equations and actually performing the elimination.

The single asymptotic equality which results from eliminating all the hypergeometric integral functions in the expression Π except the first is actually the relation, valid when $R(x) < 0$,

$$\begin{aligned} \prod_{t=1}^p \left\{ \frac{\Gamma(a_t)}{\Gamma(\rho_t)} \right\} {}_pF_p\{a_1, \dots, a_p; \rho_1, \dots, \rho_p; x\} \\ = \sum_{r=1}^p (-x)^{-a_r} \frac{\Gamma(a_r) \prod_{t=1}^p \Gamma(a_t - a_r)}{\prod_{t=1}^p \Gamma(\rho_t - a_r)} {}_{p+1}F_{p-1}\{a_r, 1+a_r-\rho_1, \dots, 1+a_r-\rho_p; \\ 1+a_r-a_1, \dots, *, \dots, 1+a_r-a_p; -1/x\}, \end{aligned}$$

the star denoting the omission of the term $1+a_r-a_r$.

10. This may be proved directly by contour integration as follows:—
We consider the integral

$$-\frac{1}{2\pi i} \int \frac{\Gamma(-s) \prod_{t=1}^p \Gamma(a_t + s)}{\prod_{t=1}^p \Gamma(\rho_t + s)} (-x)^s ds$$

taken round a contour which embraces the positive half of the real axis

[†] The reader may compare the more complex case of § 48, where the various ranges of $\arg x$ are given in detail.

[‡] *Cambridge Philosophical Transactions*, Vol. xx., pp. 253-279, § 12.

and encloses only the poles $s = 0, 1, 2, \dots$ of the subject of integration. It is evidently, by Cauchy's theory of residues, equal to

$$\prod_{t=1}^p \left\{ \frac{\Gamma(a_t)}{\Gamma(\rho_t)} \right\} {}_pF_p \{ a_1, \dots, a_p; \rho_1, \dots, \rho_p; x \}.$$

But, if $|\arg(-x)| < \frac{1}{2}\pi$, we may swing the integral back as in § 8. We thus see that it is asymptotically equal to

$$\sum_{r=1}^p (-x)^{-a_r} \sum_{n=0}^{\infty} \frac{x^{-n}}{n!} \frac{\Gamma(a_r+n) \prod_{t=1}^p \Gamma(a_t-a_r-n)}{\prod_{t=1}^p \Gamma(\rho_t-a_r-n)}.$$

We thus have the given result.

11. When $\mu > 1$ there is no direct analogue of the asymptotic equality just obtained: in fact, there is no region of the plane at infinity over which ${}_pF_q\{x\}$ can be represented by an asymptotic expansion whose dominant term is algebraically infinite (or zero).†

To take the simplest case, suppose that $\mu = 2$ so that $q = p+1$. Then we have p equalities of the type (A), § 7, each of which gives the asymptotic expansion of a sum of $(p+2)$ hypergeometric integral functions valid when $|\arg x| < 2\pi$. These are equivalent to $2p$ relations between the $(p+2)$ functions when $|\arg x| < \pi$. But only $(p+1)$ of these relations are independent. Therefore the sum of suitable multiples of two hypergeometric integral functions ($p = q-1 > 0$) can be expressed as an asymptotic expansion with its dominant term algebraic, and the equality will hold all round infinity; but a single hypergeometric integral function ($p = q-1$) does not admit of such an expansion for any portion of the plane at infinity.

In the case $p = 0, q = 1$, discussed in my previous memoir, no asymptotic expansion exists whose dominant term is algebraic.

More generally, when $\mu \geq 2$, the p equalities of the type (A), § 7, give rise to $(p+1)$ independent asymptotic equalities, each involving $(q+1)$ hypergeometric integral functions. Hence the sum of suitable multiples of $q+1-p = \mu$ such functions can be represented by an asymptotic expansion whose dominant term is algebraic, but no combination of less than μ is capable of such representation.

† A reference to Parts II. and III. will shew the connection between this theorem and the fact that e^x tends to zero as $|x|$ tends to infinity when $|\arg(-x)| < \frac{1}{2}\pi$, while $\exp(\mu/x^{1/\mu})$ where μ is an integer > 1 does not possess a similar property when $|\arg x| < \pi$.

12. The fundamental combination of this type may be obtained by considering the integral

$$-\frac{1}{2\pi i} \int \frac{\Gamma(-s) \prod_{r=1}^p \Gamma(a_r+s) \prod_{r=p+1}^q \Gamma(1-\rho_r-s)}{\prod_{r=1}^p \Gamma(\rho_r+s)} (-x)^s ds.$$

If the integral be taken round a contour which embraces the positive half of the real axis and excludes the poles of $\prod_{r=1}^p \Gamma(a_r+s)$, it will be equal to

$$\begin{aligned} & \frac{\prod_{r=1}^p \Gamma(a_r)}{\prod_{r=1}^p \Gamma(\rho_r)} \prod_{r=p+1}^q \Gamma(1-\rho_r) {}_pF_q \{ a_1, \dots, a_p; \rho_1, \dots, \rho_q; (-)^{q-p} x \} \\ & + \sum_{r=p+1}^q (-x)^{1-\rho_r} \frac{\Gamma(\rho_r-1) \prod_{t=1}^p \Gamma(1+a_t-\rho_r) \prod_{t=p+1}^q \Gamma(\rho_r-\rho_t)}{\prod_{t=1}^p \Gamma(\rho_t+1-\rho_r)} \\ & \times {}_pF_q \{ 1+a_1-\rho_r, \dots, 1+a_p-\rho_r; \\ & 2-\rho_r, 1+\rho_1-\rho_r, \dots, {}^* \dots, 1+\rho_q-\rho_r; (-)^{q-p} x \}. \end{aligned}$$

If $|\arg(-x)| < \frac{1}{2}\mu\pi$, the integral may be swung back in the previous manner and we shall obtain, as the asymptotic expansion of the linear combination of the μ generalised hypergeometric integral functions just written down, the series

$$\begin{aligned} & \sum_{r=1}^p (-x)^{-a_r} \frac{\Gamma(a_r) \prod_{t=1}^p \Gamma(a_t-a_r) \prod_{t=p+1}^q \Gamma(1+a_r-\rho_t)}{\prod_{t=1}^p \Gamma(\rho_t-a_r)} \\ & \times {}_{p+1}F_{q-1} \{ a_r, a_r+1-\rho_1, \dots, a_r+1-\rho_q; \\ & 1+a_r-a_1, \dots, {}^* \dots, 1+a_r-a_p; -1/x \}. \end{aligned}$$

The sum of these p series is asymptotic in Poincaré's sense.

In the same way, by considering other contour integrals which may with ease be written down, we can obtain suitable combinations of any number [less than $(q+2)$ and greater than $(\mu-1)$] of hypergeometric integral functions whose asymptotic expansions have their dominant terms algebraic.

We thus get all the results which can be derived from the $(p+1)$ independent asymptotic expansions contained in the formula (A) of § 7 by eliminating any possible number of the hypergeometric integral functions involved.

Other combinations of the hypergeometric integral functions involved have their dominant terms exponentially infinite: to the development of such expansions we now proceed.

PART II.

The Exponentially Infinite Asymptotic Expansions of Linear Combinations of Generalised Hypergeometric Integral Functions for which $p = q$.

13. In the ensuing investigation we limit ourselves to the case $p = q$, $\mu = 1$, partly because the theory may be developed in a more simple manner than the most general theory when $\mu > 1$, partly because the results differ somewhat in character from those of that theory.

The differential equation of the functions considered is now

$$[(\mathfrak{S} + \alpha_1) \dots (\mathfrak{S} + \alpha_p) - \frac{d}{dx} (\mathfrak{S} + \rho_1 - 1) \dots (\mathfrak{S} + \rho_p - 1)] y = 0.$$

It is of order $(p+1)$, and the independent primitives are

$$\begin{aligned} & {}_pF_p\{ \alpha_1, \dots, \alpha_p; \rho_1, \dots, \rho_p; x \}, \\ & x^{1-\rho_m} {}_pF_p\{ \alpha_1 - \rho_m + 1, \dots, \alpha_p - \rho_m + 1; \\ & \quad 2 - \rho_m, \rho_1 - \rho_m + 1, \dots, * \dots, \rho_p - \rho_m + 1; x \} \\ & \qquad \qquad \qquad (m = 1, 2, \dots, p). \end{aligned}$$

We have seen in §§ 9 and 10 that when $R(x) < 0$ each of these functions can, near $|x| = \infty$, be represented by an asymptotic expansion whose dominant term is algebraic. We now proceed to find asymptotic expansions of these functions, valid when $R(x) > 0$, and to find Orr's linear combination of the functions which has an asymptotic expansion of the same exponential type.

For this purpose we construct the function ${}_pS_p(s; \alpha_1, \dots, \alpha_p; \rho_1, \dots, \rho_p)$ or, as we shall briefly write it, ${}_pS_p(s)$, which, when $R(s) > R(\Sigma \alpha - \Sigma \rho)$, is represented by the convergent series of gamma functions

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t)}{\Gamma(t+1)} \prod_{r=1}^p \left(\frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-\alpha_r+t-s)} \right).$$

I have previously discussed the corresponding function for which $p = 1$, and have shewn (*loc. cit.*, p. 263) that

$${}_1S_1(s; \alpha; \rho) = \frac{\Gamma(-s) \Gamma(-s+1-\rho) \Gamma(s+\rho-\alpha)}{\Gamma(1-\alpha) \Gamma(\rho-\alpha)}.$$

This equality corresponds to Gauss's well known theorem. It shews that ${}_1S_1(s)$ admits of analytic continuation over the whole finite portion of the s -plane; that ${}_1S_1(s)$ has simple poles at $s = a - \rho - r$ ($r = 0, 1, 2, \dots, \infty$), and no other finite singularities except those of $\Gamma(-s)$ and $\Gamma(1-s-\rho)$; and that, if $s = u + iv$ and u be finite, $|{}_1S_1(s)| \exp\{(\frac{3}{2}\pi - \epsilon)|v|\}$ tends to zero as $|v|$ tends to infinity, if $\epsilon > 0$. Unfortunately, it does not seem possible to obtain any such simple expression for ${}_pS_p(s)$ when $p > 1$. We must therefore employ other methods to obtain the corresponding properties of ${}_pS_p(s)$ which have been already stated in § 5.

14. Let ${}_pS_p(s)$ denote the sum of all terms of the series by which ${}_pS_p(s)$ has been defined when $R(s) > R(\Sigma a - \Sigma \rho)$ except the first k of such terms. Choose the positive integer k so that $R(-s + k + 1 - \rho_r) > 0$ ($r = 1, 2, \dots, p$). Let l be a straight contour parallel to the imaginary axis passing to the left of the point $-s + k$ and to the right of the points $\rho_r - 1$. Then I say that

$$\frac{\sin \pi s}{\pi} {}_pS_p(s) = \frac{1}{2\pi i} \int_l \frac{\Gamma(-\phi-s)}{\Gamma(1-\phi)} \prod_{r=1}^p \frac{\Gamma(1-\rho_r+\phi)}{\Gamma(1-\alpha_r+\phi)} d\phi,$$

provided $R(s) > R(\Sigma a - \Sigma \rho)$.

For under this limitation the integral is convergent, since the subject of integration behaves at infinity like $(-\phi)^{-s-1} \phi^{\Sigma \alpha_r - \Sigma \rho_r}$.

Also the integral will vanish when taken round an infinite circular contour to the right of the axis l if this contour pass between the poles of $\Gamma(-\phi-s)$.

Hence, by Cauchy's theory of residues, the integral is equal to

$$\begin{aligned} \sum_{t=k}^{\infty} \prod_{r=1}^p \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-\alpha_r+t-s)} \frac{(-)^{t-1}}{\Gamma(t+1)\Gamma(1+s-t)} \\ = \sum_{t=k}^{\infty} \prod_{r=1}^p \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-\alpha_r+t-s)} \frac{\Gamma(t-s)}{\Gamma(t+1)} \frac{\sin \pi s}{\pi} \\ = \frac{\sin \pi s}{\pi} {}_pS_p(s). \end{aligned}$$

15. We will next shew that $\frac{\sin \pi s}{\pi} {}_pS_p(s)$ can be expressed as the sum of multiples of p series of the form

$$\sum_{n=0}^{\infty} \frac{\Gamma(1-\rho_r+n-s)}{\Gamma(n+1)} \prod_{t=1}^p \frac{\Gamma({}_rb_t+n)}{\Gamma({}_rc_t+n)} \quad (r = 1, 2, \dots, p),$$

the b 's and c 's being linear combinations of the a 's and ρ 's which differ for different values of r .

Provided $R(s) > R(\Sigma\alpha - \Sigma\rho)$, the previous integral vanishes when taken round an infinite circular contour to the left of the axis l if this contour pass between the poles of $\Gamma(1 - \rho_r + \phi)$.

Hence $\pi^{-1} \sin \pi s {}_pS_p(s)$ is equal to *minus* the sum of the residues of $\frac{\Gamma(-\phi-s)}{\Gamma(1-\phi)} \prod_{r=1}^p \frac{\Gamma(1-\rho_r+\phi)}{\Gamma(1-\alpha_r+\phi)}$ at its poles which lie to the left of the axis l .

We thus have, if $R(s) > R(\Sigma\alpha - \Sigma\rho)$,

$$\pi^{-1} \sin \pi s {}_pS_p(s) = \sum_{r=1}^p T_r(s)$$

$$\text{where } T_r(s) = \sum_{n=0}^{\infty} \frac{\prod_{t=1}^p \Gamma(\rho_r - \rho_t - n)}{\prod_{t=1}^p \Gamma(\rho_r - \alpha_t - n)} \frac{\Gamma(n+1 - \rho_r - s)}{\Gamma(2 - \rho_r + n)} \frac{(-)^{n-1}}{\Gamma(n+1)},$$

the accent denoting that the term corresponding to $t = r$ is to be omitted in the product.

Thus

$$T_r(s) = \sum_{n=0}^{\infty} \frac{\prod_{t=1}^p \Gamma(1 - \rho_r + \alpha_t + n)}{\prod_{t=1}^p \Gamma(1 - \rho_r + \rho_t + n) \Gamma(2 - \rho_r + n)} \times \frac{\Gamma(1 - \rho_r + n - s)}{\Gamma(n+1)} \frac{\prod_{t=1}^p \sin \pi(\alpha_t - \rho_r)}{\pi \prod_{t=1}^p \sin \pi(\rho_t - \rho_r)}.$$

Now this series for $T_r(s)$ is convergent if

$$R(s + \Sigma\rho - \Sigma\alpha) > 0,$$

that is to say, under precisely the same limitation as the series by which ${}_pS_p(s)$ was defined.

Therefore, if $T_r(s)$ denote the analytic continuation of the corresponding series, we have, over the whole of the s -plane, the equality

$$\pi^{-1} \sin \pi s {}_pS_p(s) = \sum_{r=1}^p T_r(s).$$

We have therefore reduced the consideration of the function ${}_pS_p(s)$ to the consideration of functions of the type defined, when $R(s + \Sigma\rho - \Sigma\alpha) > 0$, by the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(1 - \rho_r + n - s)}{\Gamma(n+1)} \prod_{t=1}^p \frac{\Gamma(b_t + n)}{\Gamma(c_t + n)}.$$

Our typical series we shall write

$$T(s) = \sum_{n=0}^{\infty} \frac{\Gamma(a + n - s)}{\Gamma(n+1)} \prod_{t=1}^p \frac{\Gamma(b_t + n)}{\Gamma(c_t + n)}.$$

It is convergent if $R(s - a + \Sigma c - \Sigma b) > 0$.

The relation of the present paragraph is readily verified when $p = 1$. In this case we have, if $R(s+b-a) > 0$,

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t)}{\Gamma(t+1)} \frac{\Gamma(a+t-s)}{\Gamma(b+t-s)} = \frac{\sin \pi(a-b)}{\sin \pi s} \sum_{t=0}^{\infty} \frac{\Gamma(a+t-s)}{\Gamma(t+1)} \frac{\Gamma(a-b+1+t)}{\Gamma(a+1+t)}$$

or

$$\frac{\Gamma(-s)}{\Gamma(b-s)} {}_2F_1(a-s, -s; b-s; 1) = \frac{\sin \pi(a-b)}{\sin \pi s} \frac{\Gamma(a-b+1)}{\Gamma(a+1)} {}_2F_1(a-s, a-b+1; a+1; 1).$$

By Gauss's theorem this equality may be written

$$\frac{\Gamma(-s) \Gamma(b-a+s)}{\Gamma(b-a) \Gamma(b)} = \frac{\sin \pi(a-b)}{\sin \pi s} \frac{\Gamma(a-b+1) \Gamma(b-a+s)}{\Gamma(1+s) \Gamma(b)}.$$

It is therefore obviously true.

16. Let λ be a contour parallel to the imaginary axis passing to the left of the point k and to the right of the points $-b_t$ ($t = 1, 2, \dots, p$), where k is a positive integer so chosen that $k+R(b_t) > 0$.

Then, if ${}_kT(s)$ denote the sum of all terms after the first k of the series by which $T(s)$ has been defined when $R(s-a+\Sigma c-\Sigma b) > 0$, we will shew that

$$\frac{\sin \pi(s-a)}{\pi} {}_kT(s) = \frac{1}{2\pi i} \int_{\lambda} \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \prod_{t=1}^p \frac{\Gamma(b_t+\phi)}{\Gamma(c_t+\phi)} dy.$$

For, as before, the integral is equal to the sum of the residues of the subject of integration to the right of the axis λ . It is therefore equal to

$$\sum_{n=k}^{\infty} \frac{(-)^{n-1}}{\Gamma(n+1) \Gamma(1-a-n+s)} \prod_{t=1}^p \frac{\Gamma(b_t+n)}{\Gamma(c_t+n)} = \frac{\sin \pi(s-a)}{\pi} \sum_{n=k}^{\infty} \frac{\Gamma(a+n-s)}{\Gamma(n+1)} \prod_{t=1}^p \frac{\Gamma(b_t+n)}{\Gamma(c_t+n)}.$$

We thus have the given theorem.

17. The Analytic Continuation of ${}_kT(s)$.—We proceed now to shew that, if k_t be the smallest integer such that $R(k_t+b_t) > 0$,

$$\begin{aligned} \frac{\sin \pi(s-a)}{\pi} {}_kT(s) &= - \sum_{t=1}^p \sum_{n=0}^{k_t-1} \frac{\Gamma(b_t+n)}{\Gamma(1-a+b_t+n-s)} \frac{\prod_{m=1}^{p'} \Gamma(b_m-b_t-n)}{\prod_{m=1}^p \Gamma(c_m-b_t-n)} \frac{(-)^n}{\Gamma(n+1)} \\ &\quad - \sum_{r=0}^{k-1} \frac{1}{\pi} \sum_{t=1}^p \frac{\prod_{m=1}^{p'} \sin \pi(c_m-b_t)}{\prod_{m=1}^p \sin \pi(b_m-b_t)} \\ &\quad \times \zeta(s-a+\Sigma c-\Sigma b+r+1, k_t+b_t)+I \end{aligned}$$

where

$$I = \frac{1}{2\pi i} \int_{\lambda'} \left\{ \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \prod_{t=1}^p \frac{\Gamma(1-c_t-\phi)}{\Gamma(1-b_t-\phi)} - \sum_{r=0}^R \frac{f_r(s)}{(-\phi)^{\sigma+r}} \right\} \\ \times \prod_{t=1}^p \frac{\sin \pi(c_t+\phi)}{\sin \pi(b_t+\phi)} d\phi,$$

$\sigma = s-a+\Sigma c-\Sigma b+1$, and where λ' is a contour parallel to the imaginary axis which cuts the real axis just to the left of the origin. The equality is valid for all values of s such that

$$R(s-a+\Sigma c-\Sigma b+R) > 0;$$

it is therefore valid for any finite value of u if we take R sufficiently large.

By the asymptotic expansion of the gamma function we know that, if

$$|\arg(-\phi)| < \pi,$$

$$\frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \prod_{t=1}^p \frac{\Gamma(1-c_t-\phi)}{\Gamma(1-b_t-\phi)} = \sum_{r=0}^R \frac{f_r(s)}{(-\phi)^{\sigma+r}} + \frac{J_k(s, \phi)}{(-\phi)^{\sigma+R}}$$

where $f_0(s) = 1$, $f_r(s)$ is a polynomial in s , and $|J_k(s, \phi)|$ tends uniformly to zero as $|\phi|$ tends to infinity.

Hence the integral I is convergent provided $R(\sigma+R-1) > 0$. Consider now the integral

$$I_r = \frac{1}{2\pi i} \int_{\lambda'} \frac{1}{(-\phi)^{\sigma+r}} \prod_{t=1}^p \frac{\sin \pi(c_t+\phi)}{\sin \pi(b_t+\phi)} d\phi.$$

That value of $(-\phi)^{\sigma+r}$ is to be taken which is equal to

$$\exp \{(\sigma+r) \log(-\phi)\},$$

where $\log(-\phi)$ is real when ϕ is real and negative, and has a cross-cut along the positive half of the real axis. Hence the subject of integration is one-valued in the area to the left of λ' . Also, if $R(\sigma)+r > 1$, the integral vanishes when taken round an infinite contour enclosing this area. Hence, if $R(\sigma)+r > 1$,

$$I_r = \sum_{t=1}^p \frac{1}{\pi} \sum_{n=k_t}^{\infty} \frac{(-)^{n-1}}{(b_t+n)^{\sigma+r}} \frac{\prod_{m=1}^{p'} \sin \pi(c_m-b_t-n)}{\prod_{m=1}^p \sin \pi(b_m-b_t-n)},$$

the accent denoting that the term corresponding to $m=t$ is to be omitted.

Thus

$$I_r = - \sum_{t=1}^p \frac{1}{\pi} \xi(\sigma+r, b_t+k_t) \frac{\prod_{m=1}^p \sin \pi(c_m-b_t)}{\prod_{m=1}^p \sin \pi(b_m-b_t)}$$

where $\zeta(s, a)$ denotes the Riemann ζ function defined, when $R(s) > 1$, by the equality

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(a+n)^s}.$$

Now, by the result of § 16, we see that

$$\begin{aligned} \frac{\sin \pi(s-a)}{\pi} {}_kT(s) \\ = \frac{1}{2\pi i} \int_{\lambda} \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \prod_{t=1}^p \left\{ \frac{\Gamma(1-c_t-\phi)}{\Gamma(1-b_t-\phi)} \frac{\sin \pi(c_t+\phi)}{\sin \pi(b_t+\phi)} \right\} dy. \end{aligned}$$

Hence, if we take account of the poles of the subject of integration between the contours λ and λ' , we have the equality given in the enunciation, provided $R(\sigma) > 1$. But all the quantities on the right-hand side of the equality are continuous functions (except for isolated poles), analytic if $R(\sigma+R) > 1$.

We thus have the given theorem.

18. We may now shew that, if $s = u + iv$, and u have any finite value,

$$|T(s)| \exp \left\{ \left(\frac{1}{2}\pi - \epsilon \right) |v| \right\},$$

where $\epsilon > 0$, can be made as small as we please by taking $|v|$ greater than an assignable large positive quantity V .

We take the result of the preceding paragraph.

Then, in the first place, the integral I is such that

$$|I| \exp \left\{ \left(-\frac{1}{2}\pi - \epsilon \right) |v| \right\}$$

tends uniformly to zero as $|v|$ tends to infinity. For $(-\phi)^{\sigma+r}$ is such that the argument of $-\phi$ has its modulus $\leq \frac{1}{2}\pi$ on the line of integration. Also for any finite value of $|\phi|$ the expression

$$|\Gamma(1-a-\phi+s)| \exp \left\{ \left(\frac{1}{2}\pi - \epsilon \right) |v| \right\}$$

tends uniformly to zero as $|v|$ increases; and therefore

$$\left| \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \right| \exp \left\{ \left(-\frac{1}{2}\pi - \epsilon \right) |v| \right\}$$

will tend uniformly to zero for all values of ϕ , finite or infinite (including even values for which $\phi-s$ is finite when $|v|$ is very large) on the line of integration. Hence, since the integral is uniformly convergent with respect to s , we have the given statement.

In the second place, the modulus of each term of the finite double series of gamma functions of s which occurs first on the right-hand side

of the equality of § 17 will evidently tend uniformly to zero when multiplied by $\{\exp(-\frac{1}{2}\pi - \epsilon) |v|\}$.

Finally, so far as the expressions I_r depend upon s , they depend upon the Riemann ζ functions $\zeta(\sigma + r, k_t + b_t)$. Now the integer k_t was so chosen that $|\arg(k_t + b_t)| < \frac{1}{2}\pi$. Hence, by the theory of the Riemann ζ function,[†] $|\zeta(\sigma + r, k_t + b_t)| \exp\{(-\frac{1}{2}\pi - \epsilon) |v|\}$ tends uniformly to zero as $|v|$ tends to infinity.

Hence, by the equality of § 17, we see that, for all finite values of u , $|\pi^{-1} \sin \pi(s-a) {}_kT(s)| \exp\{-\frac{1}{2}\pi - \epsilon |v|\}$ will tend uniformly to zero as $|v|$ tends to infinity. Therefore ${}_kT(s) \exp\{(\frac{1}{2}\pi - \epsilon) |v|\}$ possesses the same property. Now ${}_kT(s)$ only differs from $T(s)$ by the sum of constant multiples of a finite number of terms of the type $\Gamma(n+a-s)$; therefore $|T(s)| \exp\{(\frac{1}{2}\pi - \epsilon) |v|\}$ will tend uniformly to zero as $|v|$ tends to infinity.

We thus have the given theorem.

19.† We may now shew that, if u is large and positive, and $|v|$ be large or finite,

$$|T(s)| < K \exp\{(-\frac{1}{2}\pi + \epsilon) |v| - ku\},$$

where k is any finite positive quantity, $\epsilon > 0$, and K is a finite quantity independent of u and $|v|$. We assume that s is not in the immediate vicinity of the large poles $a+n$, n a positive integer, of $T(s)$.

We have seen in § 16 that, when u is sufficiently large,

$$\frac{\sin \pi(s-a)}{\pi} {}_mT(s) = \frac{1}{2\pi i} \int_{\lambda} \frac{\Gamma(-\phi)}{\Gamma(1-a-\phi+s)} \prod_{t=1}^p \frac{\Gamma(b_t+\phi)}{\Gamma(c_t+\phi)} d\phi,$$

where λ passes to the left of the point m , and to the right of the points $-b_t$ ($t = 1, 2, \dots, p$), and m is the smallest positive integer which makes

$$m + R(b_t) > 0.$$

Now, if $R(u-m-a) > 0$ and $R(m-\phi) > -1$,

$$\begin{aligned} & |\Gamma(u-m-a)| |\Gamma(1+\nu+m-\phi)| \\ &= |\Gamma(u-a+1+\nu-\phi)| \left| \int_0^1 x^{u-a-m-1} (1-x)^{m+\nu-\phi} dx \right|. \end{aligned}$$

[†] See, for instance, a paper by the author, "The Maclaurin Sum Formula," *Proc. London Math. Soc.*, Ser. 2, Vol. 2, p. 256.

[‡] The author expresses his thanks to the referees for pointing out an arithmetical error in the original statement of this theorem.

Also, if u is sufficiently large, and $R(m-\phi) > 0$,

$$\left| \int_0^1 x^{u-a-m-1} (1-x)^{m+v-\phi} dx \right| \leq \int_0^1 x^{u-R(a)-m-1} (1-x)^{m-R(\phi)} dx < 1.$$

Hence $|1/\Gamma(1-a-\phi+s)| < 1/|\Gamma(u-m-a)| |\Gamma(1+m+v-\phi)|$.

Therefore

$$\begin{aligned} & \left| \frac{\sin \pi(s-a)}{\pi} {}_mT(s) \right| \\ & < \frac{1}{2\pi |\Gamma(u-m-a)|} \int_{\lambda} \left| \frac{\Gamma(-\phi)}{\Gamma(1+m+v-\phi)} \prod_{i=1}^p \frac{\Gamma(b_i+\phi)}{\Gamma(c_i+\phi)} \right| |d\phi| \\ & < K \exp \left\{ \frac{1}{2} \pi (1+\epsilon) |v| - ku \right\}, \end{aligned}$$

as in § 18, where $\epsilon > 0$, K is finite and independent of u and $|v|$, and k is any finite positive quantity, however large.

Hence, if s be not in the immediate vicinity of one of the poles $a+n$ of $T(s)$,

$$|{}_mT(s)| < K \exp \left\{ \left(-\frac{1}{2}\pi + \epsilon\right) |v| - ku \right\}.$$

The same inequality is evidently true, under the same limitation, of the finite number of terms by which $T(s)$ differs from ${}_mT(s)$.

We therefore have the given theorem.

20. We can now shew that the function ${}_pS_p(s)$ is such that, if $s = u+iv$, and u be finite,

$$|{}_pS_p(s)| e^{(\frac{3}{2}\pi - \epsilon)|v|},$$

where $\epsilon > 0$, tends uniformly to zero as $|v|$ tends to infinity.

For we have seen in § 15 that

$$\pi^{-1} \sin \pi s {}_pS_p(s) = \sum_{r=1}^p T_r(s).$$

Therefore $|\pi^{-1} \sin \pi s| e^{(\frac{3}{2}\pi - \epsilon)|v|} |{}_pS_p(s)| < \sum_{r=1}^p |T_r(s)| e^{(\frac{3}{2}\pi - \epsilon)|v|}$,

that is to say, is less than a quantity which tends uniformly to zero.

We therefore have the given theorem.

Again, we can show that ${}_pS_p(s)$ admits of analytic continuation over the whole of the finite portion of the s -plane.

For we have seen in § 17 that this is true of the functions $T_r(s)$; it is therefore true of the function ${}_pS_p(s)$.

It is similarly evident that, if u be large and positive and $|v|$ be very large or finite,

$$|{}_pS_p(s)| < K \exp \left\{ \left(-\frac{3}{2}\pi + \epsilon\right) |v| - ku \right\}$$

where K is finite and independent of u and $|v|$, and k is any finite positive quantity, however large, provided s be not in the immediate vicinity of one of the poles $s = n$ or $1 - \rho_r + n$ ($r = 1, 2, \dots, p$) of ${}_pS_p(s)$.

21. We proceed now to shew that ${}_pS_p(s)$ has, except for those poles which are poles of $\Gamma(-s)$ or $\Gamma(1 - \rho_r + s)$ ($r = 1, 2, \dots, p$), as its sole finite singularities poles at the points

$$s = a_1 + \dots + a_p - \rho_1 - \dots - \rho_p - r \quad (r = 0, 1, 2, \dots, \infty),$$

and that the residues at these poles may be obtained with sufficient labour.

By the definition of § 13, ${}_pS_p(s)$ is represented, when $R(s) > R(\Sigma a - \Sigma \rho)$, by the convergent series of gamma functions

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t)}{\Gamma(t+1)} \prod_{r=1}^p \left\{ \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-a_r+t-s)} \right\}.$$

By the asymptotic expansion of the gamma function we know that, when t is large,

$$\frac{\Gamma(-s+t)}{\Gamma(1+t)} \prod_{r=1}^p \left\{ \frac{\Gamma(1-\rho_r+t-s)}{\Gamma(1-a_r+t-s)} \right\} = \frac{1}{t^{s+1+\Sigma\rho-\Sigma a}} \exp \left\{ \sum_{m=1}^{\mu} \frac{(-)^{m-1}}{m t^m} V_m + \frac{J_{\mu}}{t^{\mu}} \right\}$$

where, for any finite value of μ , $|J_{\mu}|$ can be made as small as we please by taking t sufficiently large, and where

$$V_m = \sum_{r=1}^p \{ S_m(1-\rho_r-s) - S_m(1-a_r-s) \} + S_m(-s) - S_m(1).$$

$S_m(x)$ denotes the m -th simple Bernoullian function of x , and is a polynomial of degree $(m+1)$ in x .

When t is large we see, then, that the above term may be asymptotically written

$$\frac{1}{t^{s+1+\Sigma\rho-\Sigma a}} \left\{ \sum_{r=0}^R \frac{f_r(s)}{t^r} + \frac{J_R(s)}{t^R} \right\}$$

where $f_0(s) = 1$, and $f_r(s)$ is a polynomial in s whose value for any assigned value of r can be determined with sufficient labour.

Consider now the function

$${}_pS_p(s) - \sum_{r=0}^R f_r(s) \zeta(s+r+1+\Sigma\rho-\Sigma a).$$

When $R(s+\Sigma\rho-\Sigma a) > 0$ this function may be expressed by the series

$$\sum_{t=0}^{\infty} \left\{ \frac{\Gamma(-s+t)}{\Gamma(1+t)} \prod_{r=1}^p \frac{\Gamma(1-\rho_r-s+t)}{\Gamma(1-a_r-s+t)} - \sum_{r=0}^R \frac{f_r(s)}{t^{r+s+1+\Sigma\rho-\Sigma a}} \right\}$$

(the double accent denoting that the summation $\sum_{r=0}^R$ does not exist when $t = 0$), and this series is equal to

$$\sum_{t=0}^{\infty} \frac{J_R(s)}{t^{s+1+R+\Sigma\rho-\Sigma\alpha}}.$$

But this series is convergent, provided

$$R(s+R+\Sigma\rho-\Sigma\alpha) > 0.$$

Hence the function

$${}_pS_p(s) - \sum_{r=0}^R f_r(s) \zeta(s+r+1+\Sigma\rho-\Sigma\alpha)$$

has no singularities when

$$R(s) > \Sigma\alpha - \Sigma\rho - R.$$

Now the sole finite singularity of $\zeta(s)$ is a pole at the point $s = 1$ at which the residue is unity.

Hence the sole finite singularities of ${}_pS_p(s)$ apart from those poles which are poles of $\Gamma(-s)$ or $\Gamma(1-\rho_r+s)$ ($r = 1, 2, \dots, p$) are poles at the points

$$s = \Sigma\alpha - \Sigma\rho - r \quad (r = 0, 1, 2, \dots, \infty),$$

At the point $s = \Sigma\alpha - \Sigma\rho - r$ the residue is $f_r(\Sigma\alpha - \Sigma\rho - r)$, and this quantity can be calculated with sufficient labour, should necessity arise.

22. We may now prove that, if $|\arg(-x)| < \frac{3}{2}\pi$,

$$\begin{aligned} & \prod_{r=1}^p \left\{ \frac{\Gamma(1-\rho_r)}{\Gamma(1-\alpha_r)} \right\} {}_pF_p \{ a_1, \dots, a_p; \rho_1, \dots, \rho_p; x \} \\ & + \sum_{r=1}^p (-x)^{1-\rho_r} \frac{\Gamma(\rho_r-1) \prod_{m=1}^p \Gamma(\rho_r-\rho_m)}{\prod_{m=1}^p \Gamma(\rho_r-\alpha_m)} \\ & \times {}_pF_p \{ 1+\alpha_1-\rho_r, \dots, 1+\alpha_p-\rho_r; \\ & \quad 2-\rho_r, 1+\rho_1-\rho_r, \dots, *, \dots, 1+\rho_p-\rho_r; x \} \\ & = e^x (-x)^{\Sigma\alpha-\Sigma\rho} \left\{ \sum_{r=0}^R \frac{P_r}{x^r} + \frac{J_R}{x^R} \right\} \end{aligned}$$

where $|J_R|$ tends uniformly to zero as $|x|$ tends to infinity. The quantity $(-)^r P_r$ is the residue of ${}_pS_p(s)$ at its pole $s = \Sigma\alpha - \Sigma\rho - r$.

We consider the integral

$$-\frac{1}{2\pi i} \int {}_pS_p(s) x^s ds,$$

taken round a contour which encloses all the poles of ${}_pS_p(s)$ except the poles

$$s = \Sigma \alpha - \Sigma \rho - r \quad (r = 0, 1, 2, \dots, \infty),$$

and embraces the positive half of the real axis. The integral is convergent by § 20 for all finite values of $|x|$, and by Cauchy's theory of residues is equal to

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(-1)^n}{n! \Gamma(t+1)} \prod_{r=1}^p \left\{ \frac{\Gamma(1-\rho_r-n)}{\Gamma(1-\alpha_r-n)} \right\} x^{t+n} \\ & + \sum_{r=1}^p x^{1-\rho_r} \sum_{n=0}^{\infty} \sum_{t=0}^{\infty} \frac{(-1)^n}{n! \Gamma(t+1)} \frac{\Gamma(\rho_r-1-n) \prod_{m=1}^p \Gamma(\rho_r-\rho_m-n)}{\prod_{m=1}^p \Gamma(\rho_r-\alpha_m-n)} x^{t+n} \\ & = e^x \prod_{r=1}^p \left\{ \frac{\Gamma(1-\rho_r)}{\Gamma(1-\alpha_r)} \right\} {}_pF_p \{ \alpha_1, \dots, \alpha_p; \rho_1, \dots, \rho_p; -x \} \\ & + e^x \sum_{r=1}^p x^{1-\rho_r} \frac{\Gamma(\rho_r-1) \prod_{m=1}^p \Gamma(\rho_r-\rho_m)}{\prod_{m=1}^p \Gamma(\rho_r-\alpha_m)} \\ & \times {}_pF_p \{ 1+\alpha_1-\rho_r, \dots, 1+\alpha_p-\rho_r; \\ & \quad 2-\rho_r, 1+\rho_1-\rho_r, \dots, *, \dots, 1+\rho_p-\rho_r; -x \}. \end{aligned}$$

Again, by the results of § 20, if $|\arg x| < \frac{3}{2}\pi$, the contour of integration may be swung back as in § 8, and we get for the value of the integral the asymptotic series

$$\sum_{r=0}^R (-1)^r P_r x^{\Sigma \alpha - \Sigma \rho - r} + J_R / x^{\Sigma \alpha - \Sigma \rho + R}$$

where $|J_R|$ tends uniformly to zero as $|x|$ tends to infinity. Changing x into $(-x)$, we have the given result.

The result of this paragraph is a generalisation of that previously obtained† for the case $p = 1$. In that case the coefficients P_r admit of a simple expression, and we have the theorem that, if $|\arg(-x)| < \frac{3}{2}\pi$,

$$\frac{\Gamma(1-\rho)}{\Gamma(1-\alpha)} {}_1F_1 \{ \alpha; \rho; x \} + \frac{\Gamma(\rho-1)}{\Gamma(\rho-\alpha)} (-x)^{1-\rho} {}_1F_1 \{ \alpha-\rho+1; 2-\rho; x \}$$

admits the asymptotic expansion

$$e^x (-x)^{\alpha-\rho} {}_2F_0 \{ \rho-\alpha; 1-\alpha; 1/x \}.$$

† *Loc. cit.*, § 1, Paper η , § 15.

23. We now see that the $(p+1)$ asymptotic solutions of the differential equation of § 13,

$$\left[(\mathfrak{S} + a_1) \dots (\mathfrak{S} + a_p) - \frac{d}{dx} (\mathfrak{S} + \rho_1 - 1) \dots (\mathfrak{S} + \rho_p - 1) \right] y = 0, \quad (1)$$

are the series which have been obtained in § 7,

$$(-x)^{-a_r} {}_{p+1}F_{p-1} \{ a_r, 1+a_r-\rho_1, \dots, 1+a_r-\rho_p; \\ 1+a_r-a_1, \dots, *, \dots, 1+a_r-a_p; -1/x \} \quad (A)$$

when $r = 1, 2, \dots, p$, and the series

$$e^x (-x)^{\Sigma a - \Sigma \rho} \sum_{r=0}^{\infty} \frac{P_r}{x^r}, \quad (B)$$

which is obtained in § 22.

In § 7 we have connected each of the first series with the hypergeometric integral functions which exist as principal solutions of the differential equation in the finite part of the plane, and in § 22 we have connected the last series with the same solutions.

If $R(x) > 0$, the result of § 22, which is valid if $|\arg(-x)| < \frac{3}{2}\pi$, is therefore equivalent to two relations. But in this case the series (B) outweighs all the series (A), and therefore each hypergeometric integral function which is a principal solution of the differential equation (1) can be expressed, when $R(x) > 0$, as a multiple of the series (B). The different principal solutions are therefore expressible as multiples of the same asymptotic series, though, of course, they are not corresponding multiples of one another. We have an illustration of Poincaré's axiom that different functions may admit the same asymptotic expansion.

24. When $R(x) > 0$ the actual expression of, let us say, ${}_pF_p\{x\}$ as such an asymptotic expansion is not best obtained by elimination between the results of §§ 7 and 22. It may be obtained directly by considering the contour integral

$$-\frac{1}{2\pi i} \int T(s) x^s ds,$$

where $T(s)$ denotes the series defined when $R(s) > R(\Sigma a - \Sigma \rho)$ by the convergent series of gamma functions

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} \prod_{r=1}^p \left\{ \frac{\Gamma(a_r+n)}{\Gamma(\rho_r+n)} \right\}.$$

It is evident that this function is a particular case of the function $T(s)$ considered in §§ 15-19 for which $a = 0$.

Hence $T(s)$ has poles when $s = n$ ($n = 0, 1, \dots, \infty$), and when

$$s = \Sigma \alpha - \Sigma \rho - n.$$

If the integral be taken round a contour C which embraces the positive half of the real axis and encloses all the poles $s = n$, but none of the poles $s = \Sigma \alpha - \Sigma \rho - n$, it is evidently convergent for all finite values of $|x|$, and equal, by Cauchy's theory of residues, to

$$\sum_{n=0}^{\infty} \sum_{t=0}^{\infty} \prod_{r=1}^p \left\{ \frac{\Gamma(\alpha_r + n)}{\Gamma(\rho_r + n)} \right\} \frac{(-)^t x^{n+t}}{t! \Gamma(n+1)} = \prod_{r=1}^p \left\{ \frac{\Gamma(\alpha_r)}{\Gamma(\rho_r)} \right\} e^{-x} {}_pF_p \{ \alpha; \rho; x \}.$$

Now the residue at $s = \Sigma \alpha - \Sigma \rho$ is unity; and the residue at

$$s = \Sigma \alpha - \Sigma \rho - r$$

can be calculated with sufficient labour from the asymptotic expansion of the gamma function: let it be M_r .

If $|\arg x| < \frac{1}{2}\pi$, we see by § 18 that the contour of the integral can be swung back as in § 8, and we get for the value of the integral the asymptotic series

$$x^{\Sigma \alpha - \Sigma \rho} \left\{ 1 + \sum_{r=1}^R \frac{M_r}{x^r} + \frac{J_R}{x^R} \right\}$$

where, for all finite values of R , $|J_R|$ tends uniformly to zero as $|x|$ tends to infinity.

Therefore, if $|\arg x| < \frac{1}{2}\pi$, we have the asymptotic equality

$$\prod_{r=1}^p \left\{ \frac{\Gamma(\alpha_r)}{\Gamma(\rho_r)} \right\} {}_pF_p \{ x \} = e^x x^{\Sigma \alpha - \Sigma \rho} \left\{ 1 + \sum_{r=1}^R \frac{M_r}{x^r} + \frac{J_R}{x^R} \right\}.$$

This result evidently includes that obtained by Stokes for real positive values of the variable (*vide* § 3).

Evidently we can obtain similar exponentially infinite asymptotic expansions, when $R(x) > 0$, for the other principal solutions

$$x^{1-\rho_r} {}_pF_p \{ \alpha_1 - \rho_r + 1, \dots, \alpha_p - \rho_r + 1; \\ 2 - \rho_r, \rho_1 - \rho_r + 1, \dots, *, \dots, \rho_p - \rho_r + 1; x \}$$

of the differential equation (1) of § 23.

25. When $R(x) < 0$ the result of § 22 is equivalent to but one identity.

We thus see that, when $R(x) < 0$, although, as in §§ 9 and 10, each hypergeometric integral function which is a principal solution in the finite part of the plane of the differential equation (1) of § 23 is represented near $|x| = \infty$ by linear combinations of the p asymptotic series whose dominant terms are of the type $x^{-\alpha_r}$ ($r = 1, 2, \dots, p$), yet one, and only one, particular linear combination of the principal solutions exists which is

exponentially zero when $|x|$ is very large. This combination is given in § 22.

The complete asymptotic theory of the solutions of the differential equation (1) of § 23 is now concluded. We next proceed to discuss the more difficult case when $p \neq q$.

PART III.

The Exponentially Infinite Asymptotic Expansions of Linear Combinations of Generalised Hypergeometric Integral Functions for which $p \neq q$.

26. We proceed now to consider the further theory of those hypergeometric integral functions for which $p < q$, and consequently $\mu > 1$.

We shall in the first place investigate the function

$${}_pS_q(s; a_1, \dots, a_p; \rho_1, \dots, \rho_q),$$

or, as we shall briefly write it, ${}_pS_q(s)$, which, when

$$R(s) > R\{a_1 + \dots + a_p - \rho_1 - \dots - \rho_q + \tfrac{1}{2}(\mu - 1)\}/\mu, \quad (1)$$

is represented by the convergent series of gamma functions

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu) \prod_{r=1}^q \Gamma(1-\rho_r+t/\mu-s)}{\prod_{r=1}^p \Gamma\left(\frac{t+r}{\mu}\right) \prod_{r=1}^p \Gamma(1-a_r+t/\mu-s)}.$$

The number $\mu = q + 1 - p$ is an integer.

By the multiplication formula for the gamma function, we have equally under the fundamental restriction (1)

$${}_pS_q(s) (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} = \sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu) \prod_{r=1}^q \Gamma(1-\rho_r+t/\mu-s)}{\Gamma(t+1) \prod_{r=1}^p \Gamma(1-a_r+t/\mu-s)} \mu^t.$$

We shall shew that ${}_pS_q(s)$ possesses properties which are analogous to those of ${}_pS_p(s)$ obtained in Part II., and which have already been stated in § 6.

These properties are direct generalisations of the properties of the function ${}_0S_1(x)$ defined when $R(2s+\rho-\frac{1}{2}) > 0$ by the series

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+\frac{1}{2}t) \Gamma(1-\rho-s+\frac{1}{2}t)}{\Gamma(\frac{1}{2}t+1) \Gamma(\frac{1}{2}t+\frac{1}{2})}.$$

It is easy to see that, by an application of Gauss's theorem, ${}_0S_1(s)$ could be written in the form

$$\pi^{-1} \sin \pi(\rho - \tfrac{1}{2}) 2^{2\rho+4s} \Gamma(-2s) \Gamma(2-2\rho-2s) \Gamma(2s+\rho-\tfrac{1}{2}).$$

27. We note in the first place that ${}_pS_q(s)$ can be written as the sum of μ series in the form

$$\sum_{n=0}^{\mu-1} \sum_{t=0}^{\infty} \frac{\Gamma(-s+t+n/\mu) \prod_{r=1}^q \Gamma(1-\rho_r+t+n/\mu-s)}{\Gamma(t+1) \prod_{m=1}^{\mu} \Gamma\left(t+\frac{m+n}{\mu}\right) \prod_{r=1}^p \Gamma(1-\alpha_r+t+n/\mu-s)},$$

the double accent denoting that the term corresponding to $m = \mu - n$ is to be omitted.

Each of these series is of the same type as the series ${}_pS_\rho(s)$ previously considered in Part II., with the exception that s does not occur in μ gamma functions in the denominator of the typical term of the series.

Let, now, ${}_p^kS_q(s)$ denote the sum of all terms of the series by which ${}_pS_q(s)$ has been defined under the condition (1) except the first k terms. Choose the positive integer k so that

$$R(-s+k/\mu+1-\rho_r) > 0 \quad (r = 1, 2, \dots, q).$$

Let l be a straight contour parallel to the imaginary axis passing to the left of the point $k/\mu-s$ and to the right of the points ρ_r-1 ($r = 1, 2, \dots, q$), and also of the point $(k-1)/\mu-s$.

Then, exactly as in § 14, we see that

$$\begin{aligned} & {}_p^kS_q(s) \\ &= \sum_{n=0}^{\mu-1} \frac{\pi}{\sin \pi(s-n/\mu)} \frac{1}{2\pi i} \int_l \frac{\Gamma(-\phi-s+n/\mu)}{\Gamma(1-\phi) \prod_{m=1}^{\mu} \Gamma(m/\mu+\phi+s)} \frac{\prod_{r=1}^q \Gamma(1-\rho_r+\phi)}{\prod_{r=1}^p \Gamma(1-\alpha_r+\phi)} d\phi; \end{aligned} \quad (A)$$

for the terms of the series ${}_p^kS_q(s)$ which arise from poles of $\Gamma(-\phi-s+n/\mu)$ occur when $\phi+s-n/\mu = t$ or $\phi+s = t+n/\mu$, when these points belong to the series $k/\mu, (k+1)/\mu, \dots$; that is to say, when

$$\mu t + n \geq k.$$

28. We may now shew that, under the condition (1) of § 26,

$$\begin{aligned} & {}_p^kS_q(s) \\ &= -\frac{1}{2\pi i} \int_l \frac{\pi \mu \sin \pi \mu \phi \Gamma(\phi)}{\sin \pi \mu s \sin \pi \mu (s+\phi)} \frac{\prod_{r=1}^q \Gamma(1-\rho_r+\phi)}{\prod_{m=1}^{\mu} \Gamma(m/\mu+\phi+s) \prod_{r=1}^p \Gamma(1-\alpha_r+\phi)} d\phi. \end{aligned}$$

For the previous integral (A) is equal to

$$\frac{1}{2\pi i} \int_L \frac{\prod_{r=1}^q (1-\rho_r+\phi)}{\prod_{r=1}^p \Gamma(1-\alpha_r+\phi) \prod_{m=1}^{\mu} \Gamma(m/\mu+\phi+s)} \Theta d\phi$$

where $\Theta = \frac{\sum_{n=0}^{\mu-1} \pi}{\sin \pi(s-n/\mu)} \frac{\Gamma(1-n/\mu+\phi+s) \Gamma(-\phi-s+n/\mu)}{\Gamma(1-\phi)}$

$$= -\frac{1}{\Gamma(1-\phi)} \sum_{n=0}^{\mu-1} \frac{\pi^2}{\sin \pi(s-n/\mu) \sin(\pi\phi+s-n/\mu)}$$

$$= -\frac{\pi^2}{\Gamma(1-\phi) \sin \pi\phi} \sum_{n=0}^{\mu-1} \{\cot \pi(s-n/\mu) - \cot \pi(\phi+s-n/\mu)\}$$

$$= -\frac{\pi^2 \mu}{\Gamma(1-\phi) \sin \pi\phi} \{\cot \mu s \pi - \cot \mu \pi(s+\phi)\}$$

$$= -\frac{\mu \pi \sin \pi \mu \phi \Gamma(\phi)}{\sin \pi \mu s \sin \pi \mu(s+\phi)}.$$

The integral is evidently convergent if

$$R\left(\mu s + \frac{\mu+1}{2} + \sum \rho_r - \sum \alpha_r + p - q\right) > 1,$$

i.e., if $R(s) > R\left\{\frac{1}{2}(\mu-1) + \sum \alpha_r - \sum \rho_r\right\} / \mu,$

which is the condition (1) of § 26.

We note that the integral just obtained may equally be written

$${}_p S_q(s) = \frac{1}{2\pi i} \int_L \frac{\prod_{r=1}^q \Gamma(1-\rho_r+\phi) \Gamma(\phi) \Gamma(-\mu s - \mu \phi)}{\prod_{r=1}^p \Gamma(1-\alpha_r+\phi)} \frac{\sin \pi \mu \phi}{\sin \pi \mu s} \frac{\mu^{\mu \phi + \mu s + \frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu-1)}} d\phi; \quad (1)$$

for by the multiplication formula for the gamma function

$$\prod_{m=1}^{\mu} \Gamma(m/\mu + \phi + s) = \frac{\Gamma(\mu \phi + \mu s + 1) (2\pi)^{\frac{1}{2}(\mu-1)}}{\mu^{\mu \phi + \mu s + \frac{1}{2}}}.$$

The equality (1) is, of course, obvious when the integral is once written down: the process adopted has been employed to shew how such an integral may be built up from others of a more elementary character.

29. We now proceed to shew that

$$\sin \pi \mu s {}_p S_q(s) = - \sum_{r=1}^q \frac{\prod_{t=1}^p \sin \pi(\rho_r - \alpha_t)}{\prod_{t=1}^q \sin \pi(\rho_r - \rho_t)} \frac{\sin \pi \mu(\rho_r - 1)}{\sin \pi(\rho_r - 1)} \frac{\mu^{\mu s + \mu + 1}}{2^{\mu-1}} U_r(s)$$

where $U_r(s)$ is the function defined when

$$R \left\{ \mu s + \Sigma \rho - \Sigma a - \frac{1}{2}(\mu - 1) \right\} > 0$$

by the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu)} \frac{\prod_{t=1}^p \Gamma(1 + a_t - \rho_r + n) \prod_{t=1}^{\mu-1} \Gamma(2 + t/\mu - \rho_r + n)}{\prod_{t=1}^q \Gamma(1 + \rho_t - \rho_r + n)}.$$

Since $\mu = q + 1 - p$, we may put

$$\begin{aligned} {}_r a_t &= 2 + t/\mu - \rho_r \quad (t = 1, 2, \dots, \mu - 1) \\ &= 1 + a_t - \rho_r \quad (t = \mu, \mu + 1, \dots, q), \\ {}_r b_t &= 1 + \rho_t - \rho_r \quad (t = 1, 2, \dots, q); \end{aligned}$$

and now we may say that $U_r(s)$ is the function defined when

$$R(\mu s + \mu + \Sigma b - \Sigma a) > 1$$

by the series
$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu)} \prod_{t=1}^q \frac{\Gamma({}_r a_t + n)}{\Gamma({}_r b_t + n)}$$

where one of the quantities ${}_r b_t$ is unity.

The theorem therefore enables us to reduce the consideration of the function ${}_p S_q(s)$ to that of the more simple function $U(s)$.

Under the condition

$$R(s) > R \left\{ \frac{1}{2}(\mu - 1) + \Sigma a - \Sigma \rho \right\} / \mu,$$

the integral of the previous paragraph will vanish when taken round that part of an infinite circular contour which lies to the left of the axis l . If, then, we apply Cauchy's theory of residues, we have

$$\begin{aligned} \sin \pi \mu s {}_p S_q(s) &= \sum_{r=1}^q \sum_{n=0}^{\infty} \frac{\prod_{t=1}^q \Gamma(\rho_r - \rho_t - n)}{\prod_{t=1}^p \Gamma(\rho_r - a_t - n)} \frac{(-)^{n-1} \sin \pi \mu (\rho_r - 1 - n)}{\Gamma(n+1) (2\pi)^{\frac{1}{2}(\mu-1)}} \\ &\quad \times \Gamma(\rho_r - 1 - n) \Gamma(-\mu s - \mu \rho_r + \mu + \mu n) \mu^{\mu \rho_r - \mu - \mu n + \mu s + \frac{3}{2}}, \end{aligned}$$

the accent denoting that the term corresponding to $t = r$ is to be omitted.

Thus

$$\sin \pi \mu s {}_p S_q(s) = - \sum_{r=1}^q \frac{\prod_{t=1}^p \sin \pi (\rho_r - a_t)}{\prod_{t=1}^q \sin \pi (\rho_r - \rho_t)} \frac{\sin \pi \mu (\rho_r - 1)}{\sin \pi (\rho_r - 1)} \frac{\mu^{\mu s + \mu + 1}}{2^{\mu-1}} U_r(s)$$

where

$$\begin{aligned} U_r(s) &= (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{\mu \rho_r - 2\mu + \frac{1}{2}} \sum_{n=0}^{\infty} \frac{\prod_{t=1}^p \Gamma(1 + a_t - \rho_r + n)}{\prod_{t=1}^q \Gamma(1 + \rho_r - \rho_t + n)} \mu^{-\mu n} \frac{\Gamma(\mu n - \mu \rho_r - \mu s + \mu)}{\Gamma(2 - \rho_r + n)} \end{aligned}$$

Now by the multiplication formula for the gamma function

$$\Gamma(\mu n - \mu \rho_r + 2\mu) = \frac{\mu^{\mu n - \mu \rho_r + 2\mu - \frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu-1)}} \prod_{t=0}^{\mu-1} \Gamma(n - \rho_r + 2 + t/\mu).$$

Hence

$$U_r(s) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu)} \frac{\prod_{t=1}^p \Gamma(1 + \alpha_t - \rho_r + n)}{\prod_{t=1}^q \Gamma(1 + \rho_t - \rho_r + n)} \frac{\prod_{t=1}^{\mu-1} \Gamma(2 + t/\mu - \rho_r + n)}{\prod_{t=1}^q \Gamma(1 + \rho_t - \rho_r + n)}.$$

Evidently the series $U_r(s)$ converges if $R\{\mu s + \Sigma \rho - \Sigma \alpha - \frac{1}{2}(\mu-1)\} > 0$, which is the limitation under which the series ${}_p S_q(s)$ was defined. Therefore the sum of the continuations of the specified multiples of the series $U_r(s)$ will give the function $\sin \pi \mu s {}_p S_q(s)$.

We thus have the given theorem.

30. We can readily verify the preceding transformation when $p = 0$, $q = 1$, and therefore $\mu = 2$.

We have

$${}_0 S_1(s) = \frac{1}{2\pi i} \int_l \Gamma(1 - \rho + \phi) \Gamma(\phi) \Gamma(-2s - 2\phi) \frac{\sin 2\pi \phi}{\sin 2\pi s} \frac{2^{2\phi + 2s + \frac{3}{2}}}{(2\pi)^{\frac{1}{2}}} d\phi,$$

and the transformation shews that

$$\sin 2\pi s {}_0 S_1(s) = - \frac{2^{2\rho + 2s - \frac{1}{2}}}{(2/\pi)^{\frac{1}{2}}} \frac{\sin 2\pi(\rho - 1)}{\sin \pi(\rho - 1)} U(s) \quad (1)$$

where

$$U(s) = \sum_{n=0}^{\infty} \frac{2^{-2n} \Gamma(2n - 2\rho - 2s + 2)}{\Gamma(1 + n) \Gamma(2 - \rho + n)},$$

the series being convergent if $R(2s + \rho) > \frac{1}{2}$.

$$\text{Now } U(s) = \frac{2^{-2\rho - 2s + \frac{3}{2}}}{(2\pi)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Gamma(n - \rho - s + 1) \Gamma(n - \rho - s + \frac{3}{2})}{\Gamma(1 + n) \Gamma(2 - \rho + n)},$$

and therefore, by Gauss's theorem, is equal to

$$\begin{aligned} & \frac{2^{-2\rho - 2s + \frac{3}{2}}}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma(1 - s - \rho) \Gamma(\frac{3}{2} - s - \rho) \Gamma(-\frac{1}{2} + \rho + 2s)}{\Gamma(1 + s) \Gamma(\frac{1}{2} + s)} \\ &= \frac{2^{2s}}{\pi^{\frac{1}{2}}} \frac{\Gamma(2 - 2\rho - 2s) \Gamma(\rho - \frac{1}{2} + 2s)}{\Gamma(1 + 2s)}. \end{aligned}$$

Now, by definition,

$${}_0 S_1(s) \pi^{\frac{1}{2}} = \sum_{t=0}^{\infty} \frac{\Gamma(-s + \frac{1}{2}t) \Gamma(1 - \rho + \frac{1}{2}t - s)}{\Gamma(t + 1)} 2^t;$$

and therefore by a previous investigation (§ 26)

$${}_0 S_1(s) = \pi^{-1} \sin \pi(\rho - \frac{1}{2}) 2^{2\rho + 4s} \Gamma(-2s) \Gamma(2 - 2\rho - 2s) \Gamma(2s + \rho - \frac{1}{2}).$$

Thus the identity (1) is immediately verified.

31. We have now to consider the nature of factorial series of the type $U_r(s)$, and especially the nature of their behaviour when $s = u + iv$, and u and $|v|$ either or both tend to positive infinity. With this object we shall first consider more elementary series from whose properties we can deduce inequalities and relations of the type which we desire to obtain.

As a simple example of the factorial series which we proceed to consider, let us take the function $S(s)$, defined when $R(s) > 0$ by the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + 1)}$$

where μ is an integer.

By the investigation of the binomial theorem due to Abel, we know that for all values of x and s for which the series is convergent*

$$\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} x^n = \Gamma(-s)(1-x)^s$$

where $(1-x)^s = \exp\{s \log(1-x)\}$ and $|\arg(1-x)| < \pi$. When $x = 1$ the value of the series is zero.

Let, now, w be a special root of the equation $y^\mu = 1$, so that $1, w, w^2, \dots, w^{\mu-1}$ are all the roots of the equation. Then we have

$$\begin{aligned} \sum_{r=0}^{\mu-1} (w^n)^r &= 0, \quad \text{unless } n = M(\mu) \\ &= \mu, \quad \text{if } n = M(\mu). \end{aligned}$$

Hence
$$\mu \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + 1)} = \sum_{r=0}^{\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+1)} (w^n)^r.$$

Therefore
$$S(s) = \mu^{-1} \Gamma(-s) \sum_{r=0}^{\mu-1} (1-w^r)^s.$$

This equality represents the function $S(s)$ over the whole of the s -plane.

The term corresponding to $r = 0$ is zero; therefore

$$S(s) = \mu^{-1} \Gamma(-s) \sum_{r=1}^{\mu-1} (1-w^r)^s.$$

In this expression we have to take such a value of $(1-w^r)$ that the modulus of its argument is $< \frac{1}{2}\pi$.

Now, if $w = e^{2\pi i/\mu}$, this being the simplest special root of $y^\mu = 1$, we have, as is readily seen from a figure,

$$\arg(1-w^r) = \pi r/\mu - \frac{1}{2}\pi, \quad \text{if } 2r < \mu.$$

* A full discussion of the theorem appears in a paper by the author, *Quarterly Journal of Mathematics*, Vol. xxxviii., pp. 108-116.

And the maximum value of $|\arg(1-w^r)|$

$$= \frac{1}{2}\pi - \pi/\mu$$

if $\mu > 1$.

Hence, if $s = u + iv$ and v tends to infinity, the expression

$$|S(s)| \exp\{(\pi/\mu - \epsilon)|v|\}$$

where $\epsilon > 0$ behaves like

$$\Gamma(-s) e^{(\frac{1}{2}\pi - \epsilon)|v|} \sum_{r=1}^{\mu-1} (1-w^r)^s e^{-(\frac{1}{2}\pi - \pi/\mu)|v|},$$

and therefore tends exponentially to zero as $|v|$ tends to infinity for all finite values of u .

A fortiori it is evident that, except at the poles of $\Gamma(-s)$, $|S(s)|$ will tend exponentially to zero for all values of $|v|$ if u tend to positive infinity.

32. It is evident that the previous discussion is only suggestive of similar results for more complex series, and that for such series other methods of proof must be discovered.

We proceed to consider the behaviour, as $|v|$ tends to infinity, of the more general function defined when $R(s+c) > 1$ by the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + c)},$$

μ being an integer > 1 .

As before, this series may be written

$$\frac{1}{\mu} \sum_{r=0}^{\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+c)} (w^r)^n.$$

Suppose now that the coefficients $e_1, \dots, e_m$ are so chosen that

$$(1-wx)^{-1} - 1 - wx - \dots - w^n x^n - e_1 x^{n+1} - \dots - e_m x^{n+m} = (1-x)^m Q(x)$$

where $Q(x)$ is finite near $x = 1$. Evidently this is always possible.

Consider now the integral

$$\int_0^1 (1-x)^{s+c-1} x^{-s-1} \left\{ \frac{(wx)^{n+1}}{1-wx} - e_1 x^{n+1} - \dots - e_m x^{n+m} \right\} dx.$$

It is convergent at $x = 0$ if $R(n-s) > -1$ and at $x = 1$ if $R(s+c+m) > 0$, provided $w \neq 1$.

Thus it is convergent at both limits if $n > R(s) > -m - R(c)$.

If $n > R(s) > -R(c)$, the integral is equal to

$$\begin{aligned} \sum_{t=n+1}^{\infty} w^t \frac{\Gamma(t-s)\Gamma(s+c)}{\Gamma(t+c)} - \sum_{t=1}^m e_t \frac{\Gamma(n+t-s)\Gamma(s+c)}{\Gamma(n+t+c)} \\ = \Gamma(s+c) \left\{ \sum_{t=0}^{\infty} \frac{\Gamma(t-s)}{\Gamma(t+c)} w^t - \sum_{t=0}^n \frac{\Gamma(t-s)}{\Gamma(t+c)} w^t - \sum_{t=1}^m e_t \frac{\Gamma(n+t-s)}{\Gamma(n+t+c)} \right\}. \end{aligned}$$

Evidently, if $w \neq 1$, when $\sum_{t=0}^{\infty} \frac{\Gamma(t-s)}{\Gamma(t+c)} w^t$ is replaced by the function which it defines when $R(s+c) > 0$, the equality holds for all values of s such that

$$n > R(s) > -m - R(c).$$

Thus, if n and m be sufficiently large, it holds for all values of $R(s)$ which are finite.

The integral may be written

$$\int_0^1 \left(\frac{1-x}{x} \right)^s x^{-1} (1-x)^{c-1} \left\{ \frac{(wx)^{n+1}}{1-wx} - \sum_{t=1}^m e_t x^{n+t} \right\} dx.$$

Instead of taking the integral along the real axis from 0 to 1, we may, since the subject of integration is one-valued, take it along any circular arc from 0 to 1 which does not enclose singularities of the subject of integration.* On such arcs $\frac{1-x}{x}$ has a constant argument.

If $w = e^{2r\pi i/\mu}$, it is evident that on the arc which passes through $1/w$, which is the sole singularity of the subject of integration other than the terminal points 0 and 1,

$$\arg \frac{1-x}{x} = \arg \frac{1-e^{-2r\pi i/\mu}}{e^{-2r\pi i/\mu}} = \arg \left\{ 2 \sin \frac{r\pi}{\mu} e^{(\frac{1}{2}\pi + r\pi/\mu)i} \right\} = \frac{1}{2}\pi + \frac{r\pi}{\mu}$$

if $2r < \mu$.

$$\text{If } 2r = \mu, \quad \arg \left(\frac{1-x}{x} \right) = \pm \pi.$$

$$\text{When } 2\mu > 2r > \mu, \quad \arg \left(\frac{1-x}{x} \right) = -\frac{1}{2}\pi - \pi \frac{\mu-r}{\mu}.$$

We see then that, if $w \neq 1$ and therefore $r \neq 0$, the modulus of the integral

$$\frac{1}{\Gamma(s+c)} \int_0^1 \left(\frac{1-x}{x} \right)^s x^{-1} (1-x)^{c-1} \left\{ \frac{(wx)^{n+1}}{1-wx} - \sum_{t=1}^m e_t x^{n+t} \right\} dx,$$

for all finite values of u as $|v|$ tends to infinity, behaves like

$$e^{-(r\pi/\mu - \epsilon)|v|} F(v), \quad \text{if } 2r \leq \mu,$$

or like

$$e^{-[(\mu-r)\pi/\mu - \epsilon]|v|} F(v), \quad \text{if } 2\mu > 2r \geq \mu,$$

where $F(v)$ tends exponentially to zero as $|v|$ tends to infinity if $\epsilon > 0$.

If $w = 1$, we have the function defined when $R(s+c) > 0$ by the series $\sum_{n=0}^{\infty} \frac{\Gamma(n-s)}{\Gamma(n+c)}$, which, for all finite values of u as $|v|$ tends to infinity, behaves like

$$e^{-(\frac{1}{2}\pi - \epsilon)|v|} F(v).$$

* A figure corresponding to this transformation and further details will be found in § 24 of the paper by the author quoted in § 1 (Paper ϵ).

Finally, therefore, the modulus of the function defined when $R(s) > -R(c)$ by the series $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + c)}$, μ an integer > 1 , behaves for all finite values of u as $|v|$ tends to infinity like $e^{-(\pi/\mu - \epsilon)|v|} F(v)$ where $F(v)$ tends exponentially to zero as $|v|$ tends to infinity if $\epsilon > 0$.

33. We will further show that, if $s = R + v$ where R is large and positive,

$$\left| \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - s)}{\Gamma(\mu n + c)} \right| < \eta_R \exp \{ (-\pi/\mu + \epsilon) |v| \}$$

where $\epsilon > 0$, and $\eta_R e^{kR}$, $k > 0$ but finite, is independent of v and can be made as small as we please by taking R sufficiently large, if s is not in the immediate vicinity of one of the points n .

We have, if R be large and positive, u be finite, and n be just so large that $n - R > R(s)$,

$$\begin{aligned} I &= \frac{1}{\Gamma(R + s + c)} \int_0^1 \left(\frac{1-x}{x} \right)^s (1-x)^{R+c-1} \frac{w^{n+1} x^{n-R}}{1-wx} dx \\ &= \sum_{t=0}^{\infty} \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t - \sum_{t=0}^n \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t. \end{aligned}$$

The integral can be taken along a circular arc as before, and its modulus is

$$< e^{-(\frac{1}{2}\pi + \pi/\mu - \epsilon)|v|} |w^{n+1}| \mu^R K$$

where μ is the maximum value of $|1-x|$ on the arc and K is independent of v and R .

Also, if R and $R(s+c)$ be positive and greater than unity,

$$|\Gamma(R)| |\Gamma(s+c)| < |\Gamma(R+s+c)|.$$

Hence
$$|I| < e^{(-\pi/\mu + \epsilon)|v|} \frac{\mu^R}{\Gamma(R)} K.$$

Again, $\left| \sum_{t=0}^n \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t \right| < \theta R$ times the modulus of the largest term in the series, where θ tends to a finite limit as R tends to infinity. It is therefore less than $\epsilon_R e^{-(\frac{1}{2}\pi - \epsilon)|v|}$ where $\epsilon_R e^{kR}$ tends to zero with $1/R$ and is independent of v , provided $R+s$ is not in the vicinity of one of the zeros of $\Gamma(-R-s)$. Thus when R is large

$$\left| \sum_{t=0}^{\infty} \frac{\Gamma(t-R-s)}{\Gamma(t+c)} w^t \right| < \eta'_R e^{(-\pi/\mu + \epsilon)|v|}$$

where η_R is independent of $|v|$ and $\eta'_R e^{kR}$ can be made as small as we please by taking R sufficiently large.

Hence
$$\left| \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - R - s)}{\Gamma(\mu n + c)} \right| < \eta_R e^{\pi/(\mu + \epsilon) |v|}$$

where η_R is independent of v and $e^{kR} \eta_R$ can be made as small as we please by taking R sufficiently large, provided $R + s$ is not in the vicinity of one of the points n .

34. THEOREM.—Let $g_\beta(x, 1)$ be the function defined when $|x| < 1$ by the series $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^\beta}$, and $\Phi_m(x)$ the function similarly defined by $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n + c)}{\Gamma(\mu n + \beta + m + c)} x^n$, the functions being made one-valued by cross-cuts along the positive half of the real axis from 1 to $+\infty$. Further, let the coefficients c_m be defined by the expansion

$$\left\{ \frac{\log(1-y)}{-y} \right\}^{\beta-1} (1-y)^{\mu-c} = \sum_{m=0}^{\infty} c_m (-y)^m,$$

which is valid when y is sufficiently small. Then the coefficients a can be so chosen that the function

$$F(x) = \mu^{-\beta} g_\beta(x^\mu, 1) - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m)}{\Gamma(\beta)} \Phi_m(x^\mu) - \sum_{n=0}^{L+N-1} a_n x^n$$

behaves at $x = 0$ like $x^N P(x)$ where $P(x)$ is finite when $x = 0$, and at $x = 1$ like $(1-x)^L Q(1-x)$ where $Q(1-x)$ is finite when $x = 1$, L , M , and N being finite positive integers, however large, such that $R(\beta) + M > L$.

I have previously shewn that* $g_\beta(x, 1)$ admits, near $x = 1$, the convergent expansion

$$\Gamma(1-\beta)(-\log x)^{\beta-1} x^{-1} + \sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta, 1),$$

the latter series being convergent near $x = 1$, and $(-\log x)^{\beta-1}$ being made one-valued by a cross-cut from 1 to $+\infty$.

Also it can be shown that the function defined when $|x| < 1$ by the series $\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\rho+n)} x^n$ admits, near $x = 1$, the convergent expansion

$$\Gamma(1+\alpha-\rho)(1-x)^{\rho-\alpha-1} x^{1-\rho} - \frac{\Gamma(1+\alpha-\rho)}{\Gamma(\rho-1)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(2+\alpha-\rho+n)} (1-x)^n,$$

wherein there is a similar cross-cut, so that $|\arg(1-x)| < \pi$.

* *Proc. London Math. Soc.*, Ser. 2, Vol. 4, p. 291.

Now

$$\begin{aligned}\Phi_m(x^\mu) &= \sum_{n=0}^{\infty} \frac{\Gamma(\mu n + c)}{\Gamma(\mu n + \beta + m + c)} x^{n\mu} = \frac{1}{\mu} \sum_{r=0}^{\mu-1} \sum_{n=0}^{\infty} \frac{\Gamma(n+c)}{\Gamma(n+\beta+m+c)} (w^r x)^n \\ &\quad \text{(where } w \text{ is a special root of } y^\mu - 1 = 0) \\ &= \frac{1}{\mu} \sum_{r=0}^{\mu-1} \left\{ \frac{\Gamma(1-\beta-m)(1-w^r x)^{\beta+m-1} (w^r x)^{1-\beta-m-c}}{\Gamma(\beta+m+c-1)} \sum_{n=0}^{\infty} \frac{\Gamma(c+n)}{\Gamma(2-\beta-m+n)} (1-w^r x)^n \right\},\end{aligned}$$

each expansion being valid near the corresponding singularity. Therefore, since near $x = 1$ all the functions in the summation $\sum_r$ except that corresponding to $r = 0$ are analytic, we have, near $x = 1$,

$$\Phi_m(x^\mu) = \frac{1}{\mu} \Gamma(1-\beta-m)(1-x)^{\beta+m-1} x^{1-\beta-m-c} + \sum_{n=0}^{\infty} d_n (1-x)^n,$$

the latter series being convergent near $x = 1$.

Hence the function $F(x)$ behaves in the vicinity of its singularity $x = 1$ like

$$\begin{aligned}&\mu^{-\beta} \Gamma(1-\beta)(-\mu \log x)^{\beta-1} x^{-\mu} - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(1-\beta-m)}{\mu \Gamma(\beta)} \\ &\quad \times (1-x)^{\beta+m-1} x^{1-\beta-m-c} + \sum_{n=0}^{\infty} e_n (1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n \\ &= \frac{\Gamma(1-\beta) x^{-c}}{\mu} \left(\frac{1-x}{x} \right)^{\beta-1} \left[\left(-\frac{x \log x}{1-x} \right)^{\beta-1} x^{c-\mu} - \sum_{m=0}^M c_m \left(\frac{1-x}{x} \right)^m \right] \\ &\quad + \sum_{n=0}^{\infty} e_n (1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n.\end{aligned}$$

Now near $x = 1$ we may put $x = 1/(1-y)$, so that $(1-x)/x = -y$; then the expression inside the square brackets becomes

$$\begin{aligned}&\left\{ \frac{-\log(1-y)}{y} \right\}^{\beta-1} (1-y)^{\mu-c} - \sum_{m=0}^M c_m (-y)^m \\ &= (-y)^{M+1} \left\{ \sum_{m=0}^{\infty} c_{M+m+1} (-y)^m \right\}\end{aligned}$$

if $|y|$ be small. Thus $F(x)$ near the singularity $x = 1$ behaves like

$$\sum_{n=0}^{\infty} e_n (1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n + (1-x)^{\beta+M} Q'(x),$$

where $Q'(x)$ is finite near $x = 1$.

Suppose now, as is evidently possible, that we choose the first N coefficients a so that

$$F(x) = x^N P(x),$$

where $P(x)$ is finite at $x = 0$. Then we may further choose the remaining L coefficients a so that

$$\sum_{n=0}^{\infty} e_n (1-x)^n - \sum_{n=0}^{L+N-1} a_n x^n = (1-x)^L Q(1-x)$$

where $Q(1-x)$ is finite near $x = 1$.

If, then, $R(\beta) + M > L$, we have the proposition stated.

35. We will next shew that, if N be a multiple of μ and if σ be such that

$$N > R(\sigma) > -L - R(c) + 1,$$

$$\begin{aligned} & \frac{1}{\Gamma(c+\sigma)} \int_0^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x) dx \\ &= - \sum_{n=N}^{L+N-1} a_n \frac{\Gamma(n-\sigma)}{\Gamma(n+c)} \\ &+ \sum_{r=N/\mu}^{\infty} \frac{\Gamma(\mu r - \sigma)}{\Gamma(\mu r + c)} \left\{ \frac{1}{(\mu r + \mu)^\beta} - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(\mu r + c)}{\Gamma(\beta) \Gamma(\mu r + \beta + m + c)} \right\}. \end{aligned}$$

We assume that all the numbers involved are defined as in the previous paragraph.

If N be a multiple of μ , and R be an integer $> N/\mu$, we have

$$\begin{aligned} & \int_0^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x) dx \\ &= \int_{1-\epsilon}^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x) dx - \sum_{n=N}^{L+N-1} a_n \int_0^{1-\epsilon} x^n (1-x)^{c+\sigma-1} x^{-\sigma-1} dx \\ &+ \sum_{r=N/\mu}^{R-1} \int_0^{1-\epsilon} (1-x)^{c+\sigma-1} x^{-\sigma-1+\mu r} \\ &\quad \times \left\{ \frac{1}{(\mu r + \mu)^\beta} - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(\mu r + c)}{\Gamma(\beta) \Gamma(\mu r + \beta + m + c)} \right\} dx \\ &+ \int_0^{1-\epsilon} (1-x)^{c+\sigma-1} x^{-\sigma-1} \sum_{r=R}^{\infty} \frac{x^{\mu r} J_M(r)}{r^{\beta+M}} dx, \end{aligned} \quad (1)$$

where $J_M(r)$ is defined by the equality

$$\frac{1}{(\mu r + \mu)^\beta} = \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(\mu r + c)}{\Gamma(\beta) \Gamma(\beta+m+\mu r + c)} + \frac{J_M(r)}{r^{\beta+M}}. \quad (2)$$

Now I have shown* that, if $|\arg \phi| < \pi$,

$$\frac{\Gamma(\beta)}{(\theta + \phi)^\beta} = \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(1+\phi)}{\Gamma(\beta+m+\phi+1)} + \frac{J_M}{\phi^{\beta+M}}$$

where $|J_M|$ tends to zero as $|\phi|$ tends to infinity, and the coefficients c'_m are given by the expansion

$$\left\{ \frac{-\log(1-y)}{y} \right\}^{\beta-1} (1-y)^{\theta-1} = \sum_{m=0}^{\infty} c'_m (-y)^m.$$

Putting $\phi = \mu r + c - 1, \quad \theta = \mu - c + 1,$

we see that in the formula (2) $|J_M(r)| < \eta$ an arbitrarily assigned small quantity η , if R be a sufficiently large positive quantity and $r \gg R$, the quantities c_m being defined as in the previous paragraph.

Now, in the formula (1) the modulus of the first integral tends to zero as ϵ tends to zero if $R(c + \sigma + L) > 0$.

The modulus of any one of the second system of integrals tends to a finite limit as ϵ tends to zero if $N > R(\sigma) > -R(c)$. Each of the third set of integrals has a modulus which tends to a finite limit if the same conditions hold.

The modulus of the final integral is

$$< \eta \int_0^{1-\epsilon} (1-x)^{R(c+\sigma)-1} \sum_{r=R}^{\infty} \frac{x^{-R(\sigma)-1+\mu r}}{r^{\beta+M}} dx.$$

Hence, if $N > R(\sigma) > -R(c)$ and $R(\beta + M) > 1$,

$$\begin{aligned} & \int_0^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x) dx \\ &= - \sum_{n=N}^{L+N-1} a_n \frac{\Gamma(c+\sigma) \Gamma(n-\sigma)}{\Gamma(c+n)} + \sum_{r=N/\mu}^{R-1} \frac{\Gamma(c+\sigma)}{\Gamma(\mu r+c)} \Gamma(\mu r-\sigma) \\ & \times \left\{ \frac{1}{(\mu r+\mu)^{\beta}} - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(\mu r+c)}{\Gamma(\beta) \Gamma(\mu r+\beta+m+c)} \right\} + I_R, \end{aligned}$$

where $|I_R|$ tends to zero as R tends to infinity.

Thus under the same conditions

$$\begin{aligned} & \frac{1}{\Gamma(c+\sigma)} \int_0^1 (1-x)^{c+\sigma-1} x^{-\sigma-1} F(x) dx \\ &= - \sum_{n=N}^{L+N-1} a_n \frac{\Gamma(n-\sigma)}{\Gamma(n+c)} + \sum_{r=N/\mu}^{\infty} \frac{\Gamma(\mu r-\sigma)}{\Gamma(\mu r+c)} \\ & \times \left\{ \frac{1}{(\mu r+\mu)^{\beta}} - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta+m) \Gamma(\mu r+c)}{\Gamma(\beta) \Gamma(\mu r+\beta+m+c)} \right\}. \end{aligned}$$

But, if $R(\beta) + M > L$, the integral on the left-hand side of this equality is an analytic function with no poles if

$$N > R(\sigma) > -L - R(c).$$

Also the series on the right-hand side is absolutely convergent if

$$R(\beta + M + \sigma + c) > 1,$$

and, *a fortiori*, if $R(\sigma + L + c) > 1$.

The equality is therefore true under the wider conditions of the enunciation.

36. Let $S(\sigma)$ be the function defined by the series $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \sigma)}{\Gamma(\mu n + c)(n+1)^\beta}$ when $R(c + \sigma + \beta) > 1$. Then, if $\sigma = u + iv$, and u be any real quantity of finite modulus, $|S(\sigma)| \exp\{(\pi/\mu - \epsilon)|v|\}$ tends to zero as $|v|$ tends to infinity if $\epsilon > 0$ and μ is an integer > 1 .

Let $\Sigma_m(\sigma)$ be the function defined by the series $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \sigma)}{\Gamma(\mu n + \beta + m + c)}$, if $R(\beta + m + c + \sigma) > 1$.

Then, by the result of the previous paragraph, we have, if

$$N > u > -L - R(c) + 1,$$

$$\begin{aligned} & \frac{1}{\Gamma(c + \sigma)} \int_0^1 \left(\frac{1-x}{x}\right)^{c+\sigma-1} x^{c-2} F(x) dx \\ &= \mu^{-\beta} S(\sigma) - \sum_{m=0}^M (-)^m c_m \frac{\Gamma(\beta + m)}{\Gamma(\beta)} \Sigma_m(\sigma) - \sum_{n=N}^{L+N-1} a_n \frac{\Gamma(n - \sigma)}{\Gamma(n + c)} \\ & \quad - \sum_{r=0}^{N/\mu-1} \frac{\Gamma(\mu r - \sigma)}{\Gamma(\mu r + c)} \left\{ \frac{1}{(\mu r + \mu)^\beta} - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta + m) \Gamma(\mu r + c)}{\Gamma(\beta) \Gamma(\mu r + \beta + m + c)} \right\}. \end{aligned}$$

Now the modulus of the first integral tends uniformly to zero, as $|v|$ tends to infinity, when multiplied by $\exp\{(\pi/\mu - \epsilon)|v|\}$, as we see by the method of the theorem of § 32. By the properties of the gamma function the same is also true of every term in the two final summations of the previous equality.

Also, if μ be an integer > 1 , $|\Sigma_m(\sigma)|$ tends to zero as $|v|$ tends to infinity when multiplied by $\exp\{(\pi/\mu - \epsilon)|v|\}$, as has been proved previously (§ 32).

Hence we have the given theorem.

37. If $S(\sigma)$ be the function defined when $R(c + \sigma + \beta) > 1$ by the series $\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \sigma)}{\Gamma(\mu n + c)(n+1)^\beta}$, the function $\frac{S(\sigma)}{\Gamma(-\sigma)}$ has for its sole finite

singularities poles at the points

$$\sigma = 1 - \beta - c - n \quad (n = 0, 1, 2, \dots, \infty).$$

When n is large we know that† $\frac{\Gamma(\mu n - \sigma)}{\Gamma(\mu n + c)(\mu n + \mu)^\beta}$ admits the asymptotic expansion.

$$\sum_{r=0}^R \frac{c_r(\sigma)}{(\mu n)^{\sigma + \beta + c + r}} + \frac{J_R(n)}{(\mu n)^{\sigma + \beta + c + R}},$$

where the coefficients $c_r(\sigma)$ are polynomials in σ and $|J_R(n)|$ can for any finite value of R be made as small as we please by taking n sufficiently large.

Hence the function

$$\frac{S(\sigma)}{\mu^\beta} - \sum_{r=0}^R \frac{c_r(\sigma)}{\mu^{\sigma + \beta + c + r}} \zeta(\sigma + \beta + c + r)$$

admits, when $R(c + \sigma + \beta) > 1$, the expansion

$$\sum_{n=0}^{\infty} \frac{J_R(n)}{(\mu n)^{\sigma + \beta + c + R}},$$

the double accent denoting that, when $n = 0$, the term is $\frac{\Gamma(-\sigma)}{\Gamma(c) \mu^\beta}$. The series is convergent when $R(\sigma + \beta + c) > 1 - R$. And hence for this wider range the function

$$\frac{S(\sigma)}{\mu^\beta} - \sum_{r=0}^R \frac{c_r(\sigma)}{\mu^{\sigma + \beta + r + c}} \zeta(\sigma + \beta + c + r)$$

has no singularities except poles at the points

$$\sigma = n \quad (n = 0, 1, 2, \dots, \infty).$$

Now $\zeta(s)$ has for its sole finite singularity a pole at the point $s = 1$, at which the residue is unity.

Hence the sole finite singularities of $S(\sigma)/\Gamma(-\sigma)$ are poles at the points

$$\sigma = 1 - \beta - c - n \quad (n = 0, 1, 2, \dots, \infty);$$

and at

$$\sigma = 1 - \beta - c - n$$

the residue is $\mu^{-1} c_n(1 - \beta - c - n)$.

† With this theorem the reader may compare the similar proposition established in § 5 of the author's paper "The Use of Factorial Series in an Asymptotic Expansion," *Quarterly Journal of Mathematics*, Vol. xxxviii., pp. 116-140.

38. We proceed now to consider the behaviour of the function $S(\sigma)$, defined when $R(c+\sigma+\beta) > 1$ by the series

$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \sigma)}{\Gamma(\mu n + c)(n+1)^{\beta}},$$

for values of $\sigma = u + iv$ for which u is large, real, and positive when $|v|$ does or does not tend to infinity.

We will shew that

$$S(\sigma) < \eta_v \exp \{ (-\pi/\mu + \epsilon) |v| \},$$

where η_v is independent of v , and $e^{kv} \eta_v$, $k > 0$ and finite, can be made as small as we please by taking u sufficiently large and positive, unless σ be in the immediate vicinity of one of the points n .

Suppose that $\sigma = u + iv$, where u is finite. Choose M such that $R(\beta + M) > 0$. Let R be very large, real, and positive, so that

$$R(c + \sigma + R) > 0, \quad R(c + \sigma + R + \beta) > 1,$$

$$\text{and} \quad R(\beta + m + c + \sigma + R) > 1 \quad (0 \leq m \leq M).$$

Choose N a multiple of μ so large that $N > u + R - 1$. Then, with the previous notation,

$$S(\sigma + R) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \sigma - R)}{\Gamma(\mu n + c)(n+1)^{\beta}}$$

$$\text{and} \quad \Sigma_m(\sigma + R) = \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \sigma - R)}{\Gamma(\mu n + \beta + m + c)}.$$

Let the a 's be so chosen that, near $x = 0$,

$$F(x) \equiv \mu^{-\beta} g_{\beta}(x^{\mu}, 1) - \sum_{m=0}^M (-)^m c_m \frac{\Gamma(\beta + m)}{\Gamma(\beta)} \Phi_m(x^{\mu}) - \sum_{n=0}^{N-1} a_n x^n$$

is of the form $x^N P(x)$ where $P(x)$ is finite when $x = 0$. Then, by the result of § 35,

$$\begin{aligned} & \frac{1}{\Gamma(c + \sigma + R)} \int_0^1 \left(\frac{1-x}{x} \right)^{\sigma} (1-x)^{c+R-1} x^{-R-1} F(x) dx \\ &= \mu^{-\beta} S(\sigma + R) - \sum_{m=0}^M (-)^m c_m \frac{\Gamma(\beta + m)}{\Gamma(\beta)} \Sigma_m(\sigma + R) \\ & \quad - \sum_{r=0}^{N/\mu-1} \frac{\Gamma(\mu r - \sigma - R)}{\Gamma(\mu r + c)} \left\{ \frac{1}{(\mu r + \mu)^{\beta}} - \sum_{m=0}^M \frac{(-)^m c_m \Gamma(\beta + m) \Gamma(\mu r + c)}{\Gamma(\beta) \Gamma(\mu r + \beta + m + c)} \right\}. \end{aligned} \tag{1}$$

Now we have seen in § 33 that, when $|v|$ is large,

$$\Sigma_m(\sigma + R) < {}_m\eta_R \exp \{(-\pi/\mu + \epsilon) |v|\},$$

where ${}_m\eta_R$ is independent of v , and $e^{kR} {}_m\eta_R$ can be made as small as we please by taking R sufficiently large.

Also, if we employ the transformation of § 32, and take the integral along a suitable circular arc, its modulus is less than

$$\frac{1}{|\Gamma(c + \sigma + R)|} \exp \{-(\frac{1}{2}\pi + \pi/\mu - \epsilon) |v|\} \mu_1^R K,$$

where μ_1 is the maximum value of $|(1-x)|$ on the arc, and K is finite and independent of v and R . The modulus of the integral is thus less than

$$\frac{\mu_1^R}{\Gamma(R)} K \exp \{(-\pi/\mu + \epsilon) |v|\} < \eta'_R \exp \{-\pi/\mu + \epsilon |v|\},$$

where η'_R is independent of v , and $e^{kR} \eta'_R$ can be made as small as we please by taking R sufficiently large.

The final N/μ terms in the expression (1) have a sum whose modulus is less than N/μ times the modulus of the largest term, *i.e.*,

$$< \eta''_R \exp \{(-\frac{1}{2}\pi + \epsilon) |v|\}$$

where η''_R is independent of v , and $e^{kR} \eta''_R$ can be made as small as we please by taking R sufficiently large.

Hence, finally, when $|v|$ is large,

$$|S(\sigma + R)| < \eta_R \exp \{(-\pi/\mu + \epsilon) |v|\}$$

where η_R is independent of v , and $e^{kR} \eta_R$ can be made as small as we please by taking R sufficiently large.

This is equivalent to the given theorem.

39. *Let*

$${}_qF_{q-1} \{a_1, \dots, a_q; b_1, \dots, b_q; x\} = \sum_{n=0}^{\infty} \prod_{t=1}^q \frac{\Gamma(a_t + n)}{\Gamma(b_t + n)} x^n,$$

when $|x| < 1$, where one of the quantities $b_1, \dots, b_q$ is unity. Further, let the coefficients $f_r(b, a)$ be defined by the asymptotic expansion

$$\prod_{t=1}^q \frac{\Gamma(a_t + n)}{\Gamma(b_t + n)} = \sum_{r=0}^R \frac{f_r(b, a)}{n^{\Sigma b - \Sigma a + r}} + \frac{J_R(n, b, a)}{n^{\Sigma b - \Sigma a + R}},$$

where, for all finite values of R , $|J_R(n, b, a)|$ can be made as small as we please by taking n sufficiently large. Then, however large the finite

quantity M may be, if we take $R(\Sigma b - \Sigma a + R - 1) > M$, it is always possible to find coefficients d such that, near $x = 1$,

$${}_qF_{q-1}(x) - \sum_{r=0}^R f_r(b, a) \Gamma(1 - \theta - r) (-\log x)^{\theta + r - 1} - \sum_{m=0}^M d_m (1-x)^m = (1-x)^M Q(x),$$

where $Q(x)$ vanishes at $x = 1$, and $\theta = \Sigma b - \Sigma a$.

If the double accent denote that the terms $\sum_{r=0}^R$ do not exist when $n = 0$, we have

$$\sum_{n=0}^{\infty} \left\{ \prod_{t=1}^q \frac{\Gamma(a_t + n)}{\Gamma(b_t + n)} - \sum_{r=0}^R \frac{f_r(b, a)}{n^{\Sigma b - \Sigma a + r}} \right\} x^n = \sum_{n=0}^{\infty} \frac{J_R(n, b, a)}{n^{\Sigma b - \Sigma a + R}} x^n.$$

By the properties of $J_R(n, b, a)$ this series and its k -th derivate with regard to x , where $R(\Sigma b - \Sigma a) + R - k > 1$, is convergent when $x = 1$. But, when $|x| < 1$, the series is equal to

$${}_qF_{q-1}(x) - \sum_{r=0}^R x f_r(b, a) g_{\Sigma b - \Sigma a + r}(x, 1),$$

where, when $|x| < 1$, $g_{\beta}(x, \theta) = \sum_{n=0}^{\infty} \frac{x^n}{(n + \theta)^{\beta}}.$

But the author has shown that, near $x = 1$,

$$g_{\beta}(x, \theta) - \Gamma(1 - \beta) (-\log x)^{\beta - 1} x^{-\theta}$$

is one-valued and equal to

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{x^{n+1}} \bar{\zeta}_{n+1}(\beta, \theta).$$

Therefore, near $x = 1$, the series is equal to

$${}_qF_{q-1}(x) - \sum_{r=0}^R f_r(b, a) \left\{ \Gamma(1 - \Sigma b + \Sigma a - r) (-\log x)^{\Sigma b - \Sigma a + r - 1} + \sum_{n=0}^{\infty} \left(\frac{x-1}{x} \right)^n \bar{\zeta}_{n+1}(\Sigma b - \Sigma a + r, 1) \right\}.$$

This expression and all-derivates up to the k -th converge to a definite value as x tends to unity, the various terms being made one-valued by a cross-cut from 1 to ∞ along the positive half of the real axis.

Hence we can always find coefficients d so as to satisfy the proposition.

40. A suggestive deduction from the previous theorem may be noticed.

Since one of the constants b is unity, the function ${}_qF_{q-1}(x)$ is a hypergeometric series in x of order q , and therefore by the researches of Thomae and Pochhammer* we know that, near $x = 1$,

$${}_qF_{q-1}(x) = \sum_{m=1}^{q-1} A_m P_m(1-x) + A_q(1-x)^{\theta-1} P_q(1-x)$$

where the coefficients of the constants A are solutions of the differential equation for ${}_qF_{q-1}(x)$ near $x = 1$. Moreover the functions $P_m(1-x)$ ($m = 1, 2, \dots, q$) are one-valued power series of $(1-x)$ convergent near $x = 1$.

We must therefore have, near $x = 1$, if $R(\theta + R - 1) > M$,

$$\sum_{r=0}^R f_r(b, a) \Gamma(1-\theta-r)(-\log x)^{\theta+r-1} - A_q(1-x)^{\theta-1} P_q(1-x) = (1-x)^M Q(x),$$

where $Q(x)$ vanishes at $x = 1$.

We may therefore anticipate that

$$\sum_{r=0}^{\infty} f_r(b, a) \Gamma(1-\theta-r)(-\log x)^{\theta+r-1}$$

is either an asymptotic or a convergent expansion near $x = 1$ of that solution of the differential equation for ${}_qF_{q-1}(x)$ which is multiform near $x = 1$.

41. Let ${}_qF_{q-1}\{x\}$ denote the function represented by the series

$$\sum_{n=0}^{\infty} \prod_{t=1}^q \frac{\Gamma(a_t+n)}{\Gamma(b_t+n)} x^n$$

when $|x| < 1$, with a cross-cut from 1 to ∞ along the positive half of the real axis. As in § 29, let $U_r(s)$ be the function defined when

$$R(\mu s + \mu + \Sigma b - \Sigma a) > 1$$

by the series
$$\sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu)} \prod_{t=1}^q \frac{\Gamma(a_t+n)}{\Gamma(b_t+n)}.$$

Let R , M , and N be integers such that, if $\theta = \Sigma b - \Sigma a$,

$$R(\theta) + R - 1 > M, \quad R(\mu - \mu \rho_r + \mu N) > R(\mu s) > -M - \mu. \quad (1)$$

* Thomae, *Mathematische Annalen*, Bd. II., p. 433; Pochhammer, *Crelle*, T. CII., pp. 97, &c.

Then, if the c 's be suitably chosen, we shall have

$$\begin{aligned} & \frac{1}{\Gamma(\mu s + \mu)} \int_0^1 \left(\frac{1-x}{x} \right)^{\mu s + \mu - 1} x^{2\mu - 2 - \mu \rho_r} \\ & \quad \times \left\{ {}_q F_{q-1} \{ x^\mu \} - \sum_{\tau=0}^R x^\mu f_\tau(b, a) g_{\theta+\tau}(x^\mu, 1) - \sum_{m=0}^{M+N} c_m x^{\mu m} \right\} dx \\ & = U_\tau(s) - \sum_{\tau=0}^R f_\tau(b, a) \sum_{n=1}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu) n^{\theta+\tau}} \\ & \quad - \sum_{m=0}^{M+N} c_m \frac{\Gamma(\mu m + \mu - \mu s - \mu \rho_r)}{\Gamma(\mu m + 2\mu - \mu \rho_r)}. \end{aligned} \quad (2)$$

By § 39, if the quantities $d_0, d_1, \dots, d_M$ be suitably chosen,

$${}_q F_{q-1} \{ x^\mu \} - \sum_{\tau=0}^R x^\mu f_\tau(b, a) g_{\theta+\tau}(x^\mu, 1) - \sum_{m=0}^M d_m (1-x^\mu)^m = (1-x^\mu)^M Q(x^\mu)$$

where $Q(x^\mu)$ vanishes where $x = 1$, if $R(\theta) + R - 1 > M$.

We may now choose $d_{M+1}, d_{M+2}, \dots, d_{M+N}$ so that the previous expression minus $\sum_{m=M+1}^{M+N} d_m (1-x^\mu)^m$ is near $x = 0$ equal to $x^{\mu N} P(x^\mu)$ where $P(x^\mu)$ is finite at $x = 0$.

If, now, the coefficients c are given in terms of the quantities d thus chosen by the relation

$$\sum_{m=0}^{M+N} d_m (1-x)^m = \sum_{m=0}^{M+N} c_m x^m,$$

we see that the integral in the equality (2) is convergent at $x = 1$ if

$$R(\mu s + \mu + M) > 0,$$

and is convergent at $x = 0$ if

$$R(\mu - \mu \rho_r + \mu N - \mu s) > 0.$$

It is therefore convergent at both limits, provided the conditions (1) hold good.

Let I denote the integral in the equality (2) and suppose that L is a large positive integer $> M + N$, and let $c_n = 0$ when $n > M + N$. Then

$$I = \sum_{n=0}^L \left\{ \prod_{t=1}^q \frac{\Gamma(a_t + n)}{\Gamma(b_t + n)} - \sum_{\tau=0}^R \frac{f_\tau(b, a)}{n^{\theta+\tau}} - c_n \right\} \frac{\Gamma(\mu n + \mu - \mu \rho_r - \mu s)}{\Gamma(\mu n + 2\mu - \mu \rho_r)} + J_L$$

where $J_L = \frac{1}{\Gamma(\mu s + \mu)} \int_0^1 \left(\frac{1-x}{x} \right)^{\mu s + \mu - 1} x^{2\mu - 2 - \mu \rho_r} \left\{ \sum_{n=L}^{\infty} \frac{J_R(b, a, n)}{n^{\theta+R}} x^{\mu n} \right\} dx,$

and the double accent denotes that the summation $\sum_{r=0}^R$ does not exist when $x = 0$.

Now we may take L so large that $|J_R(b, a, n)| < \epsilon$ when $n \geq L$. The final integral J_L is then such that

$$|J_L| < \frac{\epsilon}{|\Gamma(\mu s + \mu)|} \int_0^1 (1-x)^{R(\mu s) + \mu - 1} x^{R(\mu - 1 - \mu \rho_r - \mu s + \mu L)} \sum_{n=0}^{\infty} \frac{x^{\mu n}}{(L+n)^{\mu+1}} dx.$$

Thus $|J_L|$ can be made as small as we please by taking L sufficiently large, provided

$$R(\mu - \mu \rho_r + \mu L) > R(\mu s) > -\mu.$$

If, then, $R(\mu - \mu \rho_r + \mu N) > R(\mu s) > -\mu$, we have the equality

$$I = \sum_{n=0}^{\infty} \left\{ \prod_{t=1}^q \frac{\Gamma(a_t + n)}{\Gamma(b_t + n)} - \sum_{\tau=0}^R \frac{f_{\tau}(b, a)}{n^{\theta+\tau}} - c_n \right\} \frac{\Gamma(\mu n + \mu - \mu \rho_r - \mu s)}{\Gamma(\mu n + 2\mu - \mu \rho_r)}.$$

But under the wider conditions (1) of the enunciation I is an analytic function, and so also is the series just written.

If, then, $\sum_{n=1}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu) n^{\theta+\tau}}$ denotes the function $T_{\tau}(\mu s)$, which is expressible in the form of the series when $R(\theta + \tau + \mu s + \mu) > 0$, the equality (2) will hold within the assigned range (1).

We thus have the theorem.

42. We can now shew that, if $R(s)$ be finite and if $s = u + iv$,

$$|U_{\tau}(s)| \exp \{(\pi - \epsilon) |v|\}$$

tends to zero as $|v|$ tends to infinity if $\epsilon > 0$ and μ is an integer > 1 .

Take the equality (2) established in the previous paragraph. If we employ the transformation of § 32 and take the integral along a suitable circular arc, we can shew that its modulus is less than

$$\frac{1}{|\Gamma(\mu s + \mu)|} \exp \left\{ -\left(\frac{1}{2}\pi + \pi/\mu - \epsilon\right) |\mu v| \right\} K$$

when $|v|$ is large, where K is finite and independent of v , and

$$R(\mu - \mu \rho_r + \mu N) > \mu u > -M - \mu.$$

The modulus of the integral is therefore less than

$$\exp \{(-\pi + \epsilon) |v|\}$$

if $\epsilon > 0$ and $|v|$ is sufficiently large.

Again, by the theorem already proved in § 36, when u is finite

$$|T_r(\mu s)| \exp \{(\pi - \epsilon) |v|\},$$

where $\epsilon > 0$, can be made as small as we please by taking $|v|$ sufficiently large.

Again, each term of the final series in the equality (2) of the previous paragraph has a modulus less than

$$\exp \{(-\mu\pi + \epsilon) |v|\}$$

where $|v|$ is sufficiently large.

Hence, since μ is an integer > 1 , $|U_r(s)| \exp \{(\pi - \epsilon) |v|\}$ can be made as small as we please by taking $|v|$ sufficiently large, if $\epsilon > 0$.

We thus have the given theorem, since M and N may have any finite integral values as large as we please.

43. We can now shew that, if ${}_pS_q(s)$ be the function defined when

$$R(s) > R\{\tfrac{1}{2}(\mu - 1) + \Sigma \alpha - \Sigma \rho\} / \mu$$

by the series

$$\sum_{t=0}^{\infty} \frac{\Gamma(-s+t/\mu) \prod_{r=1}^q \Gamma(1-\rho_r+t/\mu-s)}{\prod_{r=1}^{\mu} \Gamma\left(\frac{t+r}{\mu}\right) \prod_{r=1}^p \Gamma(1-\alpha_r+t/\mu-s)},$$

then, for all finite values of u , $|{}_pS_q(s)| \exp \{[(\mu+1)\pi - \epsilon] |v|\}$ can be made as small as we please by taking $|v|$ sufficiently large.

For we have seen in § 29 that $\sin \pi \mu s {}_pS_q(s)$ may be written

$$\sum_{r=1}^q A_r \mu^{\mu s} U_r(s)$$

where A_r is a numerical coefficient independent of s . Also, by the previous paragraph, $|U_r(s)| \exp \{(\pi - \epsilon) |v|\}$ can be made as small as we please, if u be finite, by taking $|v|$ sufficiently large. The same is therefore true of

$$|\sin \pi \mu s {}_pS_q(s)| \exp \{(\pi - \epsilon) |v|\}.$$

We thus have the given theorem.

44. It is now evident from the previous investigations that ${}_pS_q(s)$ is a one-valued function which can be continued over the whole of the finite portion of the s -plane.

We will now prove that it has simple poles at the points

$$s = [\Sigma a - \Sigma \rho + \frac{1}{2}(\mu - 1)]/\mu - n/\mu \quad (n = 0, 1, 2, \dots, \infty)$$

and no other finite singularities except those of $\Gamma(-\mu s)$ and

$$\Gamma(\mu - \mu \rho_r - \mu s) \quad (r = 1, 2, \dots, q).$$

We have seen that, if $R(\mu s + \mu + \Sigma b - \Sigma a) > 1$,

$$\begin{aligned} U_r(s) &= \sum_{n=0}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu)} \prod_{t=1}^q \frac{\Gamma({}_r a_t + n)}{\Gamma({}_r b_t + n)} \\ &= \sum_{n=0}^{\infty} {}_r V_n(s) \quad (\text{say}). \end{aligned}$$

Also (§ 29) the constants ${}_r a_t$ and ${}_r b_t$ are such that

$$\begin{aligned} {}_r a_t &= 2 + t/\mu - \rho_r \quad (t = 1, 2, \mu - 1), \\ &= 1 + a_t - \rho_r \quad (t = \mu, \mu + 1, \dots, q), \\ {}_r b_t &= 1 + \rho_t - \rho_r \quad (t = 1, 2, \dots, q). \end{aligned}$$

And therefore the condition under which $U_r(s)$ can be represented by the series is equivalent to

$$R[\mu s + \Sigma \rho - \Sigma a - \frac{1}{2}(\mu - 3)] > 1.$$

Now, when n is large, we know that we have an asymptotic expansion of the type

$${}_r V_n(s) = \sum_{t=0}^T \frac{c_t(s)}{n^{\mu s + \mu + \Sigma b - \Sigma a + t}} + \frac{J_T(n)}{n^{\mu s + \mu + \Sigma b - \Sigma a + T}},$$

where, for all finite values of T , $|J_T(n)|$ can be made as small as we please, provided n is greater than an assignable number N . The coefficients $c_t(s)$ are polynomials in s , and can be calculated from the known asymptotic expansion for $\frac{\Gamma(a+n)}{\Gamma(b+n)}$ by sufficient labour.

We see therefore, as in § 37, that the function

$$U_r(s) - \sum_{t=0}^T c_t(s) \xi(\mu s + \mu + \Sigma b - \Sigma a + t)$$

has no finite singularities except those of $\Gamma(\mu n + \mu - \mu \rho_r - \mu s)$ when

$$R(\mu s + \mu + \Sigma b - \Sigma a) > 1 - T.$$

Hence the sole finite singularities of $U_r(s)/\Gamma(\mu-\mu\rho_r-\mu s)$ are simple poles at the points

$$s = (1-\mu+\Sigma a-\Sigma b)/\mu-n/\mu \quad (n = 0, 1, 2, \dots, \infty),$$

$$i.e., \quad s = [\Sigma a - \Sigma \rho + \tfrac{1}{2}(\mu-1)]/\mu - n/\mu.$$

Now we have seen that

$${}_pS_q(s) \sin \pi \mu s = \sum_{r=1}^q A_r \mu^{\mu s} U_r(s),$$

where the quantities A are independent of s .

Hence all possible finite singularities of ${}_pS_q(s)$ are included in

- (1) simple poles at $s = n/\mu \quad (n = 0, 1, 2, \dots, \infty),$
- (2) „ $s = n/\mu - \rho_r + 1 \quad (n = 0, 1, 2, \dots, \infty; r = 1, 2, \dots, q),$
- (3) „ $s = -n/\mu + [\Sigma a - \Sigma \rho + \tfrac{1}{2}(\mu-1)]/\mu$
 $(n = 0, 1, 2, \dots, \infty).$

We thus have the given theorem, and we see that the residues at the last system of poles can be calculated with sufficient labour by use of the asymptotic expansion which gives $\frac{\Gamma(a+n)}{\Gamma(b+n)}$, when n is large.

45. We proceed now to show that, if $s = u + iv$, and $|v|$ be finite or infinite,

$$|{}_pS_q(s)| < \eta_u \exp \{ [-(\mu+1)\pi + \epsilon] |v| \},$$

if $\epsilon > 0$, where $\eta_u e^{ku}$ ($k > 0$ but finite), can be made as small as we please by taking u sufficiently large and positive if s be not in the immediate vicinity of one of the poles of ${}_pS_q(s)$.

We can always choose the coefficients c so that the function

$$\Psi(x) \equiv {}_qF_{q-1}\{x^\mu\} - \sum_{\tau=0}^K x^\mu f_\tau(b, a) g_{\theta+\tau}(x^\mu, 1) - \sum_{m=0}^{N-1} c_m x^{\mu m} = x^{\mu N} P(x^\mu),$$

where $P(x^\mu)$ is finite at $x = 0$. Choose N so large that

$$R(\mu - \mu\rho_r - \mu s - \mu R + \mu N) > 0$$

and let $T_\tau(\mu s)$ be the function defined when

$$R(\theta + \tau + \mu s + \mu) > 0,$$

by the series
$$\sum_{n=1}^{\infty} \frac{\Gamma(\mu n - \mu \rho_r + \mu - \mu s)}{\Gamma(\mu n - \mu \rho_r + 2\mu)} n^{\theta + \tau}.$$

Then, as in § 41, we have, if R be very large and positive,

$$\begin{aligned} & \frac{1}{\Gamma(\mu s + \mu R + \mu)} \int_0^1 \left(\frac{1-x}{x} \right)^{\mu s + \mu - 1} (1-x)^{\mu R} x^{2\mu - 2 - \mu \rho_r - \mu R} \Psi(x) dx \\ &= U_r(s+R) - \sum_{\tau=0}^K f_{\tau}(b, a) T_{\tau}(\mu s + \mu R) - \sum_{m=0}^{N-1} c_m \frac{\Gamma(\mu m + \mu - \mu s - \mu R - \mu \rho_r)}{\Gamma(\mu m + 2\mu - \mu \rho_r)}. \end{aligned}$$

When R is very large, and whether $|v|$ be large or not, the modulus of the integral is less than

$$\frac{1}{|\Gamma(\mu s + \mu R + \mu)|} \exp \left\{ -\left(\frac{1}{2}\pi + \pi/\mu - \epsilon\right) |\mu v| \right\} K' \mu_1^{\mu R}$$

where K' is finite and independent of R and $|v|$, and μ_1 is the maximum value of $|1-x|$ on the circular arc which, as in § 32, we take to be the modified form of the contour of integration.

Thus, if $s+R$ be not in the immediate vicinity of one of the poles of $U_r(s)$, we have

$$|U_r(s+R)| < {}_1\eta_R \exp \{(-\pi + \epsilon) |v|\},$$

where ${}_1\eta_R e^{kR}$ can be made as small as we please by taking the positive quantity R sufficiently large.

Now we have seen that

$$\sin \pi \mu s {}_pS_q(s) = \sum_{r=1}^q A_r \mu^{\mu s} U_r(s),$$

where the quantities A are independent of s .

Hence a similar inequality is true of $|{}_pS_q(s+R)|$, and we have the given theorem.

46. Let Φ denote the linear combination of hypergeometric integral functions

$$\begin{aligned} & \frac{\prod_{r=1}^q \Gamma(1-\rho_r)}{\prod_{r=1}^p \Gamma(1-\alpha_r)} {}_pF_q \{a_1, \dots, a_p; \rho_1, \dots, \rho_q; (-)^{\mu} x\} \\ & + \sum_{r=1}^q x^{1-\rho_r} \Gamma(\rho_r-1) \frac{\prod_{t=1}^q \Gamma(\rho_r-\rho_t)}{\prod_{t=1}^p \Gamma(\rho_r-\alpha_t)} {}_pF_q \{1+a_1-\rho_r, \dots, 1+a_p-\rho_r; \\ & \quad 2-\rho_r, \dots, \rho_q-\rho_r+1, (-)^{\mu} x\} \end{aligned}$$

wherein $\mu = q+1-p$ and $q > p$.

We proceed to shew that

$$\Phi = \exp \{ -\mu x^{1/\mu} \} (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} I$$

where

$$I = -\frac{1}{2\pi i} \int_Q {}_pS_q(s) x^s ds,$$

the integral being taken round a contour Q which embraces the positive half of the real axis and encloses the poles

$$n/\mu, \quad n/\mu - \rho_r + 1 \quad (n = 0, 1, 2, \dots, \infty; \quad r = 1, 2, \dots, q),$$

but not the poles $-n/\mu + [\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu - 1)]/\mu$

of the subject of integration.

The integral is convergent by the result of § 45 for all values of $|x|$. Let T be the greatest integer such that

$$T/\mu + n \leq N$$

and let T_r be the greatest integer such that

$$T_r/\mu + 1 - \rho_r + n \leq N.$$

Then, by Cauchy's theory of residues,

$$\begin{aligned} I = & \sum_{n=0}^N \frac{(-)^n}{n!} \sum_{t=0}^T \frac{\prod_{r=1}^q \Gamma(1 - \rho_r - n) x^{t/\mu + n}}{\prod_{r=1}^p \Gamma(1 - \alpha_r - n) \prod_{r=1}^{\mu} \Gamma\left(\frac{t+r}{\mu}\right)} \\ & + \sum_{r=1}^q \sum_{n=0}^N \frac{(-)^n}{n!} \sum_{t=0}^{T_r} \frac{\Gamma(\rho_r - 1 - n) \prod_{m=1}^q \Gamma(\rho_r - \rho_m - n)}{\prod_{m=1}^{\mu} \Gamma\left(\frac{t+m}{\mu}\right) \prod_{m=1}^p \Gamma(\rho_r - \alpha_m - n)} x^{t/\mu + 1 - \rho_r + n} + I_N. \end{aligned}$$

I_N denotes the integral $-\frac{1}{2\pi i} \int {}_pS_q(s) x^s ds$ taken round a contour on which $R(s) > N$ and which embraces the real axis and encloses poles of ${}_pS_q(s)$ whose residues have not been included in the previous summations.

By the theorem of the previous paragraph it is evident that I_N can be made as small as we please by taking N sufficiently large.

Now, by the multiplication formula for the gamma function,

$$\prod_{q=1}^{\mu} \Gamma(t/\mu + q/\mu) = (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-t-\frac{1}{2}} \Gamma(t+1).$$

Hence, on making N infinite, we have

$$I = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\prod_{r=1}^q \Gamma(1-\rho_r-n)}{\prod_{r=1}^p \Gamma(1-\alpha_r-n)} \frac{\mu^{\frac{1}{2}} x^n}{(2\pi)^{\frac{1}{2}(\mu-1)}} \sum_{t=0}^{\infty} \frac{x^{t/\mu} \mu^t}{\Gamma(t+1)} \\ + \sum_{r=1}^q \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\Gamma(\rho_r-1-n)}{\prod_{m=1}^p \Gamma(\rho_r-\alpha_m-n)} \frac{\prod_{m=1}^q \Gamma(\rho_r-\rho_m-n)}{(2\pi)^{\frac{1}{2}(\mu-1)}} \frac{\mu^{\frac{1}{2}} x^{1-\rho_r+n}}{\Gamma(t+1)} \sum_{t=0}^{\infty} \frac{x^{t/\mu} \mu^t}{\Gamma(t+1)}.$$

Thus

$$I \exp \left\{ -\mu x^{1/\mu} \right\} \mu^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}(\mu-1)} \\ = \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\prod_{r=1}^q \Gamma(1-\rho_r-n)}{\prod_{r=1}^p \Gamma(1-\alpha_r-n)} x^n \\ + \sum_{r=1}^q x^{1-\rho_r} \sum_{n=0}^{\infty} \frac{(-)^n}{n!} \frac{\Gamma(\rho_r-1-n)}{\prod_{m=1}^p \Gamma(\rho_r-\alpha_m-n)} \frac{\prod_{m=1}^q \Gamma(\rho_r-\rho_m-n)}{(2\pi)^{\frac{1}{2}(\mu-1)}} x^n = \Phi.$$

We thus have the given theorem.

47: We can now shew that, if μ be an integer > 1 and if

$$|\arg x| < (\mu+1)\pi,$$

the linear combination of hypergeometric integral functions which has been denoted by Φ admits the asymptotic expansion

$$\exp \left\{ -\mu x^{1/\mu} \right\} (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} x^{[\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu-1)]/\mu} \left\{ \sum_{n=0}^N \frac{\lambda_n}{x^{n/\mu}} + \frac{J_N}{x^{N/\mu}} \right\}$$

where the quantities λ are definite functions of the parameters α and ρ and $|J_N|$ can be made as small as we please by taking $|x|$ sufficiently large.

We have seen that, for all finite values of $|\mu|$, Φ is equal to

$$\exp \left\{ -\mu x^{1/\mu} \right\} (2\pi)^{\frac{1}{2}(\mu-1)} \mu^{-\frac{1}{2}} I$$

where I is the integral $-\frac{1}{2\pi i} \int_Q {}_pS_q(s) x^s ds$ considered in the previous paragraph.

Also, it has been established in §§ 43 and 45 that, if $|\arg x| < (\mu+1)\pi$, $|{}_pS_q(s) x^s|$ will tend exponentially to zero as $|s|$ tends to infinity if $R(s)$ is greater than a finite negative quantity.

Let L be a contour parallel to the real axis which passes between the points

$$[\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu-1)]/\mu - N/\mu \quad \text{and} \quad [\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu-1)]/\mu - (N+1)/\mu.$$

Then I is equal to $-\frac{1}{2\pi i} \int_L {}_pS_q(s) x^s ds$ together with the sum of the residues of ${}_pS_q(s) x^s$ at the points

$$[\Sigma \alpha - \Sigma \rho + \tfrac{1}{2}(\mu - 1)]/\mu - n/\mu \quad (n = 0, 1, 2, \dots, N).$$

Let λ_n be the typical residue of ${}_pS_q(s)$ at such a pole: we have seen that λ_n is a function of the parameters α and ρ which can be determined with sufficient labour.

Further,

$$-\frac{1}{2\pi i} \int_L {}_pS_q(s) x^s ds = x^{[\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu - 1)]/\mu - N/\mu} J_N,$$

and it is evident that $|J_N|$ can be made as small as we please by taking $|x|$ sufficiently large.

We thus have the given theorem.

48. We can now shew that the previous theorem is equivalent to $(\mu + 1)$ different results.

For the asymptotic equality is valid if $|\arg x| < (\mu + 1)\pi$. Thus it is valid for the $(\mu + 1)$ ranges

$$\begin{aligned} (\mu - 1)\pi &< \arg x < (\mu + 1)\pi, \\ (\mu - 1 - 2)\pi &< \arg x < (\mu + 1 - 2)\pi, \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ (\mu - 1 - 2\mu)\pi &< \arg x < (\mu + 1 - 2\mu)\pi. \end{aligned}$$

We therefore have, if $m = 0, 1, 2, \dots, \mu$, and $|\arg x| < \pi$, the asymptotic equalities

$$\begin{aligned} &\frac{\prod_{r=1}^q \Gamma(1 - \rho_r)}{\prod_{r=1}^p \Gamma(1 - \alpha_r)} {}_pF_q \{ \alpha; \rho; (-)^{\mu} x \} \\ &+ \sum_{r=1}^q e^{(\mu - 2m)(1 - \rho_r)\pi i} x^{1 - \rho_r} \frac{\Gamma(\rho_r - 1) \prod_{t=1}^q \Gamma(\rho_r - \rho_t)}{\prod_{t=1}^p \Gamma(\rho_r - \alpha_t)} \\ &\times {}_pF_q \{ 1 + \alpha - \rho_r; 2 - \rho_r, \dots, \rho_q - \rho_r + 1; (-)^{\mu} x \} \\ &= \exp \left\{ -\mu e^{(\mu - 2m)\pi i/\mu} x^{1/\mu} \right\} (2\pi)^{\frac{1}{2}(\mu - 1)} \mu^{-\frac{1}{2}} x^{\theta/\mu} e^{(\mu - 2m)\theta\pi i/\mu} \sum_{n=0}^{\infty} \frac{\lambda_n e^{n(2m - \mu)\pi i/\mu}}{x^{n/\mu}}, \end{aligned}$$

where $\theta = \Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu - 1).$

Only μ of the $(\mu+1)$ results here given are independent: those corresponding to $m=0$ and $m=\mu$ can be deduced from one another by means of other asymptotic expansions of the hypergeometric functions.

For brevity write, as in § 1,

$$Q_0(x) = \frac{\prod_{r=1}^q \Gamma(1-\rho_r)}{\prod_{r=1}^p \Gamma(1-\alpha_r)} {}_pF_q \left\{ \alpha_1, \dots, \alpha_p; \rho_1, \dots, \rho_q; (-)^{\mu} x \right\},$$

$$Q_r(x) = x^{1-\rho_r} \frac{\Gamma(\rho_r-1) \prod_{i=1}^q \Gamma(\rho_r-\rho_i)}{\prod_{i=1}^p \Gamma(\rho_r-\alpha_i)} \\ \times {}_pF_q \left\{ 1+\alpha_1-\rho_r, \dots, 1+\alpha_p-\rho_r; 2-\rho_r, \dots; \rho_q-\rho_r+1; (-)^{\mu} x \right\}.$$

Then we have, if $|\arg x| < \pi$,

$$Q_0(x) + \sum_{r=1}^q e^{(\mu-2m)(1-\rho_r)\pi i} Q_r(x) \\ = \exp \left\{ -\mu e^{(\mu-2m)\pi i/\mu} x^{1/\mu} \right\} \frac{(2\pi)^{\frac{1}{2}(\mu-1)}}{\mu^{\frac{1}{2}}} x^{\theta/\mu} e^{(\mu-2m)\theta\pi i/\mu} \sum_{n=0}^{\infty} \frac{\lambda_n e^{n(2m-\mu)\pi i/\mu}}{x^{n/\mu}}, \quad (\text{A})$$

when $m = 0, 1, \dots, \mu$.

Also, by the result of § 7, if $|\arg(-x)| < \frac{1}{2}\mu\pi + \pi$,

$$\frac{1}{\sin \pi \alpha_m} Q_0(x) + \sum_{r=1}^q \frac{1}{\sin \pi (\rho_r - \alpha_m)} e^{\pi i (1-\rho_r)} Q_r(x) \\ = \frac{\Gamma(\alpha_m) \prod_{r=1}^q \Gamma(\alpha_m - \rho_r + 1)}{\pi \prod_{r=1}^p \Gamma(1 + \alpha_m - \alpha_r)} x^{-\alpha_m} e^{-\pi i \alpha_m} \\ \times {}_{q+1}F_{p-1} \left\{ \alpha_m, \alpha_m + 1 - \rho_1, \dots, \alpha_m + 1 - \rho_q; \alpha_m - \alpha_1 + 1, \dots, \alpha_m - \alpha_p + 1; \frac{1}{x} \right\}, \quad (\text{B})$$

when $m = 1, 2, \dots, p$.

Now, in the relations (A) take $m = 0$ and $m = \mu$. We have

$$Q_0(x) + \sum_{r=1}^q e^{-\mu(\rho_r-1)\pi i} Q_r(x) = \exp[\mu x^{1/\mu}] x^{\theta/\mu} e^{\theta\pi i} \Theta,$$

$$Q_0(x) + \sum_{r=1}^q e^{\mu(\rho_r-1)\pi i} Q_r(x) = \exp[\mu x^{1/\mu}] x^{\theta/\mu} e^{-\theta\pi i} \Theta,$$

where Θ denotes the asymptotic series

$$\frac{(2\pi)^{\frac{1}{2}(\mu-1)}}{\mu^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\lambda_n (-)^n}{x^{n/\mu}}.$$

From these we deduce

$$Q_0(x) \sin \pi \theta + \sum_{r=1}^q Q_r(x) \sin \pi (\theta + \mu \rho_r - \mu) = 0.$$

This equality means that when $|\arg x| < \pi$ the particular combination of functions Q just written down admits an asymptotic expansion whose dominant term is of order less than that of $\exp \{\mu x^{1/\mu}\}$ when divided by any algebraic power of x . Thus it must be possible to form the particular combination from the q other relations contained in (A) and (B).*

Hence in (A) and (B) there are $(q+1)$ relations which are independent.

There are $(q+1)$ functions Q which, when x is replaced by $(-)^{\mu} x$, are independent solutions of the equation

$$\left[(\mathfrak{S} + a_1) \dots (\mathfrak{S} + a_p) - \frac{d}{dx} (\mathfrak{S} + \rho_1 - 1) \dots (\mathfrak{S} + \rho_q - 1) \right] y = 0 \quad (1)$$

valid over the whole of the finite part of the plane.

There are similarly $(q+1)$ asymptotic solutions valid near infinity. These are, with the same transformation, given by the expressions on the right-hand sides of equations (A) ($m = 1, 2, \dots, \mu$) and (B). The relations between these solutions are given by the equations (A) and (B).

49. But the equations (A) and (B) of the previous paragraph only express the $(q+1)$ principal asymptotic solutions near $|x| = \infty$ of the differential equation (1), transformed by writing $(-)^{\mu} x$ for x , in terms of linear combinations of the $(q+1)$ hypergeometric integral functions $Q_r(x)$ ($r = 0, 1, 2, \dots, q$), which are principal solutions in the finite part of the plane.

We have conversely to express each of the functions $Q_r(x)$ in terms of

* The same phenomenon occurs in the simple case $p = 0$, $q = 1$, $\mu = 2$, and was discussed in detail in the author's earlier paper. *Loc. cit.*, § 1, Paper (η), Part II.

linear combinations of the asymptotic expansions. In § 11 it has been shewn that it is possible to express a suitable linear combination of μ , and not less than μ , functions $Q_r(x)$ in terms of the p asymptotic solutions whose dominant terms are algebraic. The expression of a linear combination of any number less than μ of the functions $Q_r(x)$ will involve asymptotic expansions whose dominant term is exponentially infinite, and such series will entirely overshadow the other series which were obtained in Part I.

Evidently *as typical of the general converse problem we may find the asymptotic expansion of* ${}_pF_q \{a; \rho; x\}$ *when* $|\arg x| < \pi$.

For this purpose we consider the integral

$$-\frac{1}{2\pi i} \int U(s) x^s ds, \quad (1)$$

where $U(s)$ is the function of s defined, when

$$R(s) > R\{\Sigma a - \Sigma \rho + \frac{1}{2}(\mu - 1)\} / \mu,$$

by the convergent series of gamma functions

$$\sum_{n=0}^{\infty} \frac{\prod_{r=1}^p \{\Gamma(a_r + n)\} \Gamma(\mu n - \mu s)}{\prod_{r=1}^q \{\Gamma(\rho_r + n)\} \Gamma(n+1)} \mu^{\mu s - \mu n + 1}.$$

By the multiplication formula for the gamma function

$$\Gamma(\mu n - \mu s) = \frac{\mu^{\mu n - \mu s - \frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu - 1)}} \prod_{r=0}^{\mu-1} \Gamma(n - s + r/\mu).$$

Hence the above series for $U(s)$ may be written

$$U(s) = \frac{\mu^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}(\mu-1)}} \sum_{n=0}^{\infty} \frac{\prod_{r=1}^p [\Gamma(a_r + n)] \prod_{r=0}^{\mu-1} \Gamma(n - s + r/\mu)}{\Gamma(n+1) \prod_{r=1}^q (\rho_r + n)}.$$

Evidently $U(s)$ is a function of the same type as the function $U_r(s)$ introduced in § 29: it possesses the same properties.

It has, besides the poles of the functions $\Gamma(\mu n - \mu s)$, as its only singularities poles at the points

$$s = [\Sigma a - \Sigma \rho + \frac{1}{2}(\mu - 1)] / \mu - r/\mu, \quad (2)$$

and the residue at the general pole we may take to be $\mu^{-\frac{1}{2}}(2\pi)^{\frac{1}{2}(1-\mu)} l_r$. The quantity l_r can be calculated with sufficient labour from the asymptotic expansion of the general term of the series for $U(s)$ when

$$R(s) > R \left\{ \Sigma \alpha - \Sigma \rho + \frac{1}{2} (\mu - 1) \right\} / \mu.$$

In particular we readily see that $l_0 = 1$.

Let now the contour C of the integral $-\frac{1}{2\pi i} \int_C U(s) x^s ds$ embrace the real axis, enclose all the poles

$$s = n + t/\mu \quad (n = 0, 1, 2, \dots, \infty; t = 0, 1, 2, \dots, \infty),$$

and enclose none of the poles (2).

Then the integral is convergent for all finite values of $|x|$. Also

$$-\frac{1}{2\pi i} \int_C \Gamma(\mu n - \mu s) \mu^{\mu s - \mu n + 1} x^s ds = \sum_{t=0}^{\infty} \frac{(-)^t \mu^t}{t!} x^{n+t/\mu} = x^n \exp \{-\mu x^{1/\mu}\}.$$

Hence the integral is equal to

$$\exp \{-\mu x^{1/\mu}\} \frac{\prod_{r=1}^p \Gamma(\alpha_r)}{\prod_{r=1}^q \Gamma(\rho_r)} {}_pF_q \{x\}.$$

Again, by the results of §§ 42 and 45 the integral may be swung back as in § 8, provided $|\arg x| < \pi$.

We thus get for its asymptotic value

$$x^{[\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu-1)]/\mu} \mu^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}(1-\mu)} \left\{ 1 + \sum_{r=1}^R \frac{l_r}{x^{r/\mu}} + \frac{J_R}{x^{R/\mu}} \right\},$$

where, for any finite value of R , $|J_R|$ tends uniformly to zero as $|x|$ tends to infinity, and the quantities l_r can be determined with sufficient labour.

We thus obtain the asymptotic equality

$$\begin{aligned} & \frac{\prod_{r=1}^p \Gamma(\alpha_r)}{\prod_{r=1}^q \Gamma(\rho_r)} {}_pF_q \{a_1, \dots, a_p; \rho_1, \dots, \rho_q; x\} \\ &= \exp \{\mu x^{1/\mu}\} x^{[\Sigma \alpha - \Sigma \rho + \frac{1}{2}(\mu-1)]/\mu} \mu^{-\frac{1}{2}} (2\pi)^{\frac{1}{2}(1-\mu)} \left\{ 1 + \sum_{r=1}^R \frac{l_r}{x^{r/\mu}} + \frac{J_R}{x^{R/\mu}} \right\}, \end{aligned}$$

valid provided $|\arg x| < \pi$.

Evidently this includes the result of Stokes stated in § 3.

When $p = 0$, $q = 1$, $\mu = 2$ we have the particular case previously

discussed by the author. In this case the coefficients l_r take a simple form, and we have* the asymptotic equality

$${}_0F_1(\rho; x) = \frac{\Gamma(\rho)}{2\pi^{\frac{1}{2}}} x^{\frac{1}{2}-\frac{1}{2}\rho} e^{2x^{\frac{1}{2}}} {}_2F_0\left\{\rho-\frac{1}{2}, \frac{3}{2}-\rho; 1/(4x^{\frac{1}{2}})\right\}.$$

50. In a similar manner we can obtain the asymptotic expansions of any of the hypergeometric integral functions which are the principal solutions of the differential equation (1) of § 48.

The theory is evidently complete.

CORRIGENDA in above paper.—

On p. 62, line 18, for " ρ_r-1 " read " $1-\rho_r$."

„ line 19, for " $e^{(\mu-2m)\pi i}$," read " $e^{(\mu-2m)\pi i/\mu}$."

„ line 20, read " $\theta = [\Sigma\alpha - \Sigma\rho + \frac{1}{2}(\mu-1)]$."

On p. 80, line 8 from foot for " $\prod_{m=1}^n \Gamma(\rho_r - \rho_m)$ " read " $\prod_{m=1}^n \Gamma(\rho_r - \rho_m)$."

* Vide § 23 of the memoir previously cited.

PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER
HAVING INTEGRAL SYSTEMS FREE FROM PARTIAL
QUADRATURES

By A. R. FORSYTH.

[Received January 1st, 1907.—Read January 10th, 1907.]

1. In his memoir* on partial differential equations and their integrals, Ampère made a selection of one aggregate of equations of the second order. The characteristic property, defining equations which are included in this aggregate, belongs to their general integrals. These integrals are to be explicitly free (or must be capable of expression in a form that is explicitly free) from partial quadratures; and they must be expressible by an equation, or by a set of equations, which occur in finite form. In the simplest cases, the integral is given by means of a single equation: more often, and in the less simple cases, the integral is given by means of three independent equations, each of which has the specified form and which, when taken together, usually may be regarded as determining the dependent variable z , and the two independent variables x and y , in terms of two parameters and two arbitrary functions of those parameters. Thus a primitive of the equation

$$r-t=2\frac{p}{x}$$

can be exhibited in the form of the single equation

$$z = \phi(y+x) + \psi(y-x) - x\{\phi'(y+x) - \psi'(y-x)\},$$

where ϕ and ψ denote arbitrary functions; it can also be exhibited in the form of the three equations

$$\left. \begin{aligned} x &= \frac{1}{2}(u-v) \\ y &= \frac{1}{2}(u+v) \\ z &= U+V-\frac{1}{2}(u-v)(U'-V') \end{aligned} \right\},$$

where U and V denote arbitrary functions of u and of v respectively.

* *Journal de l'École Polytechnique*, cah. xvii. (1815), pp. 549-611; his definition of the "first class" of equations of the second order is given on p. 558 of the memoir.

A more general form of integral, implying a more extensive aggregate of equations, is suggested by Goursat*; as defined, it is given by means of three equations expressing x, y, z in terms of two parametric variables u and v , of m arbitrary functions of u connected by $m-1$ differential equations, and of n arbitrary functions of v connected by $n-1$ differential equations.

For the purposes of the present discussion, however, it will be sufficient to take the simpler form when there is only a single arbitrary function of u and when there is only a single arbitrary function of v . The object of the discussion is to obtain some of the relations between the integral and the equivalent partial equation of the second order.

Statement of the Problem, with the various Cases that can occur.

2. Accordingly, it is assumed that the equations of the second order to be considered are those characterised by the possession of a general integral given by means of three equations

$$\left. \begin{aligned} x &= f(u, U, U_1, \dots, v, V, V_1, \dots) \\ y &= g(u, U, U_1, \dots, v, V, V_1, \dots) \\ z &= h(u, U, U_1, \dots, v, V, V_1, \dots) \end{aligned} \right\}.$$

In these equations, f, g, h denote specific functions of their arguments; U denotes an arbitrary function of u , and $U_1, \dots$ are its successive derivatives; V similarly denotes an arbitrary function of v , and $V_1, \dots$ are its successive derivatives. The integral equations are to be finite in form; so that only a finite number of derivatives of U and a finite number of derivatives of V occur. It will be assumed that U_m is the derivative of U of highest order which occurs in the three equations, and that V_n is the derivative of V of highest order which occurs; but it is not assumed (and it is not, in fact, always the case) that U_m must occur in each of the equations or that V_n must occur in each of them.[†]

* *Leçons sur l'intégration des équations aux dérivées partielles du second ordre*, t. II. (1898), p. 217.

† [In the most familiar examples of equations which have integrals of this type, such as the equation of minimal surfaces, the highest derivatives of U and of V that occur are of the second order. Transformations can be effected which make the highest derivatives appear to be of higher order; in such cases the inference is that, in an extended sense of the word, the transformed expressions are reducible. Instances of equations having integral equivalents, which are not reducible, are provided by Laplace's linear equations

$$s + ap + bq + cz = 0$$

when these are of finite rank in either variable or in both variables. An instance of an equation,

Specialised forms of integrals may arise. Thus we may have a primitive of an equation given by

$$\left. \begin{aligned} x &= f(u, U, U_1, \dots) + F(v, V, V_1, \dots) \\ y &= g(u, U, U_1, \dots) + G(v, V, V_1, \dots) \\ z &= h(u, U, U_1, \dots) + H(v, V, V_1, \dots) \end{aligned} \right\}$$

where $f, F; g, G; h, H$ are specific functions of their arguments; and a primitive of another equation might be given by

$$\left. \begin{aligned} x &= f(u, U, U_1, \dots) \\ y &= g(v, V, V_1, \dots) \\ z &= h(u, U, U_1, \dots, v, V, V_1, \dots) \end{aligned} \right\}.$$

Some illustrations of these two particular forms will be discussed.

But there are limitations upon the degeneration of the forms of the functions f, g, h as they occur initially in the general case. Thus the combinations

$$\left. \begin{aligned} x &= f(u, U, U_1, \dots) \\ y &= g(u, U, U_1, \dots) \\ z &= h(v, V, V_1, \dots) \end{aligned} \right\}, \quad \left. \begin{aligned} x &= f(u, U, U_1, \dots) \\ y &= g(v, V, V_1, \dots) \\ z &= h(u, U, U_1, \dots) \end{aligned} \right\}$$

are not permissible; at least one of the three functions f, g, h , which represent the values x, y, z , must involve both the parametric variables u and v . We shall therefore assume that h involves u and v ; if, in any set of equations, f or g (but not h) should involve u and v , a change of dependent variable can be effected whereby the expression for the new dependent variable does involve u and v .

Having made this explanation and this assumption as regards the expression for z , which is

$$z = h(u, U, U_1, \dots, v, V, V_1, \dots),$$

we see that there are three groups of cases, discriminated by the forms of f and of g in the equations

$$x = f, \quad y = g.$$

which is not linear and the integral equivalent of which involves non-reducible expressions containing derivatives of order higher than the second, is Ampère's equation

$$(r-pt)^2 = q^2rt;$$

a primitive is given by

$$\begin{aligned} x &= \frac{1}{u} V' + U''', & y &= uV' - (u^2 U'''' - 2u U''' + 2U''), \\ z &= \frac{1}{3} (2v - u^3) V' - \frac{2}{3} V + u^4 U'''' - 4u^3 U''' + 12u^2 U'' - 24u U' + 24U. \end{aligned}$$

Other instances can be obtained by taking $f(\lambda)$ and $g(\mu)$, in the equations at the end of § 26, to be polynomials not consisting of a single term alone.—*Added February 8th, 1907.*]

(I.) Each of the functions f and g involves both the quantities u and v , either with or without the arbitrary functions U and V .

Next, one of the two functions involves both the quantities u and v , either with or without the arbitrary functions U and V , while the other function involves only one of these quantities, again either with or without the associated arbitrary function. There really are four cases; but, by interchange of the variables x and y and by interchange of the parameters u and v , they can be assigned to the single case:—

(II.) The function f involves both u and v , and the function g involves v only: it being understood that the associated arbitrary functions can occur.

Lastly, one of the functions may involve only one of the parameters u and v , and the other may involve only the other of those parameters: the two possible cases can be merged into one by interchange of x and y . So we have:—

(III.) The function f involves u only, and the function g involves v only: with an understanding as to the possible occurrence of U and of V respectively.

Of these three cases to be discussed in succession, it is manifest that (I.) is the most general in form. Relative formal simplifications will occur initially in the remaining cases, though (as will appear) this simplification will not persist throughout the analysis: but the initial simplifications enable us to use the results of Case (I.) for the other two cases, and therefore to discuss the latter more briefly.

Also, it will be found that relative complications are caused in different sub-cases by conditions which, at first sight, might be deemed likely to simplify the analytical results.

A Lemma.

3. A preliminary lemma is required. Several of the results are made to depend upon the elimination of two variables ξ and η between three equations of the form

$$a\xi^2 + 2b\xi + c = 0, \quad a'\eta^2 + 2b'\eta + c' = 0,$$

$$k\xi\eta + l\xi + m\eta + n = 0;$$

and therefore an expression for the eliminant is needed. We have

$$a\xi + b = (b^2 - ac)^{\frac{1}{2}} = d, \text{ say,}$$

$$a'\eta + b' = (b'^2 - a'c')^{\frac{1}{2}} = d', \text{ say;}$$

$$\text{thus} \quad k(d-b)(d'-b') + la'(d-b) + ma(d'-b') + naa' = 0,$$

$$\text{that is,} \quad kdd' + Ld + Md' + N = 0,$$

where
$$L = la' - kb', \quad M = ma - kb,$$

$$N = naa' - mab' - lba' + kbb'.$$

The rationalised form of the last equation is

$$(N^2 - L^2 d^2 - M^2 d'^2 + k^2 d^2 d'^2)^2 - 4(kN - LM)^2 d^2 d'^2 = 0.$$

When the values of d^2 and d'^2 are substituted, and some slight reduction is effected, the eliminant becomes

$$\Theta^2 = 4(kn - lm)^2 (b^2 - ac)(b'^2 - a'c'),$$

where $\Theta = k^2 cc' - 2klcb' - 2kmbc' + 2knbb' + l^2 ca' + 2lmbb' - 2lnba'$
 $+ m^2 ac' - 2mnab' + n^2 aa'.$

The umbral expression of the eliminant is simpler: but it appears to be less useful.

A General Property, stated by Darboux.

4. As already stated, the main purpose of the investigation is the determination of differential equations, which possess integrals of the assigned type, as well as the construction of primitives of these equations. But when the equations, thus characterised by the kind of primitive possessed, have been determined, the construction of their primitive can sometimes be effected more directly by utilising a definite property: *every such equation is integrable by Darboux's method.** The property can be formally established as follows.

The equations
$$x = f, \quad y = g, \quad z = h$$

involve the variables u and v , the arbitrary function U and its derivatives up to U_m , and the arbitrary function V and its derivatives up to V_n . Hence, as

$$\frac{dh}{du} = p \frac{dx}{du} + q \frac{dy}{du},$$

$$\frac{dh}{dv} = p \frac{dx}{dv} + q \frac{dy}{dv},$$

where complete derivatives with regard to u and to v are taken, the quantities p and q will generally involve derivatives of U up to order $m+1$ and derivatives of V up to order $n+1$. Similarly, r, s, t will generally

* This result is proved in another (but equivalent) form by Goursat, *l.c.*, p. 227, being derived from the consideration of the characteristics; it was first stated by Darboux himself, *Comptes Rendus*, t. LXX. (1870), p. 748. For references to Darboux's method, and for a general discussion of the process, see chapter xviii. of Part IV. of my *Theory of Differential Equations*.

involve derivatives of U up to order $m+2$ and derivatives of V up to order $n+2$; and derivatives of z of order μ will generally involve derivatives of U up to order $m+\mu$ and derivatives of V up to order $n+\mu$.

Accordingly, construct all the derivatives of z up to order μ inclusive; when these are explicitly obtained, the total number of equations, expressing x, y, z and the derivatives of z , is

$$3+2+3+\dots+(\mu+1) = \frac{1}{2}\mu(\mu+3)+3.$$

These equations involve $u, U, U_1, \dots, U_{m+\mu}$, that is, $m+\mu+2$ quantities dependent upon u ; and they also involve $v, V, V_1, \dots, V_{n+\mu}$, that is, $n+\mu+2$ quantities dependent upon v .

If, then, μ denotes the smallest value of μ for which

$$\frac{1}{2}\mu(\mu+3)+3 > n+\mu+2,$$

that is, for which

$$\frac{1}{2}\mu(\mu+1) > n-1,$$

then generally the $n+\mu+2$ quantities $v, V, V_1, \dots, V_{n+\mu}$ can be eliminated: and we should have

$$\frac{1}{2}\mu(\mu+1)-n+1$$

equations, involving derivatives of z up to order μ , and involving also u and an arbitrary function of u with its derivatives. These equations manifestly are not derivatives of the supposed equation of the second order, because of the presence of the arbitrary function U ; hence they are equations of order μ , compatible with the equation of the second order, involving one arbitrary function, and therefore derivable by Darboux's method.

Similarly, if λ denote the smallest value of λ for which

$$\frac{1}{2}\lambda(\lambda+3)+3 > m+\lambda+2,$$

that is, for which

$$\frac{1}{2}\lambda(\lambda+1) > m-1,$$

then generally the $m+\lambda+2$ quantities $u, U, U_1, \dots, U_{m+\lambda}$ can be eliminated: and we should have

$$\frac{1}{2}\lambda(\lambda+1)-m+1$$

equations, involving derivatives of z up to order λ , and involving also v and an arbitrary function of v with its derivatives. As before, these are equations of order λ , compatible with the equation of the second order, involving one arbitrary function, and therefore derivable by Darboux's method.

But it is to be remembered that this generality is only formal. For particular equations, the result of eliminating $u, U, \dots, U_{m+\lambda}$ may lead to an equation that does not involve $v, V, \dots, V_{n+\mu}$: or some of the equations

may be evanescent : and, in such instances, it may be necessary to proceed to the construction of derivatives of order higher than the minimum which has been indicated. On the other hand, combinations of the derivatives of U and of V may disappear : then it may be unnecessary to proceed to equations of the orders indicated.

Preliminary Formulæ.

5. We proceed now to the formation of the values of the first and the second derivatives of z , denoted by p, q, r, s, t as usual : they will be constructed initially for the most general case, when

$$\left. \begin{aligned} x &= f(u, U, U_1, \dots, U_m, v, V, V_1, \dots, V_n) \\ y &= g(u, U, U_1, \dots, U_m, v, V, V_1, \dots, V_n) \\ z &= h(u, U, U_1, \dots, U_m, v, V, V_1, \dots, V_n) \end{aligned} \right\},$$

and the values for the restricted cases are then derivable by imposing the respective restrictions upon the forms of f, g, h . Complete derivatives with regard to u are indicated by means of a suffix 1 and with regard to v by means of a suffix 2 ; thus

$$\frac{df}{du} = f_1, \quad \frac{d^2f}{du^2} = f_{11}, \quad \frac{df}{dv} = f_2, \quad \frac{d^2f}{dv^2} = f_{22}, \quad \frac{d^2f}{du dv} = f_{12};$$

and so for other functions. With this notation, we have

$$\begin{aligned} h_1 du + h_2 dv &= dz \\ &= p dx + q dy \\ &= p(f_1 du + f_2 dv) + q(g_1 du + g_2 dv); \end{aligned}$$

and therefore $h_1 = pf_1 + qg_1, \quad h_2 = pf_2 + qg_2,$

so that $p = \frac{h_1 g_2 - h_2 g_1}{f_1 g_2 - f_2 g_1}, \quad q = \frac{-h_1 f_2 + h_2 f_1}{f_1 g_2 - f_2 g_1}.$

Now the quantity $f_1 g_2 - f_2 g_1$ cannot vanish : it does not vanish in virtue of the equations $x = f, y = g, z = h$; and, if it vanished identically, we should have

$$\frac{dx}{du} \frac{dy}{dv} - \frac{dx}{dv} \frac{dy}{du} = 0,$$

also identically—a relation which would mean that x and y are expressible in terms of one another by an equation otherwise involving constants alone. Similarly, the quantities $h_1 g_2 - h_2 g_1$ and $-h_1 f_2 + h_2 f_1$ do not vanish. Thus the values of p and q , as obtained, are neither zero, infinite, nor

indeterminate, though (in restricted cases) it may happen that the expressions for their values acquire simplified forms.

Again, from $h_1 = pf_1 + qg_1$, we have

$$h_{11} = pf_{11} + qg_{11} + (rf_1 + sg_1)f_1 + (sf_1 + tg_1)g_1;$$

and so for the other derivatives of the second order. Hence

$$\left. \begin{aligned} h_{11} - pf_{11} - qg_{11} &= rf_1^2 + 2sf_1g_1 + tg_1^2 \\ h_{12} - pf_{12} - qg_{12} &= rf_1f_2 + s(f_1g_2 + f_2g_1) + tg_1g_2 \\ h_{22} - pf_{22} - qg_{22} &= rf_2^2 + 2sf_2g_2 + tg_2^2 \end{aligned} \right\};$$

the individual values of r, s, t , if required, can be obtained simply by the resolution of these equations, or from the relations

$$\begin{aligned} r &= \frac{p_1g_2 - p_2g_1}{f_1g_2 - f_2g_1}, \\ s &= \frac{-p_1f_2 + p_2f_1}{f_1g_2 - f_2g_1} = \frac{q_1g_2 - q_2g_1}{f_1g_2 - f_2g_1}, \\ t &= \frac{-q_1f_2 + q_2f_1}{f_1g_2 - f_2g_1}. \end{aligned}$$

The three equations involving r, s, t , either in the form given or in equivalent forms, together with the two equations expressing p and q , and the three original equations expressing x, y , and z , are the fundamental relations for the determination of the problem. It is from among these eight relations that the quantities u, v, U and its derivatives, V and its derivatives, have to be eliminated.

6. Again, we have

$$\begin{aligned} f_1 &= \frac{\partial f}{\partial U_m} U_{m+1} + \sum_{\mu=0}^{m-1} \frac{\partial f}{\partial U_\mu} U_{\mu+1} + \frac{\partial f}{\partial u} \\ &= \frac{\partial f}{\partial U_m} U_{m+1} + F_1, \end{aligned}$$

say; and, similarly, $g_1 = \frac{\partial g}{\partial U_m} U_{m+1} + G_1$,

$$h_1 = \frac{\partial h}{\partial U_m} U_{m+1} + H_1,$$

where F_1, G_1, H_1 involve only derivatives of U and V which occur in x, y, z .

Also

$$f_2 = \frac{\partial f}{\partial V_n} V_{n+1} + \sum_{\mu=0}^{n-1} \frac{\partial f}{\partial V_\mu} V_{\mu+1} + \frac{\partial f}{\partial v}$$

$$= \frac{\partial f}{\partial V_n} V_{n+1} + F_2,$$

say; and, similarly,

$$g_2 = \frac{\partial g}{\partial V_n} V_{n+1} + G_2,$$

$$h_2 = \frac{\partial h}{\partial V_n} V_{n+1} + H_2,$$

where also F_2, G_2, H_2 involve only derivatives of U and V which occur in x, y, z . Let

$$\left. \begin{aligned} A &= \frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m} - q \frac{\partial g}{\partial U_m} \\ B &= \frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} \end{aligned} \right\};$$

then

$$\left. \begin{aligned} A U_{m+1} + H_1 - p F_1 - q G_1 &= 0 \\ B V_{n+1} + H_2 - p F_2 - q G_2 &= 0 \end{aligned} \right\}.$$

If A is different from zero, the first equation expresses U_{m+1} in terms of p, q , and of quantities that occur in x, y, z ; while, if A is zero, no such expression for U_{m+1} is derivable. Similarly, when B is different from zero, the second equation expresses V_{n+1} in terms of p, q , and of quantities that occur in x, y, z ; but, when B is zero, no such expression for V_{n+1} is derivable.

Again, we have

$$f_{11} = \frac{\partial f}{\partial U_m} U_{m+2} + \frac{\partial^2 f}{\partial U_m^2} U_{m+1}^2$$

$$+ \left\{ \frac{\partial f}{\partial U_{m-1}} + 2 \sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 f}{\partial U_\mu \partial U_m} \right) + 2 \frac{\partial^2 f}{\partial u \partial U_m} \right\} U_{m+1} + F_{11},$$

where F_{11} involves only quantities that occur in x, y, z . There are similar expressions for g_{11}, h_{11} ; hence, writing

$$a_0 = \frac{\partial^2 h}{\partial U_m^2} - p \frac{\partial^2 f}{\partial U_m^2} - q \frac{\partial^2 g}{\partial U_m^2},$$

$$2b_0 = \frac{\partial h}{\partial U_{m-1}} + 2 \sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 h}{\partial U_\mu \partial U_m} \right) + 2 \frac{\partial^2 h}{\partial u \partial U_m}$$

$$- p \left\{ \frac{\partial f}{\partial U_{m-1}} + 2 \sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 f}{\partial U_\mu \partial U_m} \right) + 2 \frac{\partial^2 f}{\partial u \partial U_m} \right\}$$

$$- q \left\{ \frac{\partial g}{\partial U_{m-1}} + 2 \sum_{\mu=0}^{m-1} \left(U_{\mu+1} \frac{\partial^2 g}{\partial U_\mu \partial U_m} \right) + 2 \frac{\partial^2 g}{\partial u \partial U_m} \right\},$$

$$c_0 = H_{11} - p F_{11} - q G_{11},$$

we have an expression for $h_{11}-pf_{11}-qg_{11}$. Similarly, writing

$$\begin{aligned} a'_0 &= \frac{\partial^2 h}{\partial V_n^2} - p \frac{\partial^2 f}{\partial V_n^2} - q \frac{\partial^2 g}{\partial V_n^2}, \\ 2b'_0 &= \frac{\partial h}{\partial V_{n-1}} + 2 \sum_{\mu=0}^{n-1} \left(V_{\mu+1} \frac{\partial^2 h}{\partial V_\mu \partial V_n} \right) + 2 \frac{\partial^2 h}{\partial v \partial V_n} \\ &\quad - p \left\{ \frac{\partial f}{\partial V_{n-1}} + 2 \sum_{\mu=0}^{n-1} \left(V_{\mu+1} \frac{\partial^2 f}{\partial V_\mu \partial V_n} \right) + 2 \frac{\partial^2 f}{\partial v \partial V_n} \right\} \\ &\quad - q \left\{ \frac{\partial g}{\partial V_{n-1}} + 2 \sum_{\mu=0}^{n-1} \left(V_{\mu+1} \frac{\partial^2 g}{\partial V_\mu \partial V_n} \right) + 2 \frac{\partial^2 g}{\partial v \partial V_n} \right\}, \\ c'_0 &= H_{22} - pF_{22} - qG_{22}, \end{aligned}$$

we have an expression for $h_{22}-pf_{22}-qg_{22}$. Again, we have

$$f_{12} = \frac{\partial^2 f}{\partial U_m \partial V_n} U_{m+1} V_{n+1} + \frac{\partial F_2}{\partial U_m} U_{m+1} + \frac{\partial F_1}{\partial V_n} V_{n+1} + F_{12},$$

where F_{12} involves only quantities that occur in x, y, z . There are similar expressions for g_{12} and h_{12} ; hence, writing

$$\begin{aligned} k_0 &= \frac{\partial^2 h}{\partial U_m \partial V_n} - p \frac{\partial^2 f}{\partial U_m \partial V_n} - q \frac{\partial^2 g}{\partial U_m \partial V_n}, \\ l_0 &= \frac{\partial H_2}{\partial U_m} - p \frac{\partial F_2}{\partial U_m} - q \frac{\partial G_2}{\partial U_m}, \\ m_0 &= \frac{\partial H_1}{\partial V_n} - p \frac{\partial F_1}{\partial V_n} - q \frac{\partial G_1}{\partial V_n}, \\ n_0 &= H_{12} - pF_{12} - qG_{12}, \end{aligned}$$

we have an expression for $h_{12}-pf_{12}-qg_{12}$. The three expressions thus obtained are

$$\left. \begin{aligned} h_{11}-pf_{11}-qg_{11} &= AU_{m+2} + a_0 U_{m+1}^2 + 2b_0 U_{m+1} + c_0 \\ h_{12}-pf_{12}-qg_{12} &= k_0 U_{m+1} V_{n+1} + l_0 U_{m+1} + m_0 V_{n+1} + n_0 \\ h_{22}-pf_{22}-qg_{22} &= BV_{n+2} + a'_0 V_{n+1}^2 + 2b'_0 V_{n+1} + c'_0 \end{aligned} \right\}.$$

Thus the quantity U_{m+2} occurs only (if at all) in the combination $h_{11}-pf_{11}-qg_{11}$; and V_{n+2} occurs only (if at all) in the combination $h_{22}-pf_{22}-qg_{22}$.

The final differential equation, in some form

$$F(x, y, z, p, q, r, s, t) = 0,$$

is to be the result of eliminating u, v, U , and V and their derivatives

among (i.) these three equations when the values of the left-hand sides in terms of r, s, t in § 5 are substituted, (ii.) the two equations giving p and q , and (ii.) the three original equations expressing x, y, z .

The various cases, depending (as in § 2) upon the forms of f, g, h , will be considered in turn.

CASE I.

7. In this case, all the three functions f, g, h involve u and v . Arbitrary functions U and V , as well as their derivatives, can occur in f, g, h .

There are four sub-cases, according to the non-evanescence or the evanescence of A and of B separately: we shall take

sub-case (1), when $A \neq 0, B \neq 0$;

„ (2), „ $A \neq 0, B = 0$;

„ (3), „ $A = 0, B \neq 0$;

„ (4), „ $A = 0, B = 0$.

These will be discussed in succession.

First Sub-case.

8. I., (1).—The equations from which the parametric quantities and the arbitrary functions are to be eliminated are

$$x = f, \quad y = g, \quad z = h;$$

$$\left. \begin{aligned} AU_{m+1} + H_1 - pF_1 - qG_1 &= 0 \\ BV_{m+1} + H_2 - pF_2 - qG_2 &= 0 \end{aligned} \right\};$$

$$AU_{m+2} + a_0 U_{m+1}^2 + 2b_0 U_{m+1} + c_0 = rf_1^2 + 2sf_1g_1 + tg_1^2,$$

$$BV_{n+2} + a'_0 V_{n+1}^2 + 2b'_0 V_{n+1} + c'_0 = rf_2^2 + 2sf_2g_2 + tg_2^2,$$

$$k_0 U_{m+1} V_{n+1} + l_0 U_{m+1} + m_0 V_{n+1} + n_0 = rf_1 f_2 + s(f_1 g_2 + f_2 g_1) + tg_1 g_2.$$

The last equation but two is the only equation which involves U_{m+2} ; it can be ignored simultaneously with U_{m+2} . Similarly, the last equation but one can be ignored simultaneously with V_{n+2} . To eliminate U_{m+1} and V_{n+1} , which occur explicitly on the left-hand side and implicitly on the right-hand side, we use the values of U_{m+1} and V_{n+1} in the forms

$$U_{m+1} = -\frac{1}{A} (H_1 - pF_1 - qG_1),$$

$$V_{n+1} = -\frac{1}{B} (H_2 - pF_2 - qG_2);$$

and the values of f_1, f, g_1, g_2 are then

$$\begin{aligned} f_1 &= U_{m+1} \frac{\partial f}{\partial U_m} + F_1 = \frac{1}{A} \left\{ F_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial f}{\partial U_m} + q \left(G_1 \frac{\partial f}{\partial U_m} - F_1 \frac{\partial g}{\partial U_m} \right) \right\}, \\ g_1 &= U_{m+1} \frac{\partial g}{\partial U_m} + G_1 = \frac{1}{A} \left\{ G_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial g}{\partial U_m} + p \left(F_1 \frac{\partial g}{\partial U_m} - G_1 \frac{\partial f}{\partial U_m} \right) \right\}, \\ f_2 &= V_{n+1} \frac{\partial f}{\partial V_n} + F_2 = \frac{1}{B} \left\{ F_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial f}{\partial V_n} + q \left(G_2 \frac{\partial f}{\partial V_n} - F_2 \frac{\partial g}{\partial V_n} \right) \right\}, \\ g_2 &= V_{n+1} \frac{\partial g}{\partial V_n} + G_2 = \frac{1}{B} \left\{ G_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial g}{\partial V_n} + p \left(F_2 \frac{\partial g}{\partial V_n} - G_2 \frac{\partial f}{\partial V_n} \right) \right\}. \end{aligned}$$

When all these values are substituted in the equation

$$k_0 U_{m+1} V_{n+1} + l_0 U_{m+1} + m_0 V_{n+1} + n_0 = r f_1 f_2 + s (f_1 g_2 + f_2 g_1) + t g_1 g_2,$$

and, when terms are collected, the new equation can (after some simple reductions) be expressed in the form

$$W = N(q^2 r - 2pq s + p^2 t) + 2M(qr - ps) + 2L(pt - qs) + rD_1 - 2sD_2 + tD_3,$$

where, if

$$\begin{aligned} \alpha &= H_1 \frac{\partial g}{\partial U_m} - G_1 \frac{\partial h}{\partial U_m} \Bigg\}, & \alpha' &= H_2 \frac{\partial g}{\partial V_n} - G_2 \frac{\partial h}{\partial V_n} \Bigg\}, \\ \beta &= F_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial f}{\partial U_m} \Bigg\}, & \beta' &= F_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial f}{\partial V_n} \Bigg\}, \\ \gamma &= G_1 \frac{\partial f}{\partial U_m} - F_1 \frac{\partial g}{\partial U_m} \Bigg\}, & \gamma' &= G_2 \frac{\partial f}{\partial V_n} - F_2 \frac{\partial g}{\partial V_n} \Bigg\}, \end{aligned}$$

the values of the coefficients in the equation are

$$\begin{aligned} D_3 &= \alpha\alpha', & 2M &= \gamma\beta' + \beta\gamma', \\ D_1 &= \beta\beta', & 2L &= \alpha\gamma' + \gamma\alpha', \\ N &= \gamma\gamma', & 2D_2 &= \beta\alpha' + \alpha\beta', \end{aligned}$$

and

$$\begin{aligned} W &= k_0(H_1 - pF_1 - qG_1)(H_2 - pF_2 - qG_2) \\ &\quad - l_0 B(H_1 - pF_1 - qG_1) - m_0 A(H_2 - pF_2 - qG_2) + n_0 AB. \end{aligned}$$

Now $4(M^2 - ND_1) = (\gamma\beta' - \beta\gamma')^2 = 4P^2, \quad \text{say,}$

$$4(L^2 - ND_3) = (\alpha\gamma' - \gamma\alpha')^2 = 4Q^2, \quad \text{say:}$$

then $4(ND_2 - LM) = (\gamma\beta' - \beta\gamma')(\alpha\gamma' - \gamma\alpha') = 4PQ.$

With these values, the equation has the form

$$\begin{aligned} \{ (Nq + M)^2 - P^2 \} r - 2 \{ (Np + L)(Nq + M) + PQ \} s + \{ (Np + L)^2 - Q^2 \} t \\ = NW. \end{aligned}$$

Note 1.—It is an immediate corollary that, if the equation

$$(Nq^2 + 2Mq + D_1)r - 2(Npq + Mp + Lq + D_2)s + (Np^2 + 2Lp + D_3)t = W$$

is to belong to Ampère's first class and has an integral possessing the preceding character with the latent limitations, a necessary condition is

$$\begin{vmatrix} N, & M, & L \\ L, & D_2, & D_3 \\ M, & D_1, & D_2 \end{vmatrix} = 0,$$

provided N is not zero: while, when N is zero, a necessary condition is

$$L^2D_1 - 2LMD_2 + M^2D_3 = 0.$$

But the equation $(1 + q^2)r - 2pq s + (1 + p^2)t = 0$

does not obey the test: the latent limitations are not satisfied.

Note 2.—The preceding result implicitly assumes that F_1, G_1, H_1 do not vanish simultaneously: likewise as to F_2, G_2, H_2 . But, if

$$\begin{aligned} F_1 &= 0, & G_1 &= 0, & H_1 &= 0; \\ F_2 &= 0, & G_2 &= 0, & H_2 &= 0, \end{aligned}$$

then

$$\left. \begin{aligned} A &= \frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m} - q \frac{\partial g}{\partial U_m} = 0 \\ B &= \frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0 \end{aligned} \right\};$$

and the assumption as to elimination is not justified.

Note 3.—The quantities $\alpha, \beta, \gamma; \alpha', \beta', \gamma'; k_0, l_0, m_0, n_0$ (except in so far as the last four involve p and q) contain only quantities which occur in the values of x, y, z : hence, if a differential equation is to emerge as the result of the elimination, the ratios of the quantities L, M, N, P, Q , and the coefficients of combinations of p and q in W , are functions of x, y, z alone.

9. Comparing the coefficients in the differential equation with their values in terms of $\alpha, \beta, \gamma; \alpha', \beta', \gamma'$, we have

$$\begin{aligned} (\alpha\beta' - \alpha'\beta)^2 &= 4(D_2^2 - D_1D_3) \\ &= \frac{4}{N^2} \{ (LM + PQ)^2 - (M^2 - P^2)(L^2 - Q^2) \} \\ &= \frac{4}{N^2} (LP + MQ)^2, \end{aligned}$$

so that

$$\alpha\beta' - \alpha'\beta = \frac{2}{N} (LP + MQ).$$

Hence
$$\alpha\beta' = D_2 + \frac{LP+MQ}{N} = \frac{(M+P)(L+Q)}{N},$$

$$\alpha'\beta = D_2 - \frac{LP+MQ}{N} = \frac{(M-P)(L-Q)}{N},$$

$$\gamma\beta' = M+P, \quad \beta\gamma' = M-P, \quad \alpha\gamma' = L+Q, \quad \alpha'\gamma = L-Q,$$

$$\gamma\gamma' = N, \quad \beta\beta' = D_1, \quad \alpha\alpha' = D_3;$$

so that

$$\left. \begin{aligned} \alpha : \beta : \gamma &= L+Q : M-P : N \\ \alpha' : \beta' : \gamma' &= L-Q : M+P : N \end{aligned} \right\}.$$

Also

$$\begin{aligned} \alpha U_{m+1} &= H_1(g_1 - G_1) - G_1(h_1 - H_1) \\ &= H_1g_1 - h_1G_1, \end{aligned}$$

$$\beta U_{m+1} = F_1h_1 - f_1H_1,$$

$$\gamma U_{m+1} = G_1f_1 - g_1F_1;$$

so that

$$\alpha f_1 + \beta g_1 + \gamma h_1 = 0,$$

that is,

$$(L+Q) \frac{dx}{du} + (M-P) \frac{dy}{du} + N \frac{dh}{du} = 0.$$

Similarly,

$$(L-Q) \frac{dx}{dv} + (M+P) \frac{dy}{dv} + N \frac{dh}{dv} = 0.$$

Again, having regard to the initial and the final form of the differential equation, we have

$$\theta f_1 f_2 = (Nq + M)^2 - P^2,$$

$$\theta(f_1 g_2 + f_2 g_1) = -2\{Np + L\}(Nq + M) + PQ\},$$

$$\theta g_1 g_2 = (Np + L)^2 - Q^2,$$

$$\theta(h_{12} - pf_{12} - qg_{12}) = NW.$$

From the first three, we have

$$\theta^2(f_1 g_2 - f_2 g_1)^2 = 4\{P(Np + L) + Q(Nq + M)\}^2,$$

that is,

$$\theta(f_1 g_2 - f_2 g_1) = +2\{LP + MQ + N(Pp + Qq)\}.$$

Hence

$$\begin{aligned} \theta f_1 g_2 &= -\{(Np + L)(Nq + M) + PQ\} + \{(Np + L)P + (Nq + M)Q\} \\ &= -(Np + L - Q)(Nq + M - P). \end{aligned}$$

Consequently,

$$\frac{f_2}{g_2} = \frac{\theta f_1 f_2}{\theta f_1 g_2} = -\frac{Nq+M+P}{Np+L-Q},$$

$$\frac{f_1}{g_1} = \frac{\theta f_1 g_2}{\theta g_1 g_2} = -\frac{Nq+M-P}{Np+L+Q};$$

and therefore

$$\frac{f_1}{Nq+M-P} = \frac{g_1}{-(Np+L+Q)} = \frac{h_1(=pf_1+qg_1)}{(M-P)p-(L+Q)q},$$

$$\frac{f_2}{Nq+M+P} = \frac{g_2}{-(Np+L-Q)} = \frac{h_2(=pf_2+qg_2)}{(M+P)p-(L-Q)q}.$$

Again, we have

$$\frac{h_{12}-pf_{12}-qg_{12}}{f_1g_2-f_2g_1} = \frac{\frac{1}{2}NW}{(Np+L)P+(Nq+M)Q},$$

which, on writing $W = [1, p, q]_3$,

because W is a non-homogeneous cubic in p and q , gives

$$\begin{vmatrix} h_{12} & f_{12} & g_{12} \\ h_1 & f_1 & g_1 \\ h_2 & f_2 & g_2 \end{vmatrix} = \frac{1}{2}N \begin{vmatrix} [f_1g_2-f_2g_1, \dots,]_3 \\ LP+MQ, & NP, & MQ \\ h_1, & f_1, & g_1 \\ h_2, & f_2, & g_2 \end{vmatrix}.$$

We thus have equations* for the determination of f, g, h : but their integration, even if possible, is complicated.

But we know that the equation, if it has an integral of the required form, is integrable by Darboux's method. For a compatible equation of any order (*e.g.*, if there is an intermediate integral), there are two systems of equations. If there are two integrals of each system, then

$$\left. \begin{aligned} \theta(x, y, z, p, q) &= u \\ \theta'(x, y, z, p, q) &= U \end{aligned} \right\}, \quad \left. \begin{aligned} \phi(x, y, z, p, q) &= v \\ \phi'(x, y, z, p, q) &= V \end{aligned} \right\}.$$

Expressing x, y, p, q in terms of z, u, v , substituting in

$$dz = (pf_1+qg_1)du + (pf_2+qg_2)dv,$$

and integrating, we find $z = h$.

And this consideration, as regards the practicability of the integration, is enough to justify the ignorance of the source of the differential equation after it has been constructed.

* Several of these equations can be deduced from those of the characteristics of the differential equation. It may be added that

$$(L \pm Q)dx + (M \mp P)dy + Ndz, \quad (Nq + M \pm P)dp - (Np + L \mp Q)dq + \frac{NWdy}{Np + L \pm Q}$$

are multiples of the exact differentials that occur in Monge's method of integration of the equation.

Second Sub-case.

10. I., (2).—In this case, each of the three functions f , g , h involves u and v : but the coefficient of V_{n+2} in Φ vanishes; that is,

$$B = \begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ f_1, & g_1, & h_1 \\ f_2, & g_2, & h_2 \end{vmatrix} = 0.$$

Taking account of the values of f_1 and f_2 in the form

$$f_1 = \frac{\partial f}{\partial U_m} U_{m+1} + F_1, \quad f_2 = \frac{\partial f}{\partial V_n} V_{n+1} + F_2,$$

with similar expressions for G_1 , G_2 , H_1 , H_2 , the relation is equivalent to the relation

$$\begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ \frac{\partial f}{\partial U_m} U_{m+1} + F_1, & \frac{\partial g}{\partial U_m} U_{m+1} + G_1, & \frac{\partial h}{\partial U_m} U_{m+1} + H_1 \\ F_2, & G_2, & H_2 \end{vmatrix} = 0.$$

This relation is not satisfied in virtue of relations $x = f$, $y = g$, $z = h$; it must therefore be satisfied identically; hence, noting that U_{m+1} occurs only through terms in the second row and even then only as a linear factor, we have

$$\begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ \frac{\partial f}{\partial U_m}, & \frac{\partial g}{\partial U_m}, & \frac{\partial h}{\partial U_m} \\ F_2, & G_2, & H_2 \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ F_1, & G_1, & H_1 \\ F_2, & G_2, & H_2 \end{vmatrix} = 0.$$

Further, remembering that the quantities

$$f_1 g_2 - f_2 g_1, \quad g_1 h_2 - g_2 h_1, \quad h_1 f_2 - h_2 f_1$$

do not vanish and that they are proportional to 1, $-p$, $-q$, the condition $B = 0$ can be written

$$\frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0.$$

Again, when we take the second expression for $B = 0$ in the form

$$\begin{vmatrix} \frac{\partial f}{\partial V_n}, & \frac{\partial g}{\partial V_n}, & \frac{\partial h}{\partial V_n} \\ f_1, & g_1, & h_1 \\ F_2, & G_2, & H_2 \end{vmatrix} = 0,$$

and when we use the relation

$$h_1 - pf_1 - qg_1 = 0,$$

which is general, and the particular relation

$$\frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0,$$

we have

$$H_2 - pF_2 - qG_2 = 0.$$

Consequently, the equation

$$BV_{n+1} + H_2 - pF_2 - qG_2 = 0$$

becomes evanescent.

It appears that there are two relations

$$\frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0,$$

$$H_2 - pF_2 - qG_2 = 0.$$

The first of these is not evanescent: the quantity V_n occurs certainly in one of the three equations $x = f$, $y = g$, $z = h$; and the relation shews that, save for trivial and negligible cases, it must occur in two of the three equations. The second of the relations might be evanescent, through the identical vanishing of F_2 , G_2 , H_2 : reserving this result for consideration, and now assuming that it is not the fact, we have another non-evanescent relation. In the latter event, two alternatives may occur: the two equations may be one and the same equation, not determining p and q ; or they may be distinct equations, so that they do determine p and q .

With the former alternative, we have

$$F_2 = \phi \frac{\partial f}{\partial V_n}, \quad G_2 = \phi \frac{\partial g}{\partial V_n}, \quad H_2 = \phi \frac{\partial h}{\partial V_n};$$

and the reserved case, when the second equation is evanescent, is included in this alternative by taking $\phi = 0$.

With the latter alternative, the values of p and q are given by the two independent equations

$$\frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0,$$

$$H_2 - pF_2 - qG_2 = 0;$$

obviously the values of p and q involve only those quantities which occur in f, g, h . But these values also must satisfy

$$AU_{m+1} + H_1 - pF_1 - qG_1 = 0;$$

and then neither A nor $H_1 - pF_1 - qG_1$, after substitution takes place for p and q , can involve quantities which do not occur in f, g, h . The derivative U_{m+1} of the arbitrary function U cannot be expressible in terms of these magnitudes; we therefore must have

$$A = \frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m} - q \frac{\partial g}{\partial U_m} = 0,$$

$$H_1 - pF_1 - qG_1 = 0,$$

in this case.

The alternatives must be considered in succession.

11. I., (2), (i.) We are to have

$$F_2 = \phi \frac{\partial f}{\partial V_n}, \quad G_2 = \phi \frac{\partial g}{\partial V_n}, \quad H_2 = \phi \frac{\partial h}{\partial V_n};$$

and the special case, in which the relation $H_2 - pF_2 - qG_2 = 0$ becomes evanescent, is covered by the value $\phi = 0$.

$$\text{Now} \quad f_2 = V_{n+1} \frac{\partial f}{\partial V_n} + \delta'f = V_{n+1} \frac{\partial f}{\partial V_n} + F_2,$$

$$\text{where} \quad \delta'f = V_n \frac{\partial f}{\partial V_{n-1}} + V_{n-1} \frac{\partial f}{\partial V_{n-2}} + \dots + \frac{\partial f}{\partial v}.$$

Hence, in the present case,

$$f_2 = (V_{n+1} + \phi) \frac{\partial f}{\partial V_n};$$

$$\text{and therefore} \quad f_{22} = \left(V_{n+2} + \frac{d\phi}{dv} \right) \frac{\partial f}{\partial V_n} + (V_{n+1} + \phi) \frac{d}{dv} \left(\frac{\partial f}{\partial V_n} \right).$$

There are corresponding expressions for g_{22} and h_{22} ; consequently

$$\begin{aligned} a'_0 V_{n+1}^2 + 2b'_0 V_{n+1} + c'_0 \\ = h_{22} - pf_{22} - qg_{22} \\ = (V_{n+1} + \phi) \left\{ \frac{d}{dv} \left(\frac{\partial h}{\partial V_n} \right) - p \frac{d}{dv} \left(\frac{\partial f}{\partial V_n} \right) - q \frac{d}{dv} \left(\frac{\partial g}{\partial V_n} \right) \right\}. \end{aligned}$$

It follows that $a'_0 \phi^2 - 2b'_0 \phi + c'_0 = 0$;

and therefore also

$$\frac{a'_0 \phi - b'_0}{1} = \frac{b'_0 \phi - c'_0}{\phi} = (b_0'^2 - a'_0 c'_0)^{\frac{1}{2}} = \Delta'_0,$$

say; and the quantity Δ'_0 is easily seen not to vanish.

For, by § 6, we have

$$\begin{aligned} a'_0 &= \frac{\partial^2 h}{\partial V_n^2} - p \frac{\partial^2 f}{\partial V_n^2} - q \frac{\partial^2 g}{\partial V_n^2}, \\ 2b'_0 &= \frac{\partial h}{\partial V_{n-1}} - p \frac{\partial f}{\partial V_{n-1}} - q \frac{\partial g}{\partial V_{n-1}} + 2 \left\{ \delta' \left(\frac{\partial h}{\partial V_n} \right) - p \delta' \left(\frac{\partial f}{\partial V_n} \right) - q \delta' \left(\frac{\partial g}{\partial V_n} \right) \right\}. \end{aligned}$$

Also, $\delta' f = \phi \frac{\partial f}{\partial V_n}$,

so that $\frac{\partial}{\partial V_n} (\delta' f) = \frac{\partial \phi}{\partial V_n} \frac{\partial f}{\partial V_n} + \phi \frac{\partial^2 f}{\partial V_n^2}$,

that is, $\frac{\partial f}{\partial V_{n-1}} + \delta' \left(\frac{\partial f}{\partial V_n} \right) = \frac{\partial \phi}{\partial V_n} \frac{\partial f}{\partial V_n} + \phi \frac{\partial^2 f}{\partial V_n^2}$,

with corresponding relations in g and h ; hence

$$\begin{aligned} a'_0 \phi &= \phi \left(\frac{\partial^2 h}{\partial V_n^2} - p \frac{\partial^2 f}{\partial V_n^2} - q \frac{\partial^2 g}{\partial V_n^2} \right) \\ &= \frac{\partial h}{\partial V_{n-1}} - p \frac{\partial f}{\partial V_{n-1}} - q \frac{\partial g}{\partial V_{n-1}} + \delta' \left(\frac{\partial h}{\partial V_n} \right) - p \delta' \left(\frac{\partial f}{\partial V_n} \right) - q \delta' \left(\frac{\partial g}{\partial V_n} \right) \\ &= b'_0 + \frac{1}{2} \left\{ \frac{\partial h}{\partial V_{n-1}} - p \frac{\partial f}{\partial V_{n-1}} - q \frac{\partial g}{\partial V_{n-1}} \right\}. \end{aligned}$$

Also, from the relation $\delta' f = \phi \frac{\partial f}{\partial V_n}$,

we have $\delta' (\delta' f) = \delta' \phi \frac{\partial f}{\partial V_n} + \phi \delta' \left(\frac{\partial f}{\partial V_n} \right)$,

with similar relations for g and h ; so that

$$\begin{aligned} c'_0 &= \delta'(\delta'h) - p\delta'(\delta'f) - q\delta'(\delta'g) \\ &= \phi \left\{ \delta' \left(\frac{\partial h}{\partial V_n} \right) - p\delta' \left(\frac{\partial f}{\partial V_n} \right) - q\delta' \left(\frac{\partial g}{\partial V_n} \right) \right\} \\ &= \phi \left\{ b'_0 - \frac{1}{2} \left(\frac{\partial h}{\partial V_{n-1}} - p \frac{\partial f}{\partial V_{n-1}} - q \frac{\partial g}{\partial V_{n-1}} \right) \right\}; \end{aligned}$$

hence we have
$$\frac{a'_0\phi - b'_0}{1} = \frac{b'_0\phi - c'_0}{\phi}$$

$$= \frac{1}{2} \left(\frac{\partial h}{\partial V_{n-1}} - p \frac{\partial f}{\partial V_{n-1}} - q \frac{\partial g}{\partial V_{n-1}} \right).$$

Consequently the value of Δ'_0 is given by

$$2\Delta'_0 = \frac{\partial h}{\partial V_{n-1}} - p \frac{\partial f}{\partial V_{n-1}} - q \frac{\partial g}{\partial V_{n-1}};$$

and there is no general reason in the form of the equations why Δ'_0 should vanish.

Again, if we write
$$f_1 = \frac{\partial f}{\partial U_m} U_{m+1} + \delta f,$$

where
$$\delta f = U_m \frac{\partial f}{\partial U_{m-1}} + \dots + U_1 \frac{\partial f}{\partial U} + \frac{\partial f}{\partial u},$$

the operators δ and δ' are permutable; that is,

$$\delta\delta' = \delta'\delta.$$

Now, in the present case,
$$\delta'f = \phi \frac{\partial f}{\partial V_n},$$

so that
$$\frac{\partial}{\partial U_m} (\delta'f) = \phi \frac{\partial^2 f}{\partial U_m \partial V_n} + \frac{\partial \phi}{\partial U_m} \frac{\partial f}{\partial V_n},$$

with similar equations for g and h ; hence

$$\begin{aligned} l_0 &= \frac{\partial}{\partial U_m} (\delta'h) - p \frac{\partial}{\partial U_m} (\delta'f) - q \frac{\partial}{\partial U_m} (\delta'g) \\ &= \phi \left\{ \frac{\partial^2 h}{\partial U_m \partial V_n} - p \frac{\partial^2 f}{\partial U_m \partial V_n} - q \frac{\partial^2 g}{\partial U_m \partial V_n} \right\} \\ &= \phi k_0. \end{aligned}$$

Similarly,

$$\begin{aligned}\delta \cdot \delta' f &= \delta \phi \frac{\partial f}{\partial V_n} + \phi \cdot \delta \left(\frac{\partial f}{\partial V_n} \right) \\ &= \delta \phi \frac{\partial f}{\partial V_n} + \phi \frac{\partial}{\partial V_n} (\delta f) \\ &= \delta \phi \frac{\partial f}{\partial V_n} + \phi \frac{\partial F_1}{\partial V_n},\end{aligned}$$

with corresponding equations for g and h ; thus

$$\begin{aligned}n_0 &= \delta \delta' h - p \delta \delta' f - q \delta \delta' g \\ &= \phi \left\{ \frac{\partial H_1}{\partial V_n} - p \frac{\partial F_1}{\partial V_n} - q \frac{\partial G_1}{\partial V_n} \right\} \\ &= \phi m_0.\end{aligned}$$

Proceeding now to construct the differential equation, we have to perform some eliminations. As A is not zero, we have

$$\begin{aligned}U_{m+1} &= -\frac{1}{A} (H_1 - pF_1 - qG_1), \\ f_1 &= -\frac{1}{A} (\beta + \gamma q), \\ g_1 &= -\frac{1}{A} (\alpha + \gamma p),\end{aligned}$$

with the notation of § 8. Also, as A does not vanish, the quantity U_{m+2} occurs only in the expression for $h_{11} - pf_{11} - qg_{11}$; consequently, the combination $rf_1^2 + 2sf_1g_1 + tg^2$ does not intervene in the elimination. Next, we have

$$\begin{aligned}h_{12} - pf_{12} - qg_{12} &= k_0 U_{m+1} V_{n+1} + l_0 U_{m+1} + m_0 V_{n+1} + n_0 \\ &= (k_0 U_{m+1} + m_0)(V_{n+1} + \phi) \\ &= \left\{ m_0 - \frac{k_0}{A} (H_1 - pF_1 - qG_1) \right\} (V_{n+1} + \phi).\end{aligned}$$

Further, $rf_1f_2 + s(f_1g_2 + f_2g_1) + tg_1g_2 = (V_{n+1} + \phi) \rho,$

where $\rho = rf_1 \frac{\partial f}{\partial V_n} + s_1 \left(f_1 \frac{\partial g}{\partial V_n} + g_1 \frac{\partial f}{\partial V_n} \right) + tg_1 \frac{\partial g}{\partial V_n};$

and therefore one of the equations to be used in the elimination becomes

$$\left\{ m_0 - \frac{k_0}{A} (H_1 - pF_1 - qG_1) - \rho \right\} (V_{n+1} + \phi) = 0.$$

Again, with the notation of § 6, and now using the condition $B = 0$, we have

$$h_{22} - pf_{22} - qg_{22} = a'_0 V_{n+1}^2 + 2b'_0 V_{n+1} + c'_0;$$

and

$$rf_2^2 + 2sf_2g_2 + tg_2^2 = (V_{n+1} + \phi)^2 \sigma,$$

where

$$\sigma = r \left(\frac{\partial f}{\partial V_n} \right)^2 + 2s \frac{\partial f}{\partial V_n} \frac{\partial g}{\partial V_n} + t \left(\frac{\partial g}{\partial V_n} \right)^2.$$

Hence the equation becomes

$$a'_0 V_{n+1}^2 + 2b'_0 V_{n+1} + c'_0 = (V_{n+1} + \phi)^2 \sigma.$$

The part $V_{n+1} + \phi = 0$ of the former equation reduces this equation to an identity, because

$$a'_0 \phi^2 - 2b'_0 \phi + c'_0 = 0.$$

Moreover, as ϕ does not involve r, s, t , or V_{n+1} , the relation $V_{n+1} + \phi = 0$ cannot hold. Hence the eliminant is merely the other part of the former equation: it is

$$\rho = m_0 - \frac{k_0}{A} (H_1 - pF_1 - qG_1),$$

which has the form

$$(\beta + \gamma g) \left(r \frac{\partial f}{\partial V_n} + s \frac{\partial g}{\partial V_n} \right) - (\alpha + \gamma p) \left(s \frac{\partial f}{\partial V_n} + t \frac{\partial g}{\partial V_n} \right) = \gamma'' + \alpha'' p + \beta'' q,$$

where the ratios of the coefficients of the various combinations of p, q, r, s, t are functions of x, y , and z only.

12. I., (2), (ii.). With the second alternative of § 10, we have*

$$A = 0, \quad B = 0;$$

the values of p and q satisfy (and are determined by) the four equations

$$\left. \begin{aligned} H_2 - pF_2 - qG_2 &= 0 \\ \frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} &= 0 \end{aligned} \right\}$$

and

$$\left. \begin{aligned} H_1 - pF_1 - qG_1 &= 0 \\ \frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m} - q \frac{\partial g}{\partial U_m} &= 0 \end{aligned} \right\}.$$

The first two of these equations are independent of one another. In the

* This is, in effect, part of the fourth sub-case of (I.), to be dealt with in § 15: as it arises now, the necessary analysis will be developed here.

second pair, the second equation is not evanescent; the first can be evanescent (in which case F_1, G_1, H_1 vanish); or, it can be effectively the same as the second (in which case

$$\frac{\partial F_1}{\partial U_m} = \theta F_1, \quad \frac{\partial G_1}{\partial U_m} = \theta G_1, \quad \frac{\partial H_1}{\partial U_m} = \theta H_1,$$

where θ involves only the quantities occurring in x, y, z); or it can be an independent equation. The last is the most comprehensive possibility among the three; it will be seen to include the other two.

To obtain the partial differential equation, we have first to eliminate U_{m+1} and V_{n+1} between the three equations

$$kU_{m+1}V_{n+1} + lU_{m+1} + mV_{n+1} + n = 0,$$

$$aU_{m+1}^2 + 2bU_{m+1} + c = 0,$$

$$a'V_{n+1}^2 + 2b'V_{n+1} + c' = 0,$$

which (by § 3) can be expressed in a form

$$\Theta^2 = 4(kn - lm)^2(b^2 - ac)(b'^2 - a'c');$$

and then the coefficients of the various combinations of r, s, t in that eliminant are functions of $u, U, U_1, \dots, U_m, v, V, V_1, \dots, V_n$: in the present case all these coefficients must be expressible in terms of x, y, z, p, q .

The expression for Θ in terms of r, s, t can be obtained by direct substitution; but the calculations are very long. They can be considerably abbreviated by the use of variables with umbral coefficients. For this purpose, we write

$$x_1, x_2 = \frac{\partial f}{\partial U_m}, \frac{\partial g}{\partial U_m} = \phi_1, \psi_1;$$

$$\xi_1, \xi_2 = F_1, G_1;$$

$$y_1, y_2 = \frac{\partial f}{\partial V_n}, \frac{\partial g}{\partial V_n} = \phi_2, \psi_2;$$

$$\eta_1, \eta_2 = F_2, G_2;$$

and

$$\begin{aligned} r &= a_1^2 = b_1^2 = c_1^2 = d_1^2 = \dots, \\ s &= a_1 a_2 = b_1 b_2 = c_1 c_2 = d_1 d_2 = \dots, \\ t &= a_2^2 = b_2^2 = c_2^2 = d_2^2 = \dots, \end{aligned}$$

the last being the umbral coefficients. Then

$$\left. \begin{aligned} a &= -a_0 + a_x^2 \\ b &= -b_0 + a_x a_\xi \\ c &= -c_0 + a_\xi^2 \end{aligned} \right\}, \quad \left. \begin{aligned} a' &= -a'_0 + a_y^2 \\ b' &= -b'_0 + a_y a_\eta \\ c' &= -c'_0 + a_\eta^2 \end{aligned} \right\},$$

$$\left. \begin{aligned} k &= -k_0 + a_x a_y \\ l &= -l_0 + a_x a_\eta \\ m &= -m_0 + a_y a_\xi \\ n &= -n_0 + a_\xi a_\eta \end{aligned} \right\}.$$

As regards the expression for Θ , we note that the interchanges

$$\left. \begin{aligned} k \\ m \end{aligned} \right\}, \quad \left. \begin{aligned} n \\ l \end{aligned} \right\}, \quad \left. \begin{aligned} c \\ a \end{aligned} \right\}, \quad \left. \begin{aligned} \phi_1 \\ F_1 \end{aligned} \right\}, \quad \left. \begin{aligned} \psi_1 \\ G_1 \end{aligned} \right\}$$

can be made simultaneously without affecting its value. Similarly, the interchanges

$$\left. \begin{aligned} k \\ l \end{aligned} \right\}, \quad \left. \begin{aligned} m \\ n \end{aligned} \right\}, \quad \left. \begin{aligned} c' \\ a' \end{aligned} \right\}, \quad \left. \begin{aligned} \phi_2 \\ F_2 \end{aligned} \right\}, \quad \left. \begin{aligned} \psi_2 \\ G_2 \end{aligned} \right\}$$

can be made simultaneously without affecting the value of Θ . Similarly, the interchanges

$$\left. \begin{aligned} a \\ a' \end{aligned} \right\}, \quad \left. \begin{aligned} b \\ b' \end{aligned} \right\}, \quad \left. \begin{aligned} c \\ c' \end{aligned} \right\}, \quad \left. \begin{aligned} l \\ m \end{aligned} \right\}, \quad \left. \begin{aligned} \phi_1 \\ \phi_2 \end{aligned} \right\}, \quad \left. \begin{aligned} \psi_1 \\ \psi_2 \end{aligned} \right\}, \quad \left. \begin{aligned} F_1 \\ F_2 \end{aligned} \right\}, \quad \left. \begin{aligned} G_1 \\ G_2 \end{aligned} \right\}$$

can also be made without affecting its value. When these properties are noted, it is possible to dispense with many of the calculations; for then many of the aggregates of terms of a particular type can be set down, when a single aggregate of that type has been actually calculated. The possibility will be utilised.

13. Then, when we adopt the customary process in the umbral calculations connected with binary forms, the terms in Θ , which are independent of $a_0, b_0, c_0, a'_0, b'_0, c'_0$, are

$$\begin{aligned} &= k^2 c_\xi^2 d_\eta^2 - 2klc_\xi^2 d_y d_\eta - 2kmc_x c_\xi d_\eta^2 + 2knc_x c_\xi d_y d_\eta + 2lmc_x c_\xi d_y d_\eta \\ &\quad + l^2 c_\xi^2 d_y^2 - 2lnc_x c_\xi d_y^2 + m^2 c_x^2 d_\eta^2 - 2mnc_x^2 d_y d_\eta + n^2 c_x^2 d_y^2 \\ &= (kc_\xi d_\eta - lc_\xi d_y - mc_x d_\eta + nc_x d_y)^2 \\ &= \mathfrak{S}^2, \end{aligned}$$

say. The part of $\mathfrak{S}$ which is independent of k_0, l_0, m_0, n_0 is

$$\begin{aligned} &= a_x a_y c_\xi \bar{d}_\eta - a_x a_\eta c_\xi \bar{d}_y - a_y a_\xi c_x \bar{d}_\eta + a_\xi a_\eta c_x \bar{d}_y \\ &= (a_x c_\xi - a_\xi c_x) (a_y \bar{d}_\eta - a_\eta \bar{d}_y) \\ &= (x^\xi)(y\eta)(ac)(a\bar{d}); \end{aligned}$$

and therefore

$$\mathfrak{S}^2 = \{ (x^\xi)(y\eta)(ac)(a\bar{d}) - (k_0 c_\xi \bar{d}_\eta - l_0 c_\xi \bar{d}_y - m_0 c_x \bar{d}_\eta + n_0 c_x \bar{d}_y) \}^2.$$

The coefficient of $(x^\xi)^2 (y\eta)^2$ in the part of $\mathfrak{S}^2$, which does not involve k_0, l_0, m_0, n_0 , is

$$= (ac)(a\bar{d})(bc)(b\bar{d});$$

or, as

$$\begin{aligned} (ac)(a\bar{d}) &= (a_1 c_2 - a_2 c_1)(a_1 \bar{d}_2 - a_2 \bar{d}_1) \\ &= r c_2 \bar{d}_2 - s(c_2 \bar{d}_1 + c_1 \bar{d}_2) + t c_1 \bar{d}_1, \end{aligned}$$

the said coefficient is

$$\begin{aligned} &= \{ r c_2 \bar{d}_2 - s(c_2 \bar{d}_1 + c_1 \bar{d}_2) + t c_1 \bar{d}_1 \}^2 \\ &= 2(rt - s^2)^2, \end{aligned}$$

on reduction: consequently, this part of $\mathfrak{S}^2$ in Θ is

$$2(rt - s^2)^2 (x^\xi)^2 (y\eta)^2.$$

Again, the coefficient of $-2k_0$ in $\mathfrak{S}^2$ is

$$= (x^\xi)(y\eta)(ac)(a\bar{d}) c_\xi \bar{d}_\eta;$$

or, since $(ac)c_\xi = (a_1 c_2 - a_2 c_1)(c_1 \xi_1 + c_2 \xi_2) = a_1(s\xi_1 + t\xi_2) - a_2(r\xi_1 + s\xi_2)$,

this coefficient is

$$\begin{aligned} (ac)c_\xi(a\bar{d})\bar{d}_\eta &= \{ a_1(s\xi_1 + t\xi_2) - a_2(r\xi_1 + s\xi_2) \} \{ a_1(s\eta_1 + t\eta_2) - a_2(r\eta_1 + s\eta_2) \} \\ &= r(s\xi_1 + t\xi_2)(s\eta_1 + t\eta_2) + t(r\xi_1 + s\xi_2)(r\eta_1 + s\eta_2) \\ &\quad - s\{ (s\xi_1 + t\xi_2)(r\eta_1 + s\eta_2) + (r\xi_1 + s\xi_2)(s\eta_1 + t\eta_2) \} \\ &= (rt - s^2) \{ rF_1 F_2 + s(F_1 G_2 + F_2 G_1) + tG_1 G_2 \}; \end{aligned}$$

hence, taking account of the interchanges which leave Θ unaltered, we have the aggregate of terms, which are linear in k_0, l_0, m_0, n_0 and do not involve $a_0, b_0, c_0, a'_0, b'_0, c'_0$, in the form

$$\begin{aligned} &-2(rt - s^2)(x^\xi)(y\eta) [k_0 \{ rF_1 F_2 + s(F_1 G_2 + F_2 G_1) + tG_1 G_2 \} \\ &\quad - l_0 \{ rF_1 \phi_2 + s(F_1 \psi_2 + G_1 \phi_2) + tG_1 \psi_2 \} \\ &\quad - m_0 \{ r\phi_1 F_2 + s(\phi_1 G_2 + \psi_1 F_2) + t\psi_1 G_2 \} \\ &\quad + n_0 \{ r\phi_1 \phi_2 + s(\phi_1 \psi_2 + \psi_1 \phi_2) + t\phi_1 \psi_2 \}] \\ &= -2(rt - s^2)(\phi_1 G_1 - \psi_1 F_1)(\phi_2 G_2 - \psi_2 F_2) \{ k_0 a_\xi a_\eta - l_0 a_\xi a_y - m_0 a_x a_\eta \\ &\quad + n_0 a_x a_y \}. \end{aligned}$$

And the part of $\mathfrak{S}^2$, which is of the second degree in k_0, l_0, m_0, n_0 , while independent of $a_0, b_0, c_0, a'_0, b'_0, c'_0$, is

$$(k_0 c_\xi d_\eta - l_0 c_\xi d_y - m_0 c_x d_\eta + n_0 c_x d_y)^2,$$

which can be expressed immediately in non-umbral forms. Hence

$$\begin{aligned} \mathfrak{S}^2 = & 2(rt-s^2)^2 (\phi_1 G_1 - \psi_1 F_1)^2 (\phi_2 G_2 - \psi_2 F_2)^2 \\ & - 2(rt-s^2) (\phi_1 G_1 - \psi_1 F_1) (\phi_2 G_2 - \psi_2 F_2) (k_0 a_\xi a_\eta - l_0 a_\xi a_y \\ & - m_0 a_x a_\eta + n_0 a_x a_y) \\ & + (k_0 c_\xi d_\eta - l_0 c_\xi d_y - m_0 c_x d_\eta + n_0 c_x d_y)^2. \end{aligned}$$

We proceed similarly with the other terms in Θ ; denoting their aggregate by ϖ , we find

$$\begin{aligned} \varpi = & -(rt-s^2) [(y\eta)^2 \{a_0 d_\xi^2 - 2b_0 d_x d_\xi + c_0 d_x^2\} \\ & + (x\xi)^2 \{a'_0 d_\eta^2 - 2b'_0 d_y d_\eta + c'_0 d_y^2\}] \\ & + 2(rt-s^2) \{(\phi_2 G_2 - \psi_2 F_2) I + (\phi_1 G_1 - \psi_1 F_1) J\} \\ & - \{a_0(n_0 a_y - m_0 a_\eta)^2 - 2(n_0 a_y - m_0 a_\eta)(l_0 a_y - k_0 a_\eta) b_0 + (l_0 a_y - k_0 a_\eta)^2 c_0\} \\ & - \{a'_0(n_0 a_x - l_0 a_\xi)^2 - 2(n_0 a_x - l_0 a_\xi)(m_0 a_x - k_0 a_\xi) b'_0 + (m_0 a_x - k_0 a_\xi)^2 c'_0\} \\ & - 2[k_0(c_0 a_x - b_0 a_\xi)(c'_0 a_y - b'_0 a_\eta) - l_0(c_0 a_x - b_0 a_\xi)(b'_0 a_y - a'_0 a_\eta) \\ & - m_0(c'_0 a_y - b'_0 a_\eta)(b_0 a_x - a_0 a_\xi) + n_0(b_0 a_x - a_0 a_\xi)(b'_0 a_y - a'_0 a_\eta)] \\ & + \{c_0 a_x b_x - b_0(a_x b_\xi + b_x a_\xi) + a_0 a_\xi b_\xi\} \{c'_0 a_y b_y - b'_0(a_y b_\eta + b_y a_\eta) + a'_0 a_\eta b_\eta\} \\ & + \Theta_0, \end{aligned}$$

where

$$\begin{aligned} I = & a_0 \{n_0(\phi_2 G_1 - \psi_2 F_1) + m_0(G_2 F_1 - F_2 G_1)\} \\ & - b_0 \{k_0(G_2 F_1 - F_2 G_1) + l_0(\phi_2 G_1 - \psi_2 F_1) + m_0(G_2 \phi_1 - F_2 \psi_1) \\ & + n_0(\phi_2 \psi_1 - \phi_1 \psi_2)\} \\ & + c_0 \{l_0(\phi_2 \psi_1 - \psi_2 \phi_1) + k_0(G_2 \phi_1 - F_2 \psi_1)\}, \\ J = & a'_0 \{n_0(\phi_1 G_2 - \psi_1 F_2) + l_0(G_1 F_2 - G_2 F_1)\} \\ & - b'_0 \{k_0(G_1 F_2 - F_1 G_2) + m_0(\phi_1 G_2 - \psi_1 F_2) + l_0(G_1 \phi_2 - F_1 \psi_2) \\ & + n_0(\phi_2 \psi_1 - \phi_1 \psi_2)\} \\ & + c'_0 \{m_0(\phi_1 \psi_2 - \phi_2 \psi_1) + k_0(G_1 \phi_2 - F_1 \psi_2)\}, \\ \Theta_0 = & k_0^2 c_0 c'_0 - 2k_0 l_0 c_0 b'_0 - 2k_0 m_0 b_0 c'_0 + 2k_0 n_0 b_0 b'_0 + 2l_0 m_0 b_0 b'_0 + l_0^2 c_0 a'_0 \\ & - 2l_0 n_0 b_0 a'_0 + m_0^2 a_0 c'_0 - 2m_0 n_0 a_0 b'_0 + n_0^2 a_0 a'_0. \end{aligned}$$

Thus $\Theta = \mathfrak{S}^2 + \varpi$.

Also

$$\begin{aligned} kn - lm &= k_0 n_0 - l_0 m_0 - (k_0 a_\xi a_\eta - l_0 a_\xi a_y - m_0 a_x a_\eta + n_0 a_x a_y) \\ &\quad + (rt - s^2)(\phi_1 G_1 - \psi_1 F_1)(\phi_2 G_2 - \psi_2 F_2), \\ b^2 - ac &= b_0^2 - a_0 c_0 + (a_0 a_\xi^2 - 2b_0 a_x a_\xi - c_0 a_x^2) - (rt - s^2)(\phi_1 G_1 - \psi_1 F_1)^2, \\ b'^2 - a'c' &= b_0'^2 - a_0' c_0' + (a_0' a_\eta^2 - 2b_0' a_y a_\eta + c_0' a_y^2) - (rt - s^2)(\phi_2 G_2 - \psi_2 F_2)^2. \end{aligned}$$

Finally, the eliminant is

$$(\mathfrak{S}^2 + \varpi)^2 = 4 (kn - lm)^2 (b^2 - ac)(b_0'^2 - a_0' c_0').$$

When expansion takes place, various sets of terms cancel: and the terms of highest and of lowest degrees in r, s, t are indicated in the equation

$$\begin{aligned} 8(rt - s^2)^3 (\phi_1 G_1 - \psi_1 F_1)^3 (\phi_2 G_2 - \psi_2 F_2)^3 \{ (\phi_2 G_2 - \psi_2 F_2) I + (\phi_1 G_1 - \psi_1 F_1) J \} \\ + \dots + \Theta_0^2 - 4(k_0 n_0 - l_0 m_0)(b_0^2 - a_0 c_0)(b_0'^2 - a_0' c_0') = 0, \end{aligned}$$

which accordingly is the form of the partial differential equation.

The temporarily omitted alternatives are the special cases of the preceding equation,

$$(\alpha) \text{ when } F_1 = G_1 = H_1 = 0;$$

$$(\beta) \text{ when } F_1 = \theta \frac{\partial F}{\partial U_m}, \quad G_1 = \theta \frac{\partial G}{\partial U_m}, \quad H_1 = \theta \frac{\partial H}{\partial U_m}.$$

Some of the special cases, when the integral system is of the form

$$x, y, z = \text{function of } u + \text{function of } v,$$

so that $h_{12} = 0$, $f_{12} = 0$, $g_{12} = 0$, and therefore $k_0 = l_0 = m_0 = n_0 = 0$, will be considered later (§ 19).

Note.—An example of the general case just considered is provided by the integral system

$$\begin{aligned} x &= f = \frac{u}{v} - V', \\ y &= g = ve^{-U'}, \\ z &= h = u + U - uU' + V - vV'. \end{aligned}$$

The quantities A and B are given by

$$A = \begin{vmatrix} 0, & -ve^{-U'}, & -u \\ \frac{1}{v}, & -ve^{-U'} U'', & 1 - uU'' \\ -\frac{u}{v^2} - V'', & e^{-U'}, & -vV'' \end{vmatrix} = 0,$$

$$B = \begin{vmatrix} -1, & 0, & -v \\ \frac{1}{v}, & -ve^{-U'} U'', & 1-uU'' \\ -\frac{u}{v^2} - V'', & e^{-U'}, & -vV'' \end{vmatrix} = 0;$$

the values of p and q are

$$p = v, \quad q = \frac{u}{v} - e^{U'};$$

and the partial differential equation of the second order, which is satisfied by the integral system, is

$$y(rt-s^2) + qr - ps = 0.$$

Third Sub-case.

14. I., (3).—In this case, we have $A = 0$ while B is not initially given as a vanishing quantity: thus

$$A = \begin{vmatrix} \frac{\partial h}{\partial U_m}, & \frac{\partial f}{\partial U_m}, & \frac{\partial g}{\partial U_m} \\ h_1, & f_1, & g_1 \\ h_2, & f_2, & g_2 \end{vmatrix} = 0.$$

The analysis follows a line of development exactly the same as in the last case; as the detailed results can be obtained by interchanging the variables u and v , they will be stated without proof.

There are three alternatives. In the first, quantities F_1 , G_1 , H_1 vanish. In the second, the equations

$$\frac{\partial h}{\partial U_m} - p \frac{\partial F}{\partial U_m} - q \frac{\partial G}{\partial U_m} = 0,$$

$$H_1 - pF_1 - qG_1 = 0$$

are effectively one and the same equation: then

$$F_1 = \theta \frac{\partial F}{\partial U_m}, \quad G_1 = \theta \frac{\partial G}{\partial U_m}, \quad H_1 = \theta \frac{\partial H}{\partial U_m},$$

and the first of the alternatives is derived from the second by making θ vanish. Clearly a non-vanishing θ involves only the same quantities as occur in x, y, z . Also $a_0\theta^2 - 2b_0\theta + c_0 = 0$,

$$\text{so that } \frac{a_0\theta - b_0}{1} = \frac{b_0\theta - c_0}{\theta} = (b_0^2 - a_0c_0)^{\frac{1}{2}}$$

$$= \Delta_0$$

$$= \frac{1}{2} \left(\frac{\partial h}{\partial U_{m-1}} - p \frac{\partial f}{\partial U_{m-1}} - q \frac{\partial g}{\partial U_{m-1}} \right).$$

Further,

$$m_0 = \theta k_0, \quad n_0 = \theta l_0;$$

and

$$V_{n-1} = -\frac{1}{B} (H_2 - pF_2 - qG_2),$$

$$f_2 = \frac{1}{B} (\beta' + \gamma'q),$$

$$g_2 = -\frac{1}{B} (\alpha' + \gamma'p),$$

with the notation of § 8. And the final differential equation, which results from the elimination, is

$$(\beta' + \gamma'q) \left(r \frac{\partial f}{\partial U_m} + s \frac{\partial g}{\partial U_m} \right) - (\alpha' + \gamma'p) \left(s \frac{\partial f}{\partial U_m} + t \frac{\partial g}{\partial U_m} \right) = \gamma''' + \alpha'''p + \beta'''q,$$

where the ratios of the coefficients of the various combinations of p , q , r , s , t are functions of x , y , and z only.

In the third alternative, the two equations

$$\frac{\partial h}{\partial U_m} - p \frac{\partial F}{\partial U_m} - q \frac{\partial G}{\partial U_m} = 0, \quad H_1 - pF_1 - qG_1 = 0$$

are independent of one another: by an argument similar to the earlier argument, we have

$$\frac{\partial h}{\partial V_n} - p \frac{\partial F}{\partial V_n} - q \frac{\partial G}{\partial V_n} = 0, \quad H_2 - pF_2 - qG_2 = 0.$$

The circumstances now are precisely the same as in the third alternative of the second case: the results are the same as before, in § 13, and need not be restated, either in general, or for the special cases, when the two new equations are one and the same, and when the second of the new equations is evanescent.

Fourth Sub-case.

15. I., (4).—In this case, we have

$$A = 0, \quad B = 0,$$

that is,

$$\left. \begin{aligned} \frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m} - q \frac{\partial g}{\partial U_m} &= 0 \\ H_1 - pF_1 - qG_1 &= 0 \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} &= 0 \\ H_2 - pF_2 - qG_2 &= 0 \end{aligned} \right\}.$$

With each pair of equations, we have the three same alternatives as before.

The two first alternatives are that the respective second equations become evanescent.

The two second alternatives are that the second equation in each

pair is effectively the same as the first equation in its own pair. In the respective cases, we have

$$\begin{aligned} F_1 &= \theta \frac{\partial f}{\partial U_m}, & G_1 &= \theta \frac{\partial g}{\partial U_m}, & H_1 &= \theta \frac{\partial h}{\partial U_m}; \\ F_2 &= \phi \frac{\partial f}{\partial V_n}, & G_2 &= \phi \frac{\partial g}{\partial V_n}, & H_2 &= \phi \frac{\partial h}{\partial V_n}. \end{aligned}$$

The vanishing of F_1, G_1, H_1 is given by a vanishing θ ; and the vanishing of F_2, G_2, H_2 by a vanishing ϕ . When the two second alternatives coexist, while θ and ϕ do not vanish, we have

$$a_0\theta^2 - 2b_0\theta + c_0 = 0, \quad a'_0\phi^2 - 2b'_0\phi + c'_0 = 0;$$

also

$$n_0 = \phi m_0 = \theta l_0 = \theta \phi k_0.$$

Again, writing

$$\begin{aligned} \rho &= r \left(\frac{\partial f}{\partial U_m} \right)^2 + 2s \frac{\partial f}{\partial U_m} \frac{\partial g}{\partial U_m} + t \left(\frac{\partial g}{\partial U_m} \right)^2, \\ \sigma &= r \frac{\partial f}{\partial U_m} \frac{\partial f}{\partial V_n} + s \left(\frac{\partial f}{\partial U_m} \frac{\partial g}{\partial V_n} + \frac{\partial f}{\partial V_n} \frac{\partial g}{\partial U_m} \right) + t \frac{\partial g}{\partial U_m} \frac{\partial g}{\partial V_n}, \\ \tau &= r \left(\frac{\partial f}{\partial V_n} \right)^2 + 2s \frac{\partial f}{\partial V_n} \frac{\partial g}{\partial V_n} + t \left(\frac{\partial g}{\partial V_n} \right)^2, \end{aligned}$$

and

$$a = -a_0 + \rho, \quad b = -b_0 + \rho\theta, \quad c = -c_0 + \rho\theta^2;$$

$$a' = -a'_0 + \tau, \quad b' = -b'_0 + \tau\phi, \quad c' = -c'_0 + \tau\phi^2;$$

$$k = -k_0 + \sigma, \quad l = -l_0 + \phi\sigma, \quad m = -m_0 + \theta\sigma, \quad n = -n_0 + \theta\phi\sigma,$$

the three equations, which lead to the partial differential equation, are

$$aU_{m+1}^2 + 2bU_{m+1} + c = 0,$$

$$a'V_{n+1}^2 + 2b'V_{n+1} + c' = 0,$$

$$kU_{m+1}V_{n+1} + lU_{m+1} + mV_{n+1} + n = 0.$$

When the values of $a, b, c; a', b', c'; k, l, m, n$ are substituted, and when the equations are re-arranged, they take the form

$$(U_{m+1} + \theta)^2 (\rho - a_0) + 2\Delta_0 (U_{m+1} + \theta) = 0,$$

$$(V_{n+1} + \phi)^2 (\tau - a'_0) + 2\Delta'_0 (V_{n+1} + \phi) = 0,$$

$$(\sigma - k_0) (U_{m+1} + \theta) (V_{n+1} + \phi) = 0,$$

where Δ_0 and Δ'_0 denote the respective quantities

$$(b_0^2 - a_0 c_0)^{\frac{1}{2}} \quad \text{and} \quad (b'^2_0 - a'_0 c'_0)^{\frac{1}{2}}.$$

Now, as θ contains only the quantities which occur in x, y, z , the relation $U_{m+1} + \theta = 0$ cannot be satisfied; similarly, $V_{n+1} + \phi$ cannot vanish.

Hence the first two equations express U_{m+1} and V_{n+1} respectively in terms of x, y, z, p, q, r, s, t ; and the partial differential equation, which survives from the third equation, is

$$\sigma - k_0 = 0,$$

that is,

$$r \frac{\partial f}{\partial U_m} \frac{\partial g}{\partial V_n} + s \left(\frac{\partial f}{\partial U_m} \frac{\partial g}{\partial V_n} + \frac{\partial f}{\partial V_n} \frac{\partial g}{\partial U_m} \right) + t \frac{\partial g}{\partial U_m} \frac{\partial g}{\partial V_n} = k_0.$$

The two third alternatives are that the two equations, in each pair of relations arising from $A = 0$ and $B = 0$, are independent of one another. When these two third alternatives co-exist, we once again have the set of circumstances considered in §§ 12, 13; the results need not be restated. When only one of these third alternatives is valid, together with the first or the second of the alternatives for the other pair of equations, once again we have particular cases of the general result just indicated.

16. In the preceding four cases which have just been considered, the discrimination has been effected by the vanishing or the non-vanishing of the quantities A and B , which are the coefficients of U_{m+2} and of V_{n+2} in $h_{11} - pf_{11} - qg_{11}$ and in $h_{22} - pf_{22} - qg_{22}$ respectively. It might seem as if a new set of cases would arise, discriminated by the vanishing or the non-vanishing of the quantity

$$\begin{vmatrix} \frac{\partial^2 h}{\partial U_m \partial V_n}, & \frac{\partial^2 f}{\partial U_m \partial V_n}, & \frac{\partial^2 g}{\partial U_m \partial V_n} \\ f_1, & g_1, & h_1 \\ f_2, & g_2, & h_2 \end{vmatrix},$$

which effectively is the coefficient of $U_{m+1}V_{n+1}$ in $h_{12} - pf_{12} - qg_{12}$; but this possibility is not actually a fact. The essential difference lies in the property that, whereas U_{m+2} occurs in the expression for $h_{11} - pf_{11} - qg_{11}$ only (so that, if $A = 0$, the form of the expression is substantially changed), and similarly for V_{n+2} and $h_{22} - pf_{22} - qg_{22}$, the quantities U_{m+1} and V_{n+1} do occur elsewhere than in a first term in $h_{12} - pf_{12} - qg_{12}$. The sole effect of the non-occurrence of the term in the last quantity, which involves $U_{m+1}V_{n+1}$, is that $k_0 = 0$; some simplification thereby arises in the preceding formulæ; but simplification, not essential change, is the sole effect of the condition.

The most obvious case arises when

$$x, y, z = \text{function of } u + \text{function of } v;$$

to its fuller consideration, not in this reference alone, we shall proceed (§ 19).

17. In several of the preceding discussions, it has been pointed out that the vanishing of F_1 , G_1 , H_1 modifies the form of the final equation obtained, and that this final form can be obtained as a limiting case of a more general final form: and similarly for the vanishing of F_2 , G_2 , H_2 . Such conditions, however, admit of a simple discussion of the problem from the beginning.

When $F_1 = 0$, $G_1 = 0$, $H_1 = 0$, then the equation

$$h_1 = pf_1 + qg_1$$

becomes, on the removal of a common factor,

$$\frac{\partial h}{\partial U_m} = p \frac{\partial f}{\partial U_m} + q \frac{\partial g}{\partial U_m}.$$

In this case, A vanishes because its value is

$$A = \begin{vmatrix} \frac{\partial h}{\partial U_m}, & \frac{\partial f}{\partial U_m}, & \frac{\partial g}{\partial U_m} \\ h_1, & f_1, & g_1 \\ h_2, & f_2, & g_2 \end{vmatrix}.$$

Thus U_{m+1} is not expressible in terms of p and q : but it is given by

$$\left\{ a_0 - r \left(\frac{\partial f}{\partial U_m} \right)^2 - 2s \frac{\partial f}{\partial U_m} \frac{\partial g}{\partial U_m} - t \left(\frac{\partial g}{\partial U_m} \right)^2 \right\} U_{m+1}^2 + 2b_0 U_{m+1} + c_0 = 0.$$

On the supposition that B does not vanish, the value of V_{n+1} is given by the equation

$$V_{n+1} = -\frac{1}{B} (H_2 - pF_2 - qG_2).$$

Moreover, as F_1 , G_1 , H_1 all vanish, we have $m_0 = 0$, $n_0 = 0$; hence

$$\begin{aligned} & k_0 U_{m+1} V_{n+1} + l_0 U_{m+1} \\ &= rf_1 f_2 + s(f_1 g_2 + f_2 g_1) + tg_1 g_2 \\ &= U_{m+1} \left[r \frac{\partial f}{\partial U_m} \left(\frac{\partial f}{\partial V_n} V_{n+1} + F_2 \right) + t \frac{\partial g}{\partial U_m} \left(\frac{\partial g}{\partial V_n} V_{n+1} + G_2 \right) \right. \\ &\quad \left. + s \left\{ \frac{\partial f}{\partial U_m} \left(\frac{\partial g}{\partial V_n} V_{n+1} + G_2 \right) + \frac{\partial g}{\partial U_m} \frac{\partial f}{\partial V_n} (V_{n+1} + F_2) \right\} \right], \end{aligned}$$

so that

$$\begin{aligned} & \left\{ k_0 - r \frac{\partial f}{\partial U_m} \frac{\partial f}{\partial V_n} - s \left(\frac{\partial f}{\partial U_m} \frac{\partial g}{\partial V_n} + \frac{\partial g}{\partial U_m} \frac{\partial f}{\partial V_n} \right) - t \frac{\partial g}{\partial U_m} \frac{\partial g}{\partial V_n} \right\} V_{n+1} \\ &= -l_0 + rF_2 \frac{\partial f}{\partial U_m} + s \left(G_2 \frac{\partial f}{\partial U_m} + F_2 \frac{\partial g}{\partial U_m} \right) + t \frac{\partial g}{\partial U_m} G_2. \end{aligned}$$

Substitution of the preceding value of V_{n+1} leads to a relation of the form

$$\alpha r + \beta s + \gamma t = \delta,$$

where $\alpha, \beta, \gamma, \delta$ are linear functions of p and q , having functions of x, y, z for their coefficients.

When $F_2 = G_2 = H_2 = 0$, while A is not zero, there is a similar form of final equation.

When $F_1 = 0 = G_1 = H_1$, $F_2 = 0 = G_2 = H_2$, the equation

$$h_{12} - pf_{12} - qg_{12} = rf_1f_2 + s(f_1g_2 + f_2g_1) + tg_1g_2$$

leads to the equation

$$\begin{aligned} \frac{\partial^2 h}{\partial U_m \partial V_n} - p \frac{\partial^2 f}{\partial U_m \partial V_n} - q \frac{\partial^2 g}{\partial U_m \partial V_n} \\ = r \frac{\partial f}{\partial U_m} \frac{\partial f}{\partial V_n} + s \left(\frac{\partial f}{\partial U_m} \frac{\partial g}{\partial V_n} + \frac{\partial f}{\partial V_n} \frac{\partial g}{\partial U_m} \right) + t \frac{\partial g}{\partial U_m} \frac{\partial g}{\partial V_n}, \end{aligned}$$

which is the final form of the equation in the supposed circumstances; the coefficients of r, s, t , and the term independent of r, s, t , are functions of x, y, z, p, q .

Note.—A simple and well known example of this last case is provided by the equations

$$\begin{aligned} x &= (1-u^2)U'' + 2uU' - 2U + (1-v^2)V'' + 2vV' - 2V, \\ y &= i[-(1+u^2)U'' + 2uU' - 2U + (1+v^2)V'' - 2vV' + 2V], \\ z &= 2uU'' - 2U' + 2vV'' - 2V. \end{aligned}$$

Here we have
$$p = \frac{u+v}{1-uv}, \quad q = i \frac{u+v}{1-uv},$$

on reduction; also

$$\frac{\partial^2 h}{\partial U_m \partial V_n} = 0 = \frac{\partial^2 f}{\partial U_m \partial V_n} = \frac{\partial^2 g}{\partial U_m \partial V_n};$$

and the final differential equation is

$$(1-u^2)(1-v^2)r - 2i(u^2-v^2)s + (1+u^2)(1+v^2)t = 0,$$

that is,
$$(1+q^2)r - 2pq s + (1+p^2)t = 0,$$

the equation of minimal surfaces.

It will be noticed that the integral equations are of the special form

$$x, y, z = \text{function of } u + \text{function of } v,$$

so that

$$h_{12} = 0, \quad f_{12} = 0, \quad g_{12} = 0.$$

18. We know that, by means of Legendre's transformation,

$$Z = px + qy - z,$$

$$X = p, \quad Y = q, \quad P = x, \quad Q = y,$$

an equation

$$f(x, y, z, p, q, r, s, t) = 0$$

is transformed into the equation

$$f\left(P, Q, PX + QY - Z, X, Y, \frac{T}{RT - S^2}, \frac{-S}{RT - S^2}, \frac{R}{RT - S^2}\right) = 0;$$

hence, when the integral of the former is known, the integral of the latter can be derived.

Thus, for example, the primitive of the equation

$$(1 + x^2)r + 2xys + (1 + y^2)t = 0$$

can be derived from the primitive of the equation

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0;$$

it is easily found to be expressible by the three equations

$$x = \frac{u + v}{1 - uv},$$

$$y = i \frac{u - v}{1 - uv},$$

$$z = \frac{1}{1 - uv} \{ (6uv - 2)(U' + V') - 4vU - 4uV \}.$$

We notice that the form of the integral equations in the primitive of the new equation is no longer

$$x, y, z = \text{function of } u + \text{function of } v,$$

which was the form of the primitive of the original equations.

Equations under Case I., having Integrals of Special Form.

19. In the preceding investigations, the principal aim has been the construction of the form of partial equations of the second order as determined by an assumed set of three equations which constitute the primitive. We now proceed (as promised at the end of § 13 and § 16) to discuss, in greater detail, some of those equations of the second order the primitive of which is constituted by a set of the form

$$x, y, z = \text{function of } u + \text{function of } v.$$

The question will be considered by dealing with the differential equations, which result from using the relation

$$h_{12}-pf_{12}-qg_{12}=rf_1f_2+s(f_1g_2+f_2g_1)+tg_1g_2,$$

without using either of the relations which respectively give the values of $h_{11}-pf_{11}-qg_{11}$, $h_{22}-pf_{22}-qg_{22}$. In the present case,

$$f_{12}=0, \quad g_{12}=0, \quad h_{12}=0;$$

so that the equation becomes

$$rf_1f_2+s(f_1g_2+f_2g_1)+tg_1g_2=0.$$

As A is not zero, and also B is not zero, the ratios of the coefficients are functions of x, y, z, p, q ; hence the equation may be written

$$ar+2bs+ct=0,$$

where a, b, c are functions of x, y, z, p, q .

Comparing the two forms of the equation, we have a quantity λ , such that

$$f_1f_2=\lambda a, \quad f_1g_2+f_2g_1=2\lambda b, \quad g_1g_2=\lambda c;$$

hence, writing $\delta=(b^2-ac)^{\frac{1}{2}}$,

we have $f_1g_2=\lambda(b+\delta); \quad f_2g_1=\lambda(b-\delta).$

Consequently, $\frac{b+\delta}{a}=\frac{g_2}{f_2}$ = function of v only,

$$\frac{b-\delta}{a}=\frac{g_1}{f_1}=\text{,,} \quad u \text{ only};$$

and therefore $f_1\frac{d}{dx}\left(\frac{b+\delta}{a}\right)+g_1\frac{d}{dy}\left(\frac{b+\delta}{a}\right)=0,$

$$f_2\frac{d}{dx}\left(\frac{b-\delta}{a}\right)+g_2\frac{d}{dy}\left(\frac{b-\delta}{a}\right)=0,$$

where $\left. \begin{aligned} \frac{d}{dx} &= \frac{\partial'}{\partial x} + r\frac{\partial}{\partial p} + s\frac{\partial}{\partial q} \\ \frac{d}{dy} &= \frac{\partial'}{\partial y} + s\frac{\partial}{\partial p} + t\frac{\partial}{\partial q} \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{\partial'}{\partial x} &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} \\ \frac{\partial'}{\partial y} &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} \end{aligned} \right\};$

that is, $a\frac{d}{dx}\left(\frac{b+\delta}{a}\right)+(b-\delta)\frac{d}{dy}\left(\frac{b+\delta}{a}\right)=0,$

$$a\frac{d}{dx}\left(\frac{b-\delta}{a}\right)+(b+\delta)\frac{d}{dy}\left(\frac{b-\delta}{a}\right)=0.$$

Adding, we have

$$a \frac{d}{dx} \left(\frac{b}{a} \right) + b \frac{d}{dy} \left(\frac{b}{a} \right) - \delta \frac{d}{dy} \left(\frac{\delta}{a} \right) = 0;$$

and, subtracting, we have

$$a \frac{d}{dx} \left(\frac{\delta}{a} \right) + b \frac{d}{dy} \left(\frac{\delta}{a} \right) - \delta \frac{d}{dy} \left(\frac{b}{a} \right) = 0.$$

The former gives

$$\begin{aligned} \frac{d}{dx} \left(\frac{b}{a} \right) + \frac{b}{a} \frac{d}{dy} \left(\frac{b}{a} \right) &= \frac{\delta}{a} \frac{d}{dy} \left(\frac{\delta}{a} \right) \\ &= \frac{b}{a} \frac{d}{dy} \left(\frac{b}{a} \right) - \frac{1}{2} \frac{d}{dy} \left(\frac{c}{a} \right), \end{aligned}$$

so that

$$\frac{d}{dx} \left(\frac{b}{a} \right) + \frac{1}{2} \frac{d}{dy} \left(\frac{c}{a} \right) = 0;$$

and similarly, after slight reduction, the second gives

$$\frac{d}{dy} \left(\frac{b}{c} \right) + \frac{1}{2} \frac{d}{dx} \left(\frac{a}{c} \right) = 0.$$

Having regard to the significance of the operations d/dx , d/dy , we notice that these conditions are to be satisfied concurrently with the original differential equation.

20. When b vanishes, so that the equation is $ar + ct = 0$, the two conditions give

$$\frac{c}{a} = \text{constant};$$

the equation is at once transformable into the form

$$r + t = 0,$$

characteristic of two-dimensional potential.

When b does not vanish (and this now will be assumed), we can divide the partial equation throughout by $2b$ (or, what is the same thing, we can take $2b = 1$): the equation then is

$$ar + s + ct = 0,$$

and the two conditions become

$$\left. \begin{aligned} \frac{d}{dx} \left(\frac{1}{a} \right) + \frac{d}{dy} \left(\frac{c}{a} \right) &= 0 \\ \frac{d}{dy} \left(\frac{1}{c} \right) + \frac{d}{dx} \left(\frac{a}{c} \right) &= 0 \end{aligned} \right\},$$

which are to be satisfied concurrently with the partial differential equation. The former of these conditions is

$$a \frac{dc}{dy} = \frac{da}{dx} + c \frac{da}{dy},$$

that is,

$$\begin{aligned} a \left\{ \frac{\partial' c}{\partial' y} - (ar + ct) \frac{\partial c}{\partial p} + t \frac{\partial c}{\partial q} \right\} \\ = \frac{\partial' a}{\partial' x} + r \frac{\partial a}{\partial p} - (ar + ct) \frac{\partial a}{\partial q} + c \left\{ \frac{\partial' a}{\partial' y} - (ar + ct) \frac{\partial a}{\partial p} + t \frac{\partial a}{\partial q} \right\}, \end{aligned}$$

concurrently with the equation; and therefore we have

$$\left. \begin{aligned} a \frac{\partial' c}{\partial' y} &= \frac{\partial' a}{\partial' x} + c \frac{\partial' a}{\partial' y} \\ a^2 \frac{\partial c}{\partial p} &= (ac - 1) \frac{\partial a}{\partial p} + a \frac{\partial a}{\partial q} \\ c^2 \frac{\partial a}{\partial p} &= ac \frac{\partial c}{\partial p} - a \frac{\partial c}{\partial q} \end{aligned} \right\},$$

apparently three relations. Similarly, the other condition leads to the three relations

$$\left. \begin{aligned} c \frac{\partial' a}{\partial' x} &= \frac{\partial' c}{\partial' y} + a \frac{\partial' c}{\partial' x} \\ c^2 \frac{\partial a}{\partial q} &= (ac - 1) \frac{\partial c}{\partial q} + c \frac{\partial c}{\partial p} \\ a^2 \frac{\partial c}{\partial q} &= ac \frac{\partial a}{\partial q} - c \frac{\partial a}{\partial p} \end{aligned} \right\}.$$

It is easy to verify that the six relations thus obtained are completely satisfied in virtue of the four relations

$$\left. \begin{aligned} a \frac{\partial' c}{\partial' y} &= \frac{\partial' a}{\partial' x} + c \frac{\partial' a}{\partial' y} \\ c \frac{\partial' a}{\partial' x} &= \frac{\partial' c}{\partial' y} + a \frac{\partial' c}{\partial' x} \\ c^2 \frac{\partial a}{\partial p} &= ac \frac{\partial c}{\partial p} - a \frac{\partial c}{\partial q} \\ a^2 \frac{\partial c}{\partial q} &= ac \frac{\partial a}{\partial q} - c \frac{\partial a}{\partial p} \end{aligned} \right\},$$

which therefore are conditions to be satisfied by a and c in order that the equation may have a primitive of the specified type.

21. Consider, first, the case (if any) in which a and c are functions of x , y , and z only. The third and the fourth of the relations are then satisfied identically: and the other two, which are the effective conditions, can be taken

$$\frac{\partial'}{\partial x} \left(\frac{1}{a} \right) + \frac{\partial'}{\partial y} \left(\frac{c}{a} \right) = 0,$$

$$\frac{\partial'}{\partial y} \left(\frac{1}{c} \right) + \frac{\partial'}{\partial x} \left(\frac{a}{c} \right) = 0.$$

As a and c do not involve p or q , these equations give

$$\frac{\partial}{\partial z} \left(\frac{1}{a} \right) = 0, \quad \frac{\partial}{\partial z} \left(\frac{c}{a} \right) = 0, \quad \frac{\partial}{\partial z} \left(\frac{1}{c} \right) = 0, \quad \frac{\partial}{\partial z} \left(\frac{a}{c} \right) = 0;$$

hence a and c are functions of x and y only. The first of the relations then shews that some function θ , of x and y , exists such that

$$\frac{c}{a} = \frac{\partial \theta}{\partial x}, \quad -\frac{1}{a} = \frac{\partial \theta}{\partial y};$$

when these values of a and c are substituted in the second relation, it takes the form

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{\partial \theta}{\partial y} \frac{\partial^2 \theta}{\partial x \partial y} + \frac{\partial \theta}{\partial x} \frac{\partial^2 \theta}{\partial y^2} = 0.$$

The primitive of this relation can be expressed* in the form

$$-x = \rho'' + \sigma'',$$

$$-y = \lambda \rho'' - \rho' + \mu \sigma'' - \sigma',$$

$$\theta = \lambda^2 \rho'' - 2\lambda \rho' + 2\rho + \mu^2 \sigma'' - 2\mu \sigma' + 2\sigma,$$

where ρ is any function of the parameter λ , and σ is any function of the parameter μ . We easily find

$$\frac{\partial \theta}{\partial x} = \lambda \mu, \quad \frac{\partial \theta}{\partial y} = -(\lambda + \mu),$$

so that

$$\frac{c}{a} = \lambda \mu, \quad \frac{1}{a} = \lambda + \mu.$$

* See my *Theory of Differential Equations*, Vol. VI., p. 343.

Consequently, the differential equation is

$$r + (\lambda + \mu)s + \lambda\mu t = 0,$$

where (changing the signs of ρ and σ) λ and μ are functions of x and y , given by the equations

$$x = \rho'' + \sigma'',$$

$$y = \lambda\rho'' - \rho' + \mu\sigma'' - \sigma';$$

and, for our purposes, ρ and σ are any functions of λ and μ .

This equation of the second order possesses two intermediate integrals, in the forms

$$p + \lambda q = \Phi'(\rho''), \quad p + \mu q = \Psi'(\sigma''),$$

where Φ and Ψ are arbitrary functions: and then

$$\begin{aligned} dz &= p dx + q dy \\ &= (p + \lambda q) \rho''' d\lambda + (p + \mu q) \sigma''' d\mu, \end{aligned}$$

so that the primitive is given by

$$\left. \begin{aligned} z &= \Phi(\rho'') + \Psi(\sigma'') \\ x &= \rho'' + \sigma'' \\ y &= \lambda\rho'' - \rho' + \mu\sigma'' - \sigma' \end{aligned} \right\},$$

where Φ and Ψ are arbitrary functions of their argument, and ρ and σ are specific functions of λ and μ .

As the equation possesses two intermediate integrals, it is known* to be transformable to $s = 0$ by contact-transformations.

22. Consider, next, the case in which a and c do not explicitly involve x , y , or z : we have

$$\begin{aligned} ac \frac{\partial c}{\partial p} - c^2 \frac{\partial a}{\partial p} &= a \frac{\partial c}{\partial q}, \\ -a^2 \frac{\partial c}{\partial q} + ac \frac{\partial a}{\partial q} &= c \frac{\partial a}{\partial p}, \end{aligned}$$

when a and c are functions of p and q only, as the full equations for the determination of a and c .

The first gives
$$\frac{\partial}{\partial p} \left(\log \frac{c}{a} \right) = - \frac{\partial}{\partial q} \left(\frac{1}{c} \right),$$

* *Theory of Differential Equations*, Vol. VI., p. 295.

and the second gives $\frac{\partial}{\partial q} \left(\log \frac{a}{c} \right) = - \frac{\partial}{\partial p} \left(\frac{1}{a} \right).$

Take a quantity θ such that

$$\frac{1}{c} = - \frac{\partial^2 \theta}{\partial p^2},$$

so that, from the modified first relation,

$$\log \frac{c}{a} = \frac{\partial^2 \theta}{\partial p \partial q}.$$

Hence

$$\log \frac{a}{c} = - \frac{\partial^2 \theta}{\partial p \partial q};$$

and therefore, from the modified second relation,

$$\frac{1}{a} = \frac{\partial^2 \theta}{\partial q^2}.$$

If now we write temporarily

$$p, q, \theta = x, y, z,$$

the equation for $\theta (= z)$, in terms of p and q as the new independent variables, is

$$e^s = \frac{c}{a} = - \frac{t}{r},$$

that is,

$$r + te^{-s} = 0.$$

23. This equation can be integrated by Darboux's method. Let λ and μ be the roots of the equation

$$\rho^2 + \rho te^{-s} + e^{-s} = 0,$$

so that

$$\lambda + \mu = -te^{-s}, \quad \lambda\mu = e^{-s}.$$

For either of the quantities λ and μ , we have

$$(2\rho + te^{-s}) \frac{\partial \rho}{\partial t} + \rho e^{-s} = 0,$$

$$(2\rho + te^{-s}) \frac{\partial \rho}{\partial s} - \rho te^{-s} - e^{-s} = 0.$$

Hence

$$\begin{aligned} (2\mu + te^{-s}) \left(\mu \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t} \right) &= \mu^2 te^{-s} + \mu e^{-s} + \mu e^{-s} \\ &= -\mu^2 (\lambda + \mu) + 2\mu \lambda \mu \\ &= \mu^2 (\lambda - \mu), \end{aligned}$$

that is,
$$\mu \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t} = -\mu^2.$$

And
$$\begin{aligned} (2\mu + te^{-s}) \left(\lambda \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t} \right) &= \lambda (ut + 1) e^{-s} + \mu e^{-s} \\ &= \lambda \mu te^{-s} + (\lambda + \mu) e^{-s} \\ &= 0, \end{aligned}$$

so that
$$\lambda \frac{\partial \mu}{\partial s} - \frac{\partial \mu}{\partial t} = 0.$$

Similarly,
$$\begin{aligned} \lambda \frac{\partial \lambda}{\partial s} - \frac{\partial \lambda}{\partial t} &= -\lambda^2, \\ \mu \frac{\partial \lambda}{\partial s} - \frac{\partial \lambda}{\partial t} &= 0. \end{aligned}$$

Let
$$u(x, y, z, p, q, r, s, t) = 0$$

be an equation compatible with

$$r + te^{-s} = 0.$$

The equations to be satisfied* by the form of u are

$$\rho^2 \frac{\partial u}{\partial r} - \rho \frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} = 0,$$

and (as the original equation does not involve x, y, z, p, q)

$$\frac{du}{dx} + \rho' \frac{du}{dy} = 0,$$

where ρ and ρ' are λ and μ in either of the two arrangements.

Now r can be removed from u by means of the original equation: hence we can take the equations for u in a form

$$\lambda \frac{\partial u}{\partial s} - \frac{\partial u}{\partial t} = 0,$$

$$\frac{du}{dx} + \mu \frac{du}{dy} = 0.$$

The latter is

$$\frac{\partial u}{\partial x} + p \frac{\partial u}{\partial z} + r \frac{\partial u}{\partial p} + s \frac{\partial u}{\partial q} + \mu \left(\frac{\partial u}{\partial y} + q \frac{\partial u}{\partial z} + s \frac{\partial u}{\partial p} + t \frac{\partial u}{\partial q} \right) = 0;$$

* *Theory of Differential Equations*, Vol. VI., § 261; the method is Darboux's.

Here F and G are arbitrary functions of their arguments: and

$$\left. \begin{aligned} \lambda + \mu &= -te^{-s} = r \\ \lambda\mu &= e^{-s} \\ \frac{1}{\lambda} + \frac{1}{\mu} &= -t \end{aligned} \right\}.$$

Now $d(rx + sy - p) = xdr + yds$

$$\begin{aligned} &= x(d\lambda + d\mu) - y\left(\frac{d\lambda}{\lambda} + \frac{d\mu}{\mu}\right) \\ &= -\frac{d\lambda}{\lambda}(y - x\lambda) - \frac{d\mu}{\mu}(y - x\mu) \\ &= -\lambda G''d\lambda - \mu F''d\mu, \end{aligned}$$

so that $rx + sy - p = -\lambda G' + G - \mu F' + F.$

Again, $d(sx + ty - q) = xds + ydt$

$$\begin{aligned} &= -x\left(\frac{d\lambda}{\lambda} + \frac{d\mu}{\mu}\right) + y\left(\frac{d\lambda}{\lambda^2} + \frac{d\mu}{\mu^2}\right) \\ &= \frac{d\lambda}{\lambda^2}(y - x\lambda) + \frac{d\mu}{\mu^2}(y - x\mu) \\ &= G''d\lambda + F''d\mu, \end{aligned}$$

so that $sx + ty - q = G' + F'.$

Further,

$$\begin{aligned} d(rx^2 + 2sxy + ty^2) \\ &= x^2dr + 2xyds + y^2dt + 2(rx + sy)dx + 2(sx + ty)dy, \end{aligned}$$

so that

$$\begin{aligned} d(rx^2 + 2sxy + ty^2 - 2z) \\ &= x^2dr + 2xyds + y^2dt + 2(rx + sy - p)dx + 2(sx + ty - q)dy. \end{aligned}$$

But $x^2dr + 2xyds + y^2dt = x^2(d\lambda + d\mu) - 2xy\left(\frac{d\lambda}{\lambda} + \frac{d\mu}{\mu}\right) + y^2\left(\frac{d\lambda}{\lambda^2} + \frac{d\mu}{\mu^2}\right)$

$$\begin{aligned} &= (y - \lambda x)^2 \frac{d\lambda}{\lambda^2} + (y - \mu x)^2 \frac{d\mu}{\mu^2} \\ &= \lambda^2 G''^2 d\lambda + \mu^2 F''^2 d\mu. \end{aligned}$$

And

$$\begin{aligned}
 (rx + sy - p)dx + (sx + ty - q)dy \\
 &= (G + F)dx - (\lambda G' + \mu F')dx + (G' + F'')dy \\
 &= (G + F)dx + G'(dy - \lambda dx) + F''(dy - \mu dx) \\
 &= (G + F)dx + (G'd\lambda + F'd\mu)x \\
 &\quad + G'(\lambda^2 G''' + 2\lambda G'')d\lambda + F''(\mu^2 F''' + 2\mu F'')d\mu \\
 &= d\{(G + F)x\} + G'(\lambda^2 G''' + 2\lambda G'')d\lambda + F''(\mu^2 F''' + 2\mu F'')d\mu.
 \end{aligned}$$

Consequently,

$$\begin{aligned}
 d(rx^2 + 2sxy + ty^2 - 2z) &= \lambda^2 G''^2 d\lambda + \mu^2 F''^2 d\mu + 2d\{(G + F)x\} \\
 &\quad + 2G'(\lambda^2 G''' + 2\lambda G'')d\lambda + 2F''(\mu^2 F''' + 2\mu F'')d\mu;
 \end{aligned}$$

hence

$$\begin{aligned}
 rx^2 + 2sxy + ty^2 - 2z - 2(G + F)x \\
 = 2\lambda^2 G'G'' + 2\mu^2 F'F'' - \int \lambda^2 G''^2 d\lambda - \int \mu^2 F''^2 d\mu.
 \end{aligned}$$

When r, s, t are eliminated from this equation by the relations

$$\lambda + \mu = r, \quad \lambda\mu = e^{-z}, \quad \frac{1}{\lambda} + \frac{1}{\mu} = -t,$$

the resulting value of z is θ : and λ, μ are expressed in terms of x and y by the equations

$$y - \lambda x = \lambda^2 G'', \quad y - \mu x = \mu^2 F''.$$

For our purpose, we want $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2},$

in this notation: where x, y stand for original p and q . Thus

$$\begin{aligned}
 \frac{1}{c} &= -\frac{\partial^2 z}{\partial x^2} = -\lambda - \mu, \\
 \frac{1}{a} &= \frac{\partial^2 z}{\partial y^2} = -\frac{\lambda + \mu}{\lambda\mu},
 \end{aligned}$$

where $q - \lambda p = \lambda^2 G'', \quad q - \mu p = \mu^2 F''.$

Thus the equation, being $ar + s + ct = 0,$

becomes $\lambda\mu r - (\lambda + \mu)s + t = 0,$

where λ and μ are expressible* in terms of p and q by relations

$$q - \lambda p = \lambda^2 G'', \quad q - \mu p = \mu^2 F''.$$

* The equation of minimal surfaces is given by

$$\lambda^2 G'' = i(1 + \lambda^2)^{\frac{1}{2}}, \quad \mu^2 F'' = i(1 + \mu^2)^{\frac{1}{2}};$$

other equations are given by taking other specific forms for F and G .

24. The primitive of this equation can be obtained in the same way, due to Ampère, as the primitive of the equation of minimal surfaces.* Substituting

$$s' = q' - ty', \quad r = p' - q'y' + ty'^2,$$

in the equation, and making it evanescent in t , we have

$$\lambda\mu(p' - q'y') - (\lambda + \mu)q' = 0,$$

$$\lambda\mu y'^2 + (\lambda + \mu)y' + 1 = 0.$$

From the second equation, we have

$$\lambda y' + 1 = 0, \quad \mu y' + 1 = 0.$$

Taking $\lambda y' + 1 = 0$, we have the first equation in the form

$$-q' + \mu p' = 0.$$

But

$$q - \mu p = \mu^2 F'',$$

so that

$$\mu' = 0;$$

thus μ is constant for the set of equations.

Similarly, the other system possesses $\lambda = \text{constant}$ as an integral. We therefore take μ and λ as new independent variables.

Now
$$y' = -\frac{1}{\lambda},$$

when μ is constant: that is,

$$\frac{\partial y}{\partial \lambda} = -\frac{1}{\lambda} \frac{\partial x}{\partial \lambda}.$$

Similarly,

$$\frac{\partial y}{\partial \mu} = -\frac{1}{\mu} \frac{\partial x}{\partial \mu}.$$

Consequently,

$$\frac{\partial}{\partial \mu} \left(\frac{1}{\lambda} \frac{\partial x}{\partial \lambda} \right) = \frac{\partial}{\partial \lambda} \left(\frac{1}{\mu} \frac{\partial x}{\partial \mu} \right),$$

that is,

$$\frac{\partial^2 x}{\partial \lambda \partial \mu} = 0;$$

hence

$$\begin{aligned} x &= \text{function of } \lambda + \text{function of } \mu \\ &= \lambda^2 \phi'(\lambda) + \mu^2 \psi'(\mu) \\ &= \lambda^2 \phi' + \mu^2 \psi', \end{aligned}$$

say. Then

$$\begin{aligned} dy &= -\frac{1}{\lambda} \frac{\partial x}{\partial \lambda} d\lambda - \frac{1}{\mu} \frac{\partial x}{\partial \mu} d\mu \\ &= -(\lambda \phi'' + 2\phi') d\lambda - (\mu \psi'' + 2\psi') d\mu, \end{aligned}$$

* See my *Theory of Differential Equations*, Vol. VI., p. 277.

so that

$$y = -\lambda\phi' - \phi - \mu\psi' - \psi.$$

Finally, we have

$$\begin{aligned} dz &= p dx + q dy \\ &= \{p(\lambda^2\phi'' + 2\lambda\phi') - q(\lambda\phi'' + 2\phi')\} d\lambda \\ &\quad + \{p(\mu^2\psi'' + 2\mu\psi') - q(\mu\psi'' + 2\psi')\} d\mu \\ &= -(\lambda\phi'' + 2\phi')\lambda^2 G'' d\lambda - (\mu\psi'' + 2\psi')\mu^2 F'' d\mu. \end{aligned}$$

Modifying the specific functions F and G , we can state the result as follows:—

The primitive of the differential equation

$$\lambda\mu r - (\lambda + \mu)s + t = 0,$$

where λ and μ are functions of p and q determined by the equations

$$q - \lambda p = f(\lambda), \quad q - \mu p = g(\mu),$$

f and g denoting specific functions, is given by the three equations

$$\begin{aligned} x &= \lambda^2\phi' + \mu^2\psi', \\ -y &= \lambda\phi' + \phi + \mu\psi' + \psi, \\ -z &= \int (\lambda\phi'' + 2\phi')f(\lambda) d\lambda + \int (\mu\psi'' + 2\psi')g(\mu) d\mu, \end{aligned}$$

in which ϕ and ψ denote arbitrary functions of λ and of μ respectively.

Note.—It is easy to verify that the equation

$$\{(Nq + M)^2 - P^2\} r - 2\{(Np + L)(Nq + M) + PQ\} s + \{Np + L\}^2 - Q^2\} t = 0,$$

where L, M, N, P, Q are constants, is a special example of the preceding equation; the functions $f(\lambda)$ and $g(\mu)$ are linear functions of λ and of μ respectively.

25. Consider, next, the case (if it can arise) in which the coefficients a and c in the equation

$$ar + s + ct = 0$$

can be functions of x, y, p, q , subject to the condition that the integral equivalent of the equation is of the specified type.

The relations to be satisfied by a and c are

$$\left. \begin{aligned} \frac{\partial}{\partial x} \left(\frac{1}{a} \right) + \frac{\partial}{\partial y} \left(\frac{c}{a} \right) &= 0 \\ \frac{\partial}{\partial y} \left(\frac{1}{c} \right) + \frac{\partial}{\partial x} \left(\frac{a}{c} \right) &= 0 \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{\partial}{\partial p} \left(\frac{1}{a} \right) + \frac{\partial}{\partial q} \left(\log \frac{a}{c} \right) &= 0 \\ \frac{\partial}{\partial q} \left(\frac{1}{c} \right) + \frac{\partial}{\partial p} \left(\log \frac{c}{a} \right) &= 0 \end{aligned} \right\}.$$

In the first pair of equations, the quantities p and q are parametric. We know that the equations are satisfied by taking

$$\frac{1}{a} = u + v, \quad \frac{1}{c} = \frac{1}{u} + \frac{1}{v},$$

where $x = f'(u) + g'(v)$, $y = uf'(u) - f(u) + vg'(v) - g(v)$,

the functions f and g being any whatever: hence, taking account of the possibility that a and c (and therefore u and v) can involve p and q , as well as x and y , and writing

$$f = f(u, p, q), \quad g = g(v, p, q),$$

then the first pair of equations are satisfied by

$$x = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v}, \quad y = u \frac{\partial f}{\partial u} - f + v \frac{\partial g}{\partial v} - g,$$

$$\frac{1}{a} = u + v, \quad \frac{1}{c} = \frac{1}{u} + \frac{1}{v}, \quad \frac{c}{a} = uv;$$

and, so far as the first pair of equations are concerned, the new functions f and g can be any whatever.

The functions f and g are to be determined so as to satisfy the second pair of equations. When the values of $1/a$, $1/c$, c/a are substituted in them, we have

$$\frac{\partial u}{\partial p} + \frac{\partial v}{\partial p} - \frac{1}{u} \frac{\partial u}{\partial q} - \frac{1}{v} \frac{\partial v}{\partial q} = 0,$$

$$- \frac{1}{u^2} \frac{\partial u}{\partial q} - \frac{1}{v^2} \frac{\partial v}{\partial q} + \frac{1}{u} \frac{\partial u}{\partial p} + \frac{1}{v} \frac{\partial v}{\partial p} = 0;$$

that is, the relations in the second pair are satisfied, if

$$u \frac{\partial u}{\partial p} = \frac{\partial u}{\partial q}, \quad v \frac{\partial v}{\partial p} = \frac{\partial v}{\partial q}.$$

Now u and v are given as functions of x , y , p , q by the preceding two equations which express x and y in terms of f and g ; hence f and g , as involving p and q , must be such that the two equations in u and v just obtained may be satisfied. But

$$0 = \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial p} + \frac{\partial^2 g}{\partial v^2} \frac{\partial v}{\partial p} + \frac{\partial^2 f}{\partial u \partial p} + \frac{\partial^2 g}{\partial v \partial p},$$

$$0 = u \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial p} + v \frac{\partial^2 g}{\partial v^2} \frac{\partial v}{\partial p} + u \frac{\partial^2 f}{\partial u \partial p} + v \frac{\partial^2 g}{\partial v \partial p} - \frac{\partial f}{\partial p} - \frac{\partial g}{\partial p},$$

$$0 = \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial q} + \frac{\partial^2 g}{\partial v^2} \frac{\partial v}{\partial q} + \frac{\partial^2 f}{\partial u \partial q} + \frac{\partial^2 g}{\partial v \partial q},$$

$$0 = u \frac{\partial^2 f}{\partial u^2} \frac{\partial u}{\partial q} + v \frac{\partial^2 g}{\partial v^2} \frac{\partial v}{\partial q} + u \frac{\partial^2 f}{\partial u \partial q} + v \frac{\partial^2 g}{\partial v \partial q} - \frac{\partial f}{\partial q} - \frac{\partial g}{\partial q}.$$

Thus there are six equations in which the four first derivatives of u and v occur linearly; when these are eliminated, we should have two equations satisfied by f and g as functions of p, q, u, v . These are easily found to be

$$u \frac{\partial^2 f}{\partial u \partial p} - \frac{\partial^2 f}{\partial u \partial q} - \frac{\partial f}{\partial p} + v \frac{\partial^2 g}{\partial v \partial p} - \frac{\partial^2 g}{\partial v \partial q} - \frac{\partial g}{\partial p} = 0,$$

$$(u-v) \left(v \frac{\partial^2 g}{\partial v \partial p} - \frac{\partial^2 g}{\partial v \partial q} \right) + v \frac{\partial f}{\partial p} - \frac{\partial f}{\partial q} + v \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} = 0,$$

which accordingly are two equations for the determination of f and g .

Remembering that f does not involve v , and that g does not involve u , we have, from the first of these two equations,

$$u \frac{\partial^2 f}{\partial u \partial p} - \frac{\partial^2 f}{\partial u \partial q} - \frac{\partial f}{\partial p} = \theta,$$

$$v \frac{\partial^2 g}{\partial v \partial p} - \frac{\partial^2 g}{\partial v \partial q} - \frac{\partial g}{\partial p} = -\theta,$$

where θ is a function of p and q alone; and then the second of the two equations becomes

$$-(u-v)\theta + v \frac{\partial f}{\partial p} - \frac{\partial f}{\partial q} + u \frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} = 0.$$

Differentiating the last with regard to u and to v separately, we have

$$-\theta + v \frac{\partial^2 f}{\partial u \partial p} - \frac{\partial^2 f}{\partial u \partial q} + \frac{\partial g}{\partial p} = 0,$$

$$\theta + u \frac{\partial^2 g}{\partial v \partial p} - \frac{\partial^2 g}{\partial v \partial q} + \frac{\partial f}{\partial p} = 0;$$

and then, differentiating the former of these with respect to v or the latter with respect to u , we have

$$\frac{\partial^2 f}{\partial u \partial p} + \frac{\partial^2 g}{\partial v \partial p} = 0,$$

that is, we may take $\frac{\partial^2 f}{\partial u \partial p} = \phi = -\frac{\partial^2 g}{\partial v \partial p}$,

where ϕ is a function of p and q only, so that the latest deduced equations are

$$\frac{\partial g}{\partial p} - \theta + v\phi - \frac{\partial^2 f}{\partial u \partial q} = 0,$$

$$\frac{\partial f}{\partial p} + \theta - u\phi - \frac{\partial^2 g}{\partial v \partial q} = 0.$$

The earlier equations give

$$\frac{\partial f}{\partial p} + \theta - u\phi + \frac{\partial^2 f}{\partial u \partial q} = 0,$$

$$\frac{\partial g}{\partial p} - \theta + v\phi + \frac{\partial^2 g}{\partial v \partial q} = 0;$$

hence

$$\frac{\partial^2 f}{\partial u \partial q} = -\frac{\partial^2 g}{\partial v \partial q} = \psi,$$

where ψ is a function of p and q only; and so we have

$$\frac{\partial f}{\partial p} = u\phi - (\psi + \theta),$$

$$\frac{\partial g}{\partial p} = -v\phi + (\psi + \theta),$$

from the preceding relations. These satisfy

$$\frac{\partial^2 f}{\partial u \partial p} = -\frac{\partial^2 g}{\partial v \partial p} = \phi.$$

Proceeding from

$$\frac{\partial^2 f}{\partial u \partial q} = \psi = -\frac{\partial^2 g}{\partial v \partial q},$$

we have

$$\frac{\partial f}{\partial q} = u\psi + \chi, \quad \frac{\partial g}{\partial q} = -v\psi + \omega,$$

where χ and ω are independent of u and v . When we substitute in the relation

$$-(u-v)\theta + v\frac{\partial f}{\partial p} - \frac{\partial f}{\partial q} + u\frac{\partial g}{\partial p} - \frac{\partial g}{\partial q} = 0,$$

we find $\chi + \omega = 0$; hence we have

$$\left. \begin{aligned} \frac{\partial f}{\partial p} &= u\phi - (\psi + \theta) \\ \frac{\partial g}{\partial p} &= -v\phi + (\psi + \theta) \end{aligned} \right\}, \quad \left. \begin{aligned} \frac{\partial f}{\partial q} &= u\psi + \chi \\ \frac{\partial g}{\partial q} &= -v\psi - \chi \end{aligned} \right\}.$$

In order that these may co-exist, we must have

$$\frac{\partial \phi}{\partial q} = \frac{\partial \psi}{\partial p} = \frac{\partial^2 \rho}{\partial p \partial q},$$

$$\frac{\partial(\psi + \theta)}{\partial q} = -\frac{\partial \chi}{\partial p} = \frac{\partial^2 \sigma}{\partial p \partial q},$$

where ρ and σ are functions of p and q only. Hence

$$f = u\rho - \sigma + F(u),$$

$$g = -v\rho + \sigma + G(v),$$

where F and G are arbitrary functions of u and of v respectively. Now

$$x = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v}$$

$$= F'(u) + G'(v),$$

$$y = u \frac{\partial f}{\partial u} - f + v \frac{\partial g}{\partial v} - g$$

$$= uF'(u) - F(u) + vG'(v) - v;$$

in other words, u and v are functions of x and y only, that is, if u and v involve x and y , they do not involve p and q .

Consequently, we have two classes of equations

$$ar + s + ct = 0$$

having an integral system of the required type, such that

$$x, y, z = \text{function of } u + \text{function of } v;$$

in one class, a and c are appropriate functions of x and y alone: in the other class, a and c are appropriate functions of p and q alone.

It is easy to see that the two equations, when subjected to Legendre's transformation, lead to equations of the same form; but, when this transformation is applied to either integral, it leads to new integral systems of a different type. An example has already (§ 18) been given.

26. We may summarise the results of the preceding investigations as regards an equation

$$ar + bs + ct = 0,$$

where a , b , c involve no derivatives of order higher than the first, the equation being supposed to have an integral system represented by

$$x, y, z = \text{function of } u + \text{function of } v;$$

as follows:—

(A.) When $b = 0$, then a and c are constant: the equation can be transformed to

$$r + t = 0.$$

(B.) When b is not zero, then we can (by division) replace it by unity.

Then a and c can be functions of x and y alone; and they can be functions of p and q alone; but they cannot be functions of x , y , p , q simultaneously.

When a and c are functions of x and y alone, then, taking two parameters λ and μ , and any two functions ρ and σ of those parameters respectively, we express λ and μ in terms of x and y by the relations

$$x = \rho'' + \sigma'', \quad y = \lambda\rho'' - \rho' + \mu\sigma'' - \sigma';$$

the partial differential equation is

$$r + (\lambda + \mu)s + \lambda\mu t = 0,$$

and its primitive is $z = \Phi(\lambda) + \Psi(\mu)$,

where Φ and Ψ are arbitrary functions.

When a and c are functions of p and q alone, we take two parameters λ and μ , and any two functions f and g of those parameters, and we determine λ and μ in terms of p and q , by the relations

$$q - \lambda p = f(\lambda), \quad q - \mu p = g(\mu);$$

the differential equation is

$$\lambda\mu r - (\lambda + \mu)s + t = 0;$$

and its primitive is

$$\begin{aligned} x &= \lambda^2\phi' + \mu^2\psi', \\ y &= \lambda\phi' + \phi + \mu\psi' + \psi, \\ -z &= \int (\lambda\phi'' + 2\phi') f(\lambda) d\lambda + \int (\mu\psi'' + 2\psi') g(\mu) d\mu, \end{aligned}$$

where ϕ and ψ are arbitrary functions.

CASE II.: *with Sub-cases.*

27. In this case, the functions f and h involve both u and v , together (possibly) with the arbitrary functions U and V ; while g involves only v and the arbitrary function V , with (possibly) the derivatives of the latter.

There are, as before, sub-cases according to the non-*evanescence* or the *evanescence* of the quantities A and B . Most of the initial forms of the results for the present case can be deduced as special forms of the results in the preceding sections; we shall therefore not deal with the sub-cases in the same detail as before. For all the sub-cases, we have

$$g_1 = 0, \quad g_{12} = 0, \quad g_{11} = 0;$$

also
$$p = \frac{h_1}{f_1}, \quad q = \frac{h_2}{g_2} - \frac{f_2 h_1}{f_1 g_2}.$$

II., (1).—Proceeding as in the least restricted of the former sub-cases (§ 8), in which the values of A and B do not vanish, being

$$A = \frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m}, \quad B = \frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n},$$

we have $U_{m+1} = -\frac{1}{A} (H_1 - pF_1)$, $f_1 = \frac{1}{A} \left(F_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial f}{\partial U_m} \right)$,

and V_{n+1} , f_2 have the same formal values as before. We proceed from the equation

$$h_{12} - pf_{12} - qg_{12} = rf_1f_2 + s(f_1g_2 + f_2g_1) + tg_1g_2,$$

which now is $h_{12} - pf_{12} = rf_1f_2 + sf_1g_2$;

and we find, on substitution, the equation

$$(q+b)r - (p+a)s = w,$$

where

$$\left. \begin{aligned} a \left(F_2 \frac{\partial g}{\partial V_n} - G_2 \frac{\partial f}{\partial V_n} \right) &= G_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial g}{\partial V_n} \\ b \left(F_2 \frac{\partial g}{\partial V_n} - G_2 \frac{\partial f}{\partial V_n} \right) &= H_2 \frac{\partial f}{\partial V_n} - F_2 \frac{\partial h}{\partial V_n} \end{aligned} \right\},$$

and w is a polynomial in p and q , of degree two in p and degree one in q : and where it is assumed that $\frac{1}{F_2} \frac{\partial f}{\partial V_n} - \frac{1}{G_2} \frac{\partial g}{\partial V_n}$ is not zero. The quantities a and b , as well as the coefficients in w , involve only the magnitudes which occur in x, y, z : hence, if an equation of the second order is to be the equivalent of the integral system, these quantities must be functions of x, y, z alone.

We evidently have

$$a \frac{\partial f}{\partial V_n} - b \frac{\partial g}{\partial V_n} = -\frac{\partial h}{\partial V_n},$$

$$aF_2 + bG_2 = -H_2;$$

so that

$$\frac{dz}{dv} + a \frac{dx}{dv} + b \frac{dy}{dv} = 0.$$

28. The application of Ampère's method to the equation

$$(q+b)r - (p+a)s = w$$

requires the substitution of

$$r = p' - q'y' + ty'^2, \quad s = q' - ty';$$

and the modified equation is made evanescent in t . Thus

$$(q+b)y'^2 + (p+a)y' = 0,$$

$$(q+b)(p' - q'y') - (p+a)q' = w.$$

The former equation gives either

$$y' = 0,$$

or

$$(q+b)y' + (p+a) = 0.$$

Hence there are two subsidiary systems of equations, viz.,

$$y' = 0, \quad (q+b)p' - (p+a)q' = w, \quad z' = p;$$

$$\text{and} \quad (q+b)y' + (p+a) = 0, \quad (q+b)p' = w, \quad z' = p + qy'.$$

Consider, in particular, the special case when w is zero, so that the equation is

$$(q+b)r - (p+a)s = 0;$$

and we set aside, as trivial, the instances (i.) when a and b are constants; (ii.) when a is a constant, and when b is a function of y only. The first group of subsidiary equations then gives

$$y = \text{constant},$$

when β is variable: that is, we take

$$y = a,$$

$$\text{and then} \quad (q+b)\frac{\partial p}{\partial \beta} - (p+a)\frac{\partial q}{\partial \beta} = 0,$$

$$\frac{\partial z}{\partial \beta} = p \frac{\partial x}{\partial \beta}.$$

Similarly, from the second set of equations, we have

$$p = \text{constant},$$

when a is variable: that is, we take

$$p = \beta,$$

$$\text{and then} \quad (q+b)\frac{\partial y}{\partial a} + (p+a)\frac{\partial x}{\partial a} = 0,$$

$$\frac{\partial z}{\partial a} = p \frac{\partial x}{\partial a} + q \frac{\partial y}{\partial a}.$$

Accordingly, as connected with the equation

$$(q+b)r - (p+a)s = 0,$$

we can take y and p as the independent variables: and the aggregate of equations then is

$$\left. \begin{aligned} q+b-(p+a)\frac{\partial q}{\partial p} &= 0 \\ q+b+(p+a)\frac{\partial x}{\partial y} &= 0 \\ \frac{\partial z}{\partial p} &= p \frac{\partial x}{\partial p} \\ \frac{\partial z}{\partial y} &= p \frac{\partial x}{\partial y} + q \end{aligned} \right\}.$$

From the first and the second of these, we have

$$\frac{\partial x}{\partial y} + \frac{\partial q}{\partial p} = 0,$$

which is the condition of coexistence of the third and the fourth. Consequently, a function (say ϕ) of y and p exists such that

$$x = \frac{\partial \phi}{\partial p}, \quad q = -\frac{\partial \phi}{\partial y};$$

and then, from the third and the fourth of the equations, it follows that

$$z = p \frac{\partial \phi}{\partial p} - \phi.$$

The function ϕ then satisfies the equation

$$\frac{\partial^2 \phi}{\partial y \partial p} + \frac{q+b}{p+a} = 0,$$

when for q , and for x and z in a and b , we substitute their values. From the nature of the case, ϕ must be an explicit function, expressible in a form that is free from partial quadratures; and, consequently, the equation must be integrable by Darboux's method. Now, when we change the notation for the variables, the equation can be written

$$s' + f(x', y', z', p', q') = 0;$$

and the cases in which this equation is integrable by Darboux's method have been discussed by Goursat.* we need not therefore consider the equation further, for the case in which $w = 0$. The general case still remains.

29. One special form of the case discussed arises, when the integral system is of the form

$$x, z = \text{function of } u + \text{function of } v,$$

$$y = \text{function of } v.$$

We then have $f_{12} = 0, \quad g_{12} = 0, \quad h_{12} = 0,$

so that $w = 0$; and the partial equation is

$$f_1 f_2 r + f_1 g_2 s = 0,$$

that is,

$$f_2 r + g_2 s = 0.$$

* See chapter viii. of his treatise quoted at the beginning of this paper; also *Annales de Toulouse*, 2e Sér., t. I. (1899), pp. 31-78, 439-463.

$$\begin{aligned}\text{Now} \quad \frac{g_2}{f_2} &= \text{function of } v \text{ only} \\ &= \quad \quad \quad y \text{ only} \\ &= 1/Y',\end{aligned}$$

say; so that, writing $Y' dy = dy'$,

the equation becomes reduced to

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y'} = 0,$$

and its primitive is easily obtainable in the form

$$z - F(y') = G(x - y'),$$

where F and G are arbitrary functions.

30. II., (2).—In this sub-case, we have $B = 0$; and there are two alternatives. In the first of them, the two equations

$$\frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0,$$

$$H_2 - pF_2 - qG_2 = 0,$$

are effectively one and the same, so that

$$F_2 = \phi \frac{\partial f}{\partial V_n}, \quad G_2 = \phi \frac{\partial g}{\partial V_n}, \quad H_2 = \phi \frac{\partial h}{\partial V_n};$$

the circumstances, when F_2 , G_2 , H_2 vanish, are given by $\phi = 0$. The quantities α and γ of § 8 vanish, owing to the form of g , which is a quantity independent of u ; and thus the degenerate expression of the equation of § 11 becomes*

$$r + as = c',$$

where a and c' are functions of x, y, z, p, q . But this form assumes that $\partial f / \partial V_n$ is not zero; if it should vanish, the equation is

$$s = c'.$$

In the other of the two alternatives, we have $A = 0$ as well as $B = 0$; we shall proceed to this almost immediately as the last of the sub-cases to be considered.

* An instance of this possibility is (taking account of the interchange of variables covered by the explanations in § 2) provided by

$$x = \phi''(u), \quad y = (u+v)^2 \phi''(u) - 2(u+v) \phi'(u) + 2\phi(u) + \psi(v), \quad z = u+v;$$

the equation is

$$s + tz^2 + 2q^2z = 0,$$

and its primitive is easily obtained by quadratures.

31. II., (3).—In this sub-case, we have $A = 0$: again, there are two alternatives. The second of the alternatives is, for the moment, deferred until the discussion of the similar alternative for the last sub-case.

In the first of them, the two equations

$$\frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m} = 0, \quad H_1 - pF_1 = 0,$$

can be only one and the same equation; and it gives the value of p . As B is not zero for this alternative, the partial equation is

$$(\beta' + \gamma'q)r \frac{\partial f}{\partial U_m} - (\alpha' + \gamma'p)s \frac{\partial f}{\partial U_m} = \gamma''' + \alpha'''p + \beta'''q,$$

that is,

$$(q + b)r - (p + a)s = w,$$

where now w is linear in p and in q ; but otherwise the final equation is the same as in § 27.

32. II., (4).—In this sub-case, we have $A = 0$, $B = 0$; it thus includes the two alternatives respectively deferred from the consideration of the last two cases.

$$\text{We must have} \quad H_1 = \theta \frac{\partial h}{\partial U_m}, \quad F_1 = \theta \frac{\partial g}{\partial U_m};$$

if it should happen that

$$H_2 = \phi \frac{\partial h}{\partial V_n}, \quad F_2 = \phi \frac{\partial f}{\partial V_n}, \quad G_2 = \phi \frac{\partial g}{\partial V_n},$$

then, by § 11, we easily see that the final partial equation is

$$r + s = k,$$

where k is a function of x, y, z, p, q at the utmost: the equation is transformable to

$$s + f(x, y, z, p, q) = 0,$$

and therefore (having regard to the integral system, which is derivable by Darboux's method) it belongs to the class already considered by Goursat (p. 170, foot-note).

If it should happen that the relations

$$H_2 = \phi \frac{\partial h}{\partial V_n}, \quad F_2 = \phi \frac{\partial f}{\partial V_n}, \quad G_2 = \phi \frac{\partial g}{\partial V_n}$$

are not satisfied, either for a non-zero value or for a zero value of ϕ , then we have the alternative considered in § 13; and the result for the present case can be deduced from the result there obtained by putting

$$G_1 = 0, \quad \psi_1 = 0:$$

the result is of the form

$$\{(rt-s^2)(ar+\beta s+\gamma t+\delta)+U\}^2 = \alpha'^2\beta' \{\alpha''-(rt-s^2)\beta''\},$$

where $\alpha, \beta, \gamma, \delta, \beta''$ are functions of x, y, z, p, q ; U is a polynomial in $r, s, t, 1$ of the second degree, and $\alpha', \beta', \alpha''$ are polynomials in $r, s, t, 1$ of the first degree only.

If, in particular, while g is a function of v only, and the assumptions as to the conditions under which $A = 0$ and $B = 0$ are satisfied still hold, it should happen that

$$x, z = f, h = \text{function of } u + \text{function of } v,$$

so that

$$k_0 = 0 = l_0 = m_0 = n_0,$$

it is easy to deduce (from earlier results) that the final equation is of the form

$$rt-s^2 = 0.$$

This conclusion can be verified from the fact that the two independent equations

$$H_2 - pF_2 - qG_2 = 0, \quad \frac{\partial h}{\partial V_n} - p \frac{\partial f}{\partial V_n} - q \frac{\partial g}{\partial V_n} = 0$$

determine p and q as functions of v alone, so that we have

$$\chi(p, q) = 0;$$

and therefore

$$rt-s^2 = 0.$$

CASE III. : *with Sub-cases.*

33. In this case, f is a function of u only, g is a function of v only, while h is a function of u and v , the arbitrary function U and its derivatives occurring with u , and likewise for V and v .

$$\text{We thus have } f_2 = 0, \quad f_{12} = 0, \quad f_{22} = 0;$$

$$g_1 = 0, \quad g_{12} = 0, \quad g_{11} = 0.$$

If the quantity A , which now is $\frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m}$, does not vanish, and,

if similarly B , which now is $\frac{\partial h}{\partial V_n} - q \frac{\partial g}{\partial V_n}$, which does not vanish, then

$$U_{m+1} = -\frac{1}{A}(H_1 - pF_1), \quad V_{n+1} = -\frac{1}{B}(H_2 - qG_2),$$

$$f_1 = \frac{1}{A} \left(F_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial f}{\partial U_m} \right), \quad g_2 = \frac{1}{B} \left(G_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial g}{\partial V_n} \right),$$

where p and q are given by the relations

$$h_1 = pf_1, \quad h_2 = qg_2.$$

The equation $h_{12} - pf_{12} - qg_{12} = f_1 f_2 r + (f_1 g_2 + f_2 g_1) s + g_1 g_2 t$

now becomes
$$h_{12} = f_1 g_2 s;$$

and, on substituting the values of U_{m+1} and V_{n+1} in h_{12} and reducing, we find the equation

$$s = \alpha + \beta p + \gamma q + \xi pq,$$

where $\alpha, \beta, \gamma, \xi$ are functions of x, y, z only; and the value of ξ is given by

$$\begin{aligned} & \left(F_1 \frac{\partial h}{\partial U_m} - H_1 \frac{\partial f}{\partial U_m} \right) \left(G_2 \frac{\partial h}{\partial V_n} - H_2 \frac{\partial g}{\partial V_n} \right) \xi \\ &= F_1 G_2 \frac{\partial^2 h}{\partial U_m \partial V_n} - F_1 \frac{\partial g}{\partial V_n} \frac{\partial (\delta' h)}{\partial U_m} - G_2 \frac{\partial f}{\partial U_m} \frac{\partial (\delta h)}{\partial V_n} + \frac{\partial f}{\partial U_m} \frac{\partial g}{\partial V_n} (\delta \delta' h), \end{aligned}$$

in the notation of § 11.

This differential equation is to have an integral equivalent of the form

$$x = \text{function of } u, \quad y = \text{function of } v,$$

$$z = \text{function of } u \text{ and } v.$$

The problem of the determination of the different forms of the equation with their respective integral equivalents is similar to the problem propounded by Moutard, and solved by Lloyd Tanner and Cosserat;* the necessary modifications of the analysis are omitted from the present communication.

When A is zero but not B , or when B is zero but not A , the final differential equation is easily seen to be

$$s = a + bp + cq,$$

where a, b, c are functions of x, y, z alone: it is only a particular case of the preceding form.

34. There remains the sub-case, for which $A = 0$ and $B = 0$, that is,

$$\begin{aligned} \frac{\partial h}{\partial U_m} - p \frac{\partial f}{\partial U_m} &= 0, & H_1 - p F_1 &= 0, \\ \frac{\partial h}{\partial V_n} - q \frac{\partial g}{\partial V_n} &= 0, & H_2 - q G_2 &= 0. \end{aligned}$$

* See my *Theory of Differential Equations*, Vol. VI., ch. xv.

The form of the final differential equation might be deduced from the result of § 13, by inserting the appropriate simplifications; it is as easy to proceed directly from the equations

$$h_{11} - pf_{11} = f_1^2 r, \quad h_{12} = f_1 g_2 s, \quad h_{22} - qg_{22} = g_2^2 t,$$

which, on writing

$$\left. \begin{aligned} a &= -\phi^2 r + a_0 \\ b &= -\xi \phi r + b_0 \\ c &= -\xi^2 r + c_0 \end{aligned} \right\}, \quad \left. \begin{aligned} a' &= -\psi^2 t + a'_0 \\ b' &= -\eta \psi t + b'_0 \\ c' &= -\eta^2 t + c'_0 \end{aligned} \right\},$$

$$\left. \begin{aligned} k &= -\phi \psi s + k_0 \\ l &= -\phi \eta s + l_0 \\ m &= -\psi \xi s + m_0 \\ n &= -\eta \xi s + n_0 \end{aligned} \right\},$$

where $\phi = \frac{\partial f}{\partial U_m}, \quad \psi = \frac{\partial g}{\partial V_n}, \quad \xi = F_1, \quad \eta = G_2,$

and where $a_0, b_0, c_0, a'_0, b'_0, c'_0, k_0, l_0, m_0, n_0$ have their former signification, become

$$aU_{m+1}^2 + 2bU_{m+1} + c = 0,$$

$$a'V_{n+1}^2 + 2b'V_{n+1} + c' = 0,$$

$$kU_{m+1}V_{n+1} + lU_{m+1} + mV_{n+1} + n = 0.$$

The final equation is the result of eliminating U_{m+1} and V_{n+1} between these three equations. Let

$$\alpha_1 = k_0 \xi \eta - l_0 \xi \psi - m_0 \eta \phi + n_0 \phi \psi,$$

$$\beta_1 = a_0 \xi^2 - 2b_0 \xi \phi + c_0 \phi^2,$$

$$\gamma_1 = a'_0 \eta^2 - 2b'_0 \eta \psi + c'_0 \psi^2,$$

$$E = (a'_0, -b'_0, c'_0) \chi (l_0 \xi - n_0 \phi, k_0 \xi - m_0 \phi)^2,$$

$$G = (a_0, -b_0, c_0) \chi (l_0 \eta - n_0 \psi, k_0 \eta - m_0 \psi)^2,$$

$$F = k_0 (c_0 \phi - b_0 \xi) (c'_0 \psi - b'_0 \eta) - l_0 (c_0 \phi - b_0 \xi) (b'_0 \psi - a'_0 \eta) \\ - m_0 (b_0 \phi - a_0 \xi) (c'_0 \psi - b'_0 \eta) + n_0 (b_0 \phi - a_0 \xi) (b'_0 \psi - a'_0 \eta),$$

$$\theta_0 = k_0^2 c_0 c'_0 - \dots + n_0^2 a_0 a'_0,$$

as before; then the eliminant, being the final equation, is

$$\Theta^2 = 4 (a_1 s - k_0 n_0 + l_0 m_0)^2 (b_0^2 - a_0 c_0 + \beta_1 r) (b_0'^2 - a_0' c_0' + \gamma_1 t),$$

where $\Theta = a_1^2 r t + \beta_1 \gamma s^2 - E r - 2 F s - G t + \theta_0.$

The aggregate of terms of the highest degree in this equation is made up of those of degree 4; it is

$$(a_1^2 r t - \beta_1 \gamma s^2)^2.$$

Simpler forms, however, occur for special forms of the functions. Thus, in particular, if the integral equivalent is

$$x = \text{function of } u, \quad y = \text{function of } v,$$

$$h = \text{function of } u + \text{function of } v,$$

the equation reduces to $s = 0,$

as (obviously beforehand) should be the final form.

ON THE REDUCIBILITY OF COVARIANTS OF BINARY
QUANTICS OF INFINITE ORDER

PART III.

By P. W. WOOD.

[Received January 5th, 1907.—Read January 10th, 1907.]

Introduction.

In the first part of this paper* I have shewn that, with the ordinary symbolical notation, the covariant $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ is reducible if, and only if, the coefficient of the term $z^{\lambda_1} z^{\lambda_2} \dots z^{\lambda_\delta}$ in a certain expansion $\phi(z_1, z_2, \dots, z_\delta)$ is zero: here the symbols $a_1, a_2, \dots, a_{\delta+1}$ refer to $(\delta+1)$ distinct binary forms each of infinite order and $\lambda_1 + \lambda_2 + \dots + \lambda_\delta = 2^\delta - 1$, so that the covariant is of minimum weight for irreducibility. The expansion is defined thus:

$$\phi(z_1, z_2, \dots, z_\delta) \equiv \prod_r [z_{r^{(1)}} - z_{r^{(2)}} + z_{r^{(3)}} - \dots + (-)^{\epsilon-1} z_{r^{(\epsilon)}}]$$

where $r^{(1)}, r^{(2)}, \dots, r^{(\epsilon)}$ are any ϵ of the suffixes 1, 2, ..., δ , such that

$$r^{(1)} < r^{(2)} < \dots < r^{(\epsilon)};$$

these suffixes r are to be chosen in all possible ways to satisfy these conditions, and so in all there are $(2^\delta - 1)$ factors each linear in the variables z .

For brevity I shall speak of terms having zero coefficients as zero terms, and the coefficient of the term $z_1^{\lambda_1} z_2^{\lambda_2} \dots z_\delta^{\lambda_\delta}$ in $\phi(z_1, z_2, \dots, z_\delta)$ will be denoted by $[\lambda_1, \lambda_2, \dots, \lambda_\delta]$: of course, $\lambda_1 + \lambda_2 + \dots + \lambda_\delta = 2^\delta - 1$, and any term having the sum of κ indices less than $2^\kappa - 1$ is necessarily a zero term. Clearly $[\lambda_1, \lambda_2, \dots, \lambda_\delta] = -[\lambda_\delta, \lambda_{\delta-1}, \dots, \lambda_1]$, and, if either coefficient is zero, so also is the other: it follows therefore that in enumerating zero terms the indices may be read either backwards or forwards.

I have already† found certain general classes of zero terms, namely:

(i.) if $[\lambda_1, \lambda_2, \dots, \lambda_\epsilon] \dagger = 0$, then also $[\dots, \lambda_1, \dots, \lambda_2, \dots, \lambda_\epsilon, \dots] = 0$;

* *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 2, pp. 352 *et seq.*

† *Ibid.*, pp. 361 *et seq.*

‡ $[\lambda_1, \lambda_2, \dots, \lambda_\epsilon]$ is the coefficient of $z_1^{\lambda_1} z_2^{\lambda_2} \dots z_\epsilon^{\lambda_\epsilon}$ in $\phi(z_1, z_2, \dots, z_\epsilon)$, so that

$$\lambda_1 + \lambda_2 + \dots + \lambda_\epsilon = 2^\epsilon - 1.$$

(ii.) $[\lambda_1, \lambda_2, \dots, \lambda_\delta] = 0$, if $\lambda_{r(1)} + \lambda_{r(2)} + \dots + \lambda_{r(\kappa)} = 2^\kappa - 1$, and the sum of the indices λ is odd in any of the groups into which the indices are divided by the removal of the κ indices $\lambda_{r(1)}, \lambda_{r(2)}, \dots, \lambda_{r(\kappa)}$, whose sum is $2^\kappa - 1$;

(iii.) $[\lambda_1, \lambda_2, \dots, \lambda_{\epsilon-1}, \lambda_\epsilon, \lambda_{\epsilon-1}, \dots, \lambda_2, \lambda_1] = 0$ (i.e., the indices read the same backwards and forwards).

The nature of the first two general classes suggests a further investigation of terms having the sum of κ indices equal to 2^κ or $2^\kappa + 1$; and some of the simpler cases of this kind are considered in the present paper. The treatment does not profess to be exhaustive; its chief interest is to indicate a method of finding new types of zero terms, but a consideration of the zero terms found by direct calculation and given in IV. is enough to shew that there must be a great number of zero terms belonging to none of these types. The labour involved in such direct calculations is so great that any further extension is not contemplated, although every such extension is useful as a possible indication of new general classes of zero terms.

A complete catalogue of known reducible covariants is given in IV.

CONTENTS.

- I., §§ 1-5.—Zero Coefficients $[\lambda_1, \lambda_2, \dots, \lambda_\delta]$, wherein one of the indices is equal to $2^{\delta-1} - 1$.
 II., §§ 6-13.—Zero Coefficients $[\lambda_1, \lambda_2, \dots, \lambda_\delta]$, wherein the sum of any κ of the indices is equal to 2^κ or $2^\kappa + 1$.
 III., § 14.—Zero Coefficients $[2, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$.
 IV., §§ 15, 16.—Tabulation of all known Zero Coefficients and Reducible Covariants.

I. Zero Coefficients $[\lambda_1, \lambda_2, \dots, \lambda_\delta]$; $\lambda_r = 2^{\delta-1} - 1$.

1. If we know the expansion $\phi(z_1, z_2, \dots, z_{\delta-1})$, we can at once write down all the terms of $\phi(z_1, z_2, \dots, z_\delta)$ in which any index is $2^{\delta-1}$, the greatest index possible; for it is clear that the terms of this expansion, in which the index λ_r of z_r is $2^{\delta-1}$, are given by

$$z_r^{2^{\delta-1}} \phi(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_\delta);$$

it is easily verified that the sign is positive, since z_r occurs with a negative sign in $2^{\delta-2}$ factors.

Now let us consider those terms for which $\lambda_r = 2^{\delta-1} - 1$; to obtain such terms, we must, in the product of the $2^{\delta-1} - 1$ factors, choose in all possible ways the variable z_r from all but one of the $2^{\delta-1} - 1$ factors containing z_r . Suppose O_E and O_O to represent factors of $\phi(z_1, z_2, \dots, z_{r-1})$ containing respectively an even or an odd number of the variables $z_1, z_2, \dots, z_{r-1}$, and let O' represent any factor of $\phi(z_{r+1}, z_{r+2}, \dots, z_\delta)$.

Then every factor of $\phi(z_1, z_2, \dots, z_\delta)$ containing z_r is either of the form $O_E + z_r - O'$ or of the form $O_O - z_r + O'$ where O_E and O' may be actually zero. The number of factors O_O or O_E (including zero) is 2^{r-2} , and the number of factors O' (including zero) is $2^{\delta-r}$.

2. We can easily shew that

$$\begin{aligned}\Sigma O_E &= 2^{r-3}(z_1 - z_{r-1}), \\ \Sigma O_O &= 2^{r-3}(z_1 + z_{r-1});\end{aligned}$$

for this is clearly true for two or three variables, and, assuming its truth for $(r-1)$ variables, the corresponding values for r variables are

$$\Sigma O_E + \Sigma(O_O - z_r) = 2^{r-3}(z_1 - z_{r-1}) + 2^{r-3}(z_1 + z_{r-1}) - 2^{r-2}z_r = 2^{r-2}(z_1 - z_r)$$

and

$$\Sigma O_O + \Sigma(O_E + z_r) = 2^{r-3}(z_1 + z_{r-1}) + 2^{r-3}(z_1 - z_{r-1}) + 2^{r-2}z_r = 2^{r-2}(z_1 + z_r)$$

respectively; so that the result is true in general.

It follows at once that $\Sigma O = 2^{r-2}z_1$,

and therefore $\Sigma O' = 2^{\delta-r-1}z_{r+1}$.

3. The terms in $z_r^{2^{\delta-1}-1}$ are

$$z_r^{2^{\delta-1}-1} \Sigma \Sigma \{ (O_E - O') - (O_O + O') \} \phi(z_1, z_2, \dots, z_{r-1}, z_{r+1}, \dots, z_\delta),$$

where the sign $\Sigma \Sigma$ implies that every O_E and every O_O must be taken with every O' ; these terms are

$$\begin{aligned}& z_r^{2^{\delta-1}-1} \{ 2^{\delta-r} \Sigma(O_E - O_O) - 2^{r-1} \Sigma O' \} \phi(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_\delta) \\ &= -z_r^{2^{\delta-1}-1} \{ 2^{\delta-r} 2^{r-2} z_{r-1} + 2^{r-1} 2^{\delta-r-1} z_{r+1} \} \phi(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_\delta) \\ &= -2^{\delta-2} z_r^{2^{\delta-1}-1} (z_{r-1} + z_{r+1}) \phi(z_1, \dots, z_{r-1}, z_{r+1}, \dots, z_\delta).\end{aligned}$$

Cor.—The terms in $z_1^{2^{\delta-1}-1}$ are

$$-2^{\delta-2} z_1^{2^{\delta-1}-1} z_2 \phi(z_2, z_3, \dots, z_\delta),$$

and therefore, if $[\lambda_1, \lambda_2, \dots, \lambda_{\delta-1}] = 0$,

then also $[(2^{\delta-1}-1), \lambda_1+1, \lambda_2, \dots, \lambda_{\delta-1}] = 0$

and $[\lambda_1, \lambda_2, \dots, \lambda_{\delta-1}+1, (2^{\delta-1}-1)] = 0$.

4. Hence, if the coefficients

$$\left. \begin{aligned} & [\lambda_1, \lambda_2, \dots, \lambda_{r-1}-1, \lambda_{r+1}, \dots, \lambda_\delta] \\ & \text{and} \\ & [\lambda_1, \lambda_2, \dots, \lambda_{r-1}, \lambda_{r+1}-1, \dots, \lambda_\delta] \end{aligned} \right\}, (\lambda_1 + \dots + \lambda_{r-1} + \lambda_{r+1} + \dots + \lambda_\delta = 2^{\delta-1}),$$

of $\phi(z_1, z_2, \dots, z_{r-1}, z_{r+1}, \dots, z_\delta)$ are equal and of opposite signs, then

$$[\lambda_1, \lambda_2, \dots, \lambda_{r-1}, (2^{\delta-1}-1), \lambda_{r+1}, \dots, \lambda_\delta] = 0.$$

By means of this we can find, from the tables already determined, new zero terms in expansions containing one more variable, and, in practice, this investigation is simplified by the method used previously for tabulating the coefficients. For instance, take the expansion* of $\phi(z_1, z_2, z_3, z_4)$: from this we can determine the zero terms of $\phi(z_1, z_2, z_3, z_4, z_5)$ involving z_3^{15} by examining all the diagonals from left to right downward and seeking for two adjacent coefficients which are equal and of opposite signs [cf. § 15, (G), (i.), (β)]. Again, to find the zero terms involving z_4^{15} , we need only seek in every column for two adjacent coefficients which are equal and of opposite signs [cf. § 15, (G), (i.), (α)].

From the tables for $z_1 z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4} z_5^{\lambda_5}$ (*loc. cit.*) and $z_1^2 z_2^{\lambda_2} z_3^{\lambda_3} z_4^{\lambda_4} z_5^{\lambda_5}$ (see § 14) we can find all zero coefficients such as $[2, 31, \lambda_3, \lambda_4, \lambda_5, \lambda_6]$ [cf. § 15, (G), (v.), (α)].

5. Many general classes of zero terms having one index $(2^{\delta-1}-1)$ may be constructed; the following will serve as examples:—

(i.) If $[\lambda_1, \dots, \lambda_{r-1}, 1, \lambda_{r+1}, \dots, \lambda_{\delta-1}] = 0$, then also

$$\begin{aligned} &[\lambda_1, \dots, (\lambda_{r-1}+1), (2^{\delta-1}-1), 1, \lambda_{r+1}, \dots, \lambda_{\delta-1}] \\ &= [\lambda_1, \dots, \lambda_{r-1}, 1, (2^{\delta-1}-1), (\lambda_{r+1}+1), \dots, \lambda_{\delta-1}] = 0. \end{aligned}$$

(ii.) We know† that

$$[\lambda_1, \dots, \lambda_{r-2}, 1, 2, \lambda_{r+1}, \dots, \lambda_{\delta-1}] + [\lambda_1, \dots, \lambda_{r-2}, 2, 1, \lambda_{r+1}, \dots, \lambda_{\delta-1}] = 0,$$

and therefore $[\lambda_1, \dots, \lambda_{r-2}, 2, (2^{\delta-1}-1), 2, \lambda_{r+1}, \dots, \lambda_{\delta-1}] = 0$ (see § 6).

(iii.) We know‡ that $[\lambda_1, \lambda_2, \dots, \lambda_{r-1}, 1, \lambda_{r+1}, \dots, \lambda_{\delta-1}] = 0$ if $\lambda_1 + \dots + \lambda_{r-1}$ is odd, and therefore

$$[\lambda_1, \dots, \lambda_{r-1}, 2, 2, \lambda_{r+2}, \dots, \lambda_{\delta-1}] + [\lambda_1, \dots, \lambda_{r-1}, 3, 1, \lambda_{r+2}, \dots, \lambda_{\delta-1}] = 0,$$

since the expansion contains the factor $z_r - z_{r+1}$; so it follows that

$$[\lambda_1, \dots, \lambda_{r-1}, 3, (2^{\delta-1}-1), 2, \lambda_{r+2}, \dots, \lambda_{\delta-1}] = 0$$

if $\lambda_1 + \dots + \lambda_{r-1}$ is odd.

(iv.) It will presently (§ 8) appear that $[2, \lambda_2, 3, 3, \dots] = 0$ if λ_2 is even, and we know† that $[1, \lambda_2, 3, 3, \dots] = 0$ if λ_2 is odd; it follows, therefore, that $[2, (2^{\delta-1}-1), \lambda_3, 3, 3, \dots] = 0$ if λ_3 is odd (see § 10).

* *Proc. London Math. Soc.*, Ser. 2, Vol. 3, p. 318.

† Because $z_{r-1} - z_r$ is a factor of $\phi(z_1, z_2, \dots, z_{\delta-1})$.

‡ Class (ii.) in the Introduction.

(v.) $[1, 4, \lambda_3, 2, \dots] = 0$ if λ_3 is odd, and $^*[2, 3, \lambda_3, 2, \dots] = 0$ whatever λ_3 may be; so $[2, (2^{\delta-1}-1), 4, \lambda_4, 2, \dots] = 0$ if λ_4 is odd. The extension of this last case to a much more general case is obvious.

(vi.) If $[1, \lambda_2, \lambda_3, \dots, \lambda_\delta] = 0$, then $[(2^\delta-1), 2, \lambda_2, \dots, \lambda_\delta] = 0$; but we know that $[2^\delta-1, 1, \lambda_2, \dots, \lambda_\delta] = 0$, and therefore

$$[(2^\delta-1), 2, (2^{\delta+1}-1), (\lambda_2+1), \dots, \lambda_\delta] = 0$$

if, and only if, $[1, \lambda_2, \lambda_3, \dots, \lambda_\delta] = 0$.

It may, in certain cases, be possible to pick out isolated specimens of this kind from known zero terms: for instance, by direct calculation we find that (see catalogue in § 15) $[1, 14, 3, 8, 5] = 0$ and $[1, 13, 4, 8, 5] = 0$; so $[1, 14, 31, 4, 8, 5] = 0$. In the catalogue in § 15 such cases will be omitted: it is only necessary to remember that

$$[\lambda_1, \lambda_2, \dots, \lambda_{r-1}, (2^{\delta-1}-1), \lambda_{r+1}, \dots, \lambda_\delta] = 0$$

if $[\lambda_1, \dots, (\lambda_{r-1}-1), \lambda_{r+1}, \dots, \lambda_\delta] = 0$ and $[\lambda_1, \dots, \lambda_{r-1}, (\lambda_{r+1}-1), \dots, \lambda_\delta] = 0$; and also that $[(2^{\delta-1}-1), \lambda_2, \dots, \lambda_\delta] = 0$ if, and only if,

$$[(\lambda_2-1), \lambda_3, \dots, \lambda_\delta] = 0.$$

II. *Zero Coefficients* $[\lambda_1, \lambda_2, \dots, \lambda_\delta]$; $\lambda_{r(1)} + \lambda_{r(2)} + \dots + \lambda_{r(\kappa)} = 2^\kappa$, or $2^\kappa + 1$.

6. Let us first consider how we may calculate the coefficient $[\lambda_1, \dots, 2, \lambda_r, 2, \dots, \lambda_\delta]$; the method of calculation in this simple case will be explained fully, as it is typical of the methods following.

If we take the three factors $z_{r-1}, z_{r+1}, (z_{r-1}-z_{r+1})$, involving only the variables z_{r-1}, z_{r+1} , we get

$$z_{r-1}^2 z_{r+1} - z_{r-1} z_{r+1}^2.$$

The first term requires one more z_{r+1} to be taken in all possible ways from the $(2^{\delta-1}-2)$ remaining factors of $\phi(z_1, z_2, \dots, z_\delta)$ involving z_{r+1} : any such factor may be written in some one of the following forms:—

$$\begin{aligned} & \left\{ \begin{array}{l} O_E + z_{r-1} - z_r + z_{r+1} - O', \\ O_O - z_{r-1} + z_r - z_{r+1} + O', \end{array} \right. & \left\{ \begin{array}{l} O_E + z_{r-1} - z_{r+1} + O', \\ O_O - z_{r-1} + z_{r+1} - O', \end{array} \right. \\ & \left\{ \begin{array}{l} O_E + z_r - z_{r+1} + O', \\ O_O - z_r + z_{r+1} - O', \end{array} \right. & \left\{ \begin{array}{l} O_E + z_{r+1} - O', \\ O_O - z_{r+1} + O', \end{array} \right. \end{aligned}$$

* Class (i.) in the Introduction.

where O_E and O_0 are made up (as in § 1) from the variables $z_1, z_2, \dots, z_{r-2}$, and O' is made up from the variables $z_{r+2}, \dots, z_\delta$. Now, as far as the first term is concerned, we may put $z_{r-1} = 0$, and therefore the factor from which z_{r+1} is to be chosen may be written

$$\begin{cases} O_E \pm (z_r - z_{r+1} + O'), & \begin{cases} O_E \pm (z_{r+1} - O'), \\ O_0 \pm (z_{r+1} - O'). \end{cases} \\ O_0 \pm (z_r - z_{r+1} + O'), & \end{cases}$$

If we take z_{r+1} from the factor $O_E - z_r + z_{r+1} - O'$, the resulting expression will be

$$z_{r-1}^2 z_{r+1}^2 \left[\frac{\phi(z_1, z_2, \dots, z_\delta)}{\phi(z_{r-1}, z_{r+1})} \right]_{z_{r-1}=0, z_{r+1}=0} \frac{1}{O_E - z_r - O'};$$

and, if we take z_{r+1} from the factor $O_E - z_{r+1} + O'$, the resulting expression will be

$$-z_{r-1}^2 z_{r+1}^2 \left[\frac{\phi(z_1, \dots, z_\delta)}{\phi(z_{r-1}, z_{r+1})} \right]_{z_{r-1}=0, z_{r+1}=0} \frac{1}{O_E + O'}.$$

It will be convenient to denote $\left[\frac{\phi(z_1, \dots, z_\delta)}{\phi(z_{r-1}, z_{r+1})} \right]_{z_{r-1}=0, z_{r+1}=0}$ by $\psi(z_{r-1}, z_{r+1})$:

we know* that $\psi(z_{r-1}, z_{r+1})$ contains only even powers of z_r ; therefore from the first term alone we find that the expression multiplying $z_{r-1}^2 z_{r+1}^2$ is some even function of z_r together with

$$\psi(z_{r-1}, z_{r+1}) \Sigma \Sigma \left[\frac{1}{O_E - z_r - O'} - \frac{1}{O_E + z_r + O'} + \frac{1}{O_0 - z_r - O'} - \frac{1}{O_0 + z_r + O'} \right].$$

In the same way from the second term alone we find for the expression multiplying $-z_{r-1}^2 z_{r+1}^2$ some even function of z_r together with

$$\psi(z_{r-1}, z_{r+1}) \Sigma \Sigma \left[\frac{1}{O_E - z_r - O'} + \frac{1}{O_E - z_r + O'} - \frac{1}{O_0 + z_r - O'} - \frac{1}{O_0 + z_r + O'} \right].$$

Hence in all the expression multiplying $z_{r-1}^2 z_{r+1}^2$ is some even function of z_r together with

$$\psi(z_{r-1}, z_{r+1}) \Sigma \Sigma \left[\frac{1}{O_0 + z_r - O'} + \frac{1}{O_0 - z_r - O'} - \frac{1}{O_E - z_r + O'} - \frac{1}{O_E + z_r + O'} \right],$$

which is itself an even function of z_r .

Hence every coefficient $[\lambda_1, \dots, 2, \lambda_r, 2, \dots, \lambda_\delta]$ is zero, when λ_r is odd.

It should be noted that the method fails in the investigation of coefficients like $[\lambda_1, \dots, 2, \lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_{r+\epsilon}, 2, \dots, \lambda_\delta]$, where several indices $\lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_{r+\epsilon}$ are interpolated between the two indices we are concerned with.

* *Loc. cit.*, p. 363.

7. The reasons for the preceding result may be stated in general terms: in the first place, since we are concerned only with the index of z_r , we may neglect the factors not involving z_r , and moreover we may neglect any factor containing $z_{r-1} - z_r + z_{r+1}$, since it gives the same divisor whether we require an additional z_{r-1} or an additional z_{r+1} , and we require the difference of the quotients. The only factors we are concerned with are therefore those involving $z_{r-1} - z_r$ or $z_r - z_{r+1}$, and the divisors for z_{r-1} differ from those for z_{r+1} only in having their signs changed and $-z_r$ written for z_r ; the result of taking the difference of quotients arising from all such divisors is therefore necessarily an even function of z_r .

It will not be necessary to explain the cases following with the same fullness of detail. We next take the coefficients of terms for which the sum of three indices is 8, excluding the cases where one index is unity or the sum of two indices is 4; the only possible values of the indices are 2, 3, 3. Now we know that any coefficient [... 2 ... 3 ... 2 ...] is zero, since $[2, 3, 2] = 0$, and further the indices may be read backwards or forwards. The only cases to be considered are therefore

$$[\lambda_1, \dots, \lambda_{r-2}, 2, \lambda_r, 3, 3, \dots, \lambda_s],$$

$$[\lambda_1, \dots, \lambda_{r-3}, 2, 3, \lambda_r, 3, \dots, \lambda_s],$$

and

$$[\lambda_1, \dots, \lambda_{r-2}, 2, 3, 3, \lambda_{r+2}, \dots, \lambda_s].$$

8. $[\lambda_1, \dots, \lambda_{r-2}, 2, \lambda_r, 3, 3, \dots, \lambda_s]$.—

The 7 factors in $z_{r-1}, z_{r+1}, z_{r+2}$ give us the two terms

$$2z_{r-1}z_{r+1}^3z_{r+2}^3 - 2z_{r-1}^2z_{r+1}^2z_{r+2}^3.$$

The first term requires one more z_{r-1} , and the second one more z_{r+1} ; obviously we can put $z_{r+2} = 0$ throughout. Moreover in considering our divisors we can neglect all those factors involving $z_{r-1} - z_{r+1} + z_{r+2}$ (as in § 7).

The factors containing z_{r-1} are of the types

$$\begin{cases} O_E + z_{r-1} - z_r \pm O', \\ O_O - z_{r-1} + z_r \pm O', \end{cases}$$

with a similar set obtained by putting $z_r = 0$. The factors containing z_{r+1} are of the types

$$\begin{cases} O_E + z_r - z_{r+1} \pm O', \\ O_O - z_r + z_{r+1} \pm O', \end{cases}$$

with a similar set obtained by changing the sign of z_{r+1} and putting $z_r = 0$.

So the divisors for z_{r-1} are of the types

$$\begin{cases} O_E - z_r \pm O', \\ -O_0 - z_r \pm O', \end{cases} \quad \begin{cases} O_E \pm O', \\ -O_0 \pm O', \end{cases}$$

and those for $-z_{r+1}$ are of the types

$$\begin{cases} O_E + z_r \pm O', \\ -O_0 + z_r \pm O', \end{cases} \quad \begin{cases} -O_E \pm O', \\ +O_0 \pm O'. \end{cases}$$

The same considerations as before shew, without the need of any further calculation, that the expression multiplying $\psi(z_{r-1}, z_{r+1}, z_{r+2})$ is an even function of z_r and also of even total degree in the variables $z_{r+3}, \dots, z_\delta$ involved in the factors O' . Hence the only coefficients of this kind which are *not* zero are those typified by the coefficient

$$[\lambda_1, \dots, 2, \lambda_r, 3, 3, \lambda_{r+3}, \dots, \lambda_\delta],$$

where λ_r and the sum $(\lambda_{r+3} + \lambda_{r+4} + \dots + \lambda_\delta)$ are both even.

9. $[\lambda_1, \dots, \lambda_{r-3}, 2, 3, \lambda_r, 3, \dots, \lambda_\delta]$.—

As before, the 7 factors in $z_{r-2}, z_{r-1}, z_{r+1}$ give the two terms

$$2z_{r-2}z_{r-1}^3z_{r+1}^3 - 2z_{r-2}^2z_{r-1}^2z_{r+1}^3;$$

the factors with which we are concerned are of the types

$$\begin{aligned} \text{(i.)} \quad & \begin{cases} O_E + z_{r-2} - z_{r-1} + z_r \pm O', \\ O_0 - z_{r-2} + z_{r-1} - z_r \pm O'; \end{cases} \\ \text{(ii.)} \quad & \begin{cases} O_E + z_{r-2} - z_r \pm O', \\ O_0 - z_{r-2} + z_r \pm O'; \end{cases} \\ \text{(iii.)} \quad & \begin{cases} O_E + z_{r-1} - z_r \pm O', \\ O_0 - z_{r-1} + z_r \pm O', \end{cases} \end{aligned}$$

with a similar set found by making z_r zero.

It is clear that the divisors obtained for z_{r-2} from set (ii.), and for z_{r-1} from set (iii.), are the same; so that the terms arising from them will vanish in the sequel; and without further calculation it appears that the final result must be an even function of the variables $z_{r+2}, z_{r+3}, \dots, z_\delta$ involved in O' ; and therefore

$$[\lambda_1, \dots, 2, 3, \lambda_r, 3, \dots, \lambda_\delta] = 0, \text{ if } \lambda_{r+2} + \dots + \lambda_\delta \text{ is odd.}$$

Specially we see that

$$[2, 3, \lambda_3, 3, \dots] = 0, \text{ if } \lambda_3 \text{ is even.}$$

10. $[\lambda_1, \dots, 2, 3, 3, \lambda_{r+2}, \dots, \lambda_\delta]$.—

The preceding investigations now shew that also in the present case corresponding to every divisor containing a series of terms O' there is another divisor differing from it only in having $-O'$ instead of O' (this comes about because we put $z_{r+1} = 0$ throughout); and therefore the final expression can only be of even total degree in the variables involved in the terms O' . Hence

$$[\lambda_1, \dots, 2, 3, 3, \lambda_{r+2}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_{r+2} + \lambda_{r+3} + \dots + \lambda_\delta \text{ is odd.}$$

The same method shews further that

$$[\dots, 2, \dots, 3, \dots, 3, \lambda_{r+1}, \lambda_{r+2}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_{r+2} + \lambda_{r+3} + \dots + \lambda_\delta \text{ is odd,}$$

and, in general, if $[\kappa_1, \kappa_2, \kappa_3, \dots, \kappa_\epsilon] = 0$, then* also

$$[\dots, \kappa_1, \dots, \kappa_2, \dots, \kappa_\epsilon + 1, \lambda_{r+1}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_{r+1} + \lambda_{r+2} + \dots + \lambda_\delta \text{ is odd.}$$

11. Using this result, we can, from any zero term in the expansion in ϵ variables, build up an infinite number of new zero terms in the expansions containing $(\epsilon+1)$ or more variables; in passing from ϵ to $\epsilon+1$ variables no new result is obtained; for the new coefficient is necessarily $[\kappa_1, \kappa_2, \dots, (\kappa_\epsilon+1), (2^\epsilon-1)]$, which we know to be zero (§ 3, Cor.).

From the known zero $[3, 1, 3]$ we can deduce

$$[\dots, 3, \dots, 1, \dots, 4, \lambda_{r+1}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_{r+1} + \lambda_{r+2} + \dots + \lambda_\delta \text{ is odd;}$$

and from $[7, 3, 3, 2] = 0$ we deduce

$$[\dots, 7, \dots, 3, \dots, 3, \dots, 3, \lambda_{r+1}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_{r+1} + \dots + \lambda_\delta \text{ is odd;}$$

and so on.

Note that $[2, \lambda_2, 3, \dots] = 0$, if λ_2 is odd (§ 12),

and $[2, \lambda_2, 3, 3, \dots] = 0$, if λ_2 is even (§ 9);

and therefore $[2, \lambda_2, 3, 3, \dots] = 0$, whether λ_2 is odd or even.

In dealing with the case of four indices whose sum is 16 the calculation (except in special cases) becomes more complicated and suggests no new development.

12. Next, let us consider the case where the sum of κ indices is equal to $2^\kappa+1$: even in the simpler cases the amount of algebraical computation

* It is not necessarily true that

$$[\kappa_1, \dots, \kappa_2, \dots, (\kappa_{\epsilon'}+1), \lambda_r, \dots, \lambda_s, \kappa_{\epsilon'}+1, \dots, \kappa_{\epsilon}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_r + \lambda_{r+1} + \dots + \lambda_{s-1} + \lambda_s \text{ is odd.}$$

is by no means inconsiderable. As an illustration of the principle we shall first take the coefficient $[2, \lambda_2, 3, \lambda_4, \dots, \lambda_s]$, and show that it is zero if λ_2 is odd.

The three factors $z_1, z_3, (z_1 - z_3)$ give $z_1^2 z_3 - z_1 z_3^2$; so that the first term requires two more z_3 's, and the second term requires one more z_1 and one more z_3 , from the remaining factors of $\phi(z_1, z_2, \dots, z_s)$.

The remaining factors with which we are concerned are of the following types :—

$$\left. \begin{array}{l} z_1 - z_2 + z_3 \\ z_1 - z_2 + z_3 - O \\ z_1 - z_3 + O \end{array} \right\}, \text{ containing both } z_1 \text{ and } z_3;$$

$$\left. \begin{array}{l} z_1 - z_2 \\ z_1 - z_2 + O \\ z_1 - O \end{array} \right\}, \text{ containing } z_1 \text{ but not } z_3;$$

$$\left. \begin{array}{l} z_2 - z_3 \\ z_2 - z_3 + O \\ z_3 - O \end{array} \right\}, \text{ containing } z_3 \text{ but not } z_1;$$

O being any factor of $\phi(z_4, z_5, \dots, z_s)$.

For simplicity (the principle and the form of the result are not affected) we may take the case of five variables, so that O may be any one of $z_4, z_5, z_4 - z_5$; denote these factors by O_1, O_2, O_3 . In the present case $\frac{\phi(z_1, \dots, z_5)}{\phi(z_1, z_3)}$ has to be divided by pairs of the above factors (with the appropriate signs) and then z_1 and z_3 equated to zero. To obtain one more z_1 we must divide $\psi(z_1, z_3)$ by each of the following :—

$$A \left\{ \begin{array}{ll} -z_2, & -z_2, \\ -z_2 - O_1, & -z_2 + O_1 \\ -z_2 - O_2, & -z_2 + O_2 \\ -z_2 - O_3, & -z_2 + O_3 \\ O_1, & -O_1 \\ O_2, & -O_2 \\ O_3, & -O_3 \end{array} \right\} B;$$

and to obtain one more z_3 we must divide $\psi(z_1, z_3)$ by each of the following :—

$$A \left\{ \begin{array}{ll} -z_2, & -z_2 \\ -z_2 - O_1, & -z_2 - O_1 \\ -z_2 - O_2, & -z_2 - O_2 \\ -z_2 - O_3, & -z_2 - O_3 \\ -O_1, & -O_1 \\ -O_2, & -O_2 \\ -O_3, & -O_3 \end{array} \right\} B.$$

Hence to obtain two more z_3 's we must divide by all possible pairs of the second set, and to obtain one more z_1 and one more z_3 we must divide by all possible pairs one taken from the first set and one taken from the second set, but corresponding divisors in the two columns A must not be taken together, since they both arise from the same factor originally.

After some reduction we find that the terms required are given by

$$z_1^2 z_3^3 \psi(z_1 z_3) \left\{ \frac{2}{z_2^2} - 2 \left(\Sigma \frac{1}{O} \right)^2 + 2 \Sigma \frac{1}{O^2} + 4 \Sigma \frac{1}{O} \Sigma \frac{O}{z_2^2 - O^2} \right. \\ \left. + 4 \Sigma \frac{1}{z_2^2 - O^2} + 2 \Sigma \frac{1}{z_2 + O} \Sigma \frac{1}{z_2 - O} \right\},$$

where O may be any one of $z_4, z_5, z_4 - z_5$.

This expression obviously contains only even powers of z_2 , since it is unaltered on changing the sign of z_2 , and a similar expression can be found in the same way in the case of δ variables.

It may be worthy of remark that neither of the coefficients

$$[3, \lambda_2, 2, \dots], \quad \lambda_2 \text{ odd},$$

$$[\dots, 2, \lambda_r, 3, \dots], \quad \lambda_r \text{ odd},$$

is necessarily zero.

13. Finally, we shall consider coefficients $[1, \lambda_2, 4, 4, \dots]$; the enquiry leads to a negative result, but throws further light on the method.

The seven factors involving z_1, z_3, z_4 give the terms

$$z_1^4 z_3^2 z_4^2 - 2 z_1^3 z_3^3 z_4 + z_1^2 z_3^4 z_4^4.$$

The remaining factors with which we are concerned are (since we may

put $z_1 = 0$) of the following types

$$\left. \begin{array}{l} \pm(z_2 - z_3 + z_4) \\ \pm(z_2 - z_3 + z_4 - O) \\ \pm(z_3 - z_4 + O) \end{array} \right\} \text{containing both } z_3 \text{ and } z_4,$$

$$\left. \begin{array}{l} \pm(z_2 - z_3) \\ \pm(z_2 - z_3 + O) \\ \pm(z_3 - O) \end{array} \right\} \text{containing } z_3 \text{ but not } z_4,$$

$$\left. \begin{array}{l} \pm(z_2 - z_4) \\ \pm(z_2 - z_4 + O) \\ \pm(z_4 - O) \end{array} \right\} \text{containing } z_4 \text{ but not } z_3;$$

O being any factor of $\phi(z_5, z_6, \dots, z_8)$.

For simplicity we take the case of six variables, so that O may be any one of $z_5, z_6, z_5 - z_6$; denote these factors by O_1, O_2, O_3 .

We get the following sets of divisors:—

For one more z_3 —

$$\text{A} \left\{ \begin{array}{ll} -z_2, & -z_2 \\ -z_2 + O_1, & -z_2 - O_1 \\ -z_2 + O_2, & -z_2 - O_2 \\ -z_2 + O_3, & -z_2 - O_3 \\ -z_2, & -z_2 \\ -z_2 + O_1, & -z_2 - O_1 \\ -z_2 + O_2, & -z_2 - O_2 \\ -z_2 + O_3, & -z_2 - O_3 \\ O_1, & -O_1 \\ O_2, & -O_2 \\ O_3, & -O_3 \\ O_1, & -O_1 \\ O_2, & -O_2 \\ O_3, & -O_3 \end{array} \right\} \text{B.}$$

For one more z_4 —

$$\left. \begin{array}{l} z_2, \\ z_2 - O_1, \\ z_2 - O_2, \\ z_2 - O_3, \\ z_2, \\ z_2 - O_1, \\ z_2 - O_2, \\ z_2 - O_3, \\ -O_1, \\ -O_2, \\ -O_3, \\ -O_1, \\ -O_2, \\ -O_3, \end{array} \right\} \begin{array}{l} -z_2 \\ -z_2 - O_1 \\ -z_2 - O_2 \\ -z_2 - O_3 \\ -z_2 \\ -z_2 - O_1 \\ -z_2 - O_2 \\ -z_2 - O_3 \\ -O_1 \\ -O_2 \\ -O_3 \\ -O_1 \\ -O_2 \\ -O_3 \end{array} \right\} \text{B.}$$

We must divide $\psi(z_1, z_3, z_4)$ by the products in pairs of the first set, and by the products in pairs of the second set; from the sum of these two results we must subtract twice the expression found by dividing $\psi(z_1, z_3, z_4)$ by the product in pairs, one from the first set and one from the second set, with the same restriction as before, that corresponding divisors in the two columns A must not be taken together (*cf.* § 12).

After some reduction the terms required can be written in the following form:—

$$\begin{aligned} z_1 z_3^4 z_4^4 \psi(z_1 z_3 z_4) & \left\{ 6\Sigma \left\{ \frac{1}{(z_2 - O)^2} - \frac{1}{(z_2 + O)^2} \right\} + 16\Sigma\Sigma \frac{1}{OO'} - \frac{64}{z_2} \Sigma \frac{1}{O} \right. \\ & - 8\Sigma\Sigma \left\{ \frac{1}{(z_2 + O)(z_2 + O')} - \frac{1}{(z_2 - O)(z_2 - O')} + \frac{1}{O(z_2 + O')} - \frac{1}{O(z_2 - O')} \right\} \\ & \left. + 16\Sigma\Sigma \left[\frac{1}{z_2 - O} \frac{1}{z_2 - O'} - \frac{1}{O(z_2 - O')} \right] \right\}, \end{aligned}$$

where O and O' are any two of $z_5, z_6, z_5 - z_6$ and are never equal. It will be seen that, except for the terms in square brackets [...], the whole expression changes sign with z_2 , and therefore, except for these terms, involves only odd powers of z_2 .

In the case of 5 variables the terms in [...] do not arise, and the whole expression becomes

$$z_1 z_3^4 z_4^4 \psi(z_1 z_3 z_4) \left\{ \frac{6}{(z_2 - z_5)^2} - \frac{6}{(z_2 + z_5)^2} - \frac{64}{z_2 z_5} \right\},$$

so that

$$[1, \lambda_2, 4, 4, \lambda_5] = 0, \text{ if } \lambda_2 \text{ is even,}$$

as is verified by previous calculation, but it is not necessarily true in general that

$$[1, \lambda_2, 4, 4, \dots] = 0, \text{ if } \lambda_2 \text{ is even.}$$

III. Zero Coefficients $[2, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$.

14. This complete expansion has been calculated from the known expansion $[\phi(z_2, z_3, z_4, z_5)]^2$ by the following means.

We have to choose z_1 in all possible ways from two of the factors of $\phi(z_1, \dots, z_5)$; one factor is necessarily z_1 itself, and, since any other factor involving z_1 is of the form $z_1 - O$, where O is one of the 15 factors of $\phi(z_2, z_3, z_4, z_5)$, the terms required are found by dividing $[\phi(z_2, z_3, z_4, z_5)]^2$ by each of its 15 linear factors in turn, and adding the results obtained.

The simplest way of performing these divisions is:—

(i.) To differentiate $[\phi(z_2, z_3, z_4, z_5)]^2$ with respect to z_2 : this is easily performed by virtue of the scheme of tabulation for the coefficients. The result is to give twice the sum of the quotients arising from division by each of the eight factors involving z_2 .

(ii.) To perform the divisions by the remaining seven factors synthetically upon the tables as they stand.

If the final result is tabulated in the same way as before, the same tests for accuracy may be applied; these have actually been applied and the accuracy of the expansion has been otherwise verified.

The expansion thus obtained has been made use of in accordance with § 4 to find other new reducibles; the actual coefficients are not tabulated here, but those zero terms not included in any known general class are catalogued in § 15.

The same method of calculation is not applicable to the determination of the coefficients $[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$ when λ_2, λ_3 , or λ_4 is 2.

IV. Tabulation of all known Zero Terms (Reducible Covariants).

15. The following catalogue gives a summary of the results obtained in this and the two preceding papers: it contains all the known general classes and all the isolated reducible forms (found by actual calculation)

determined up to the present. For convenience of reference the indices only of the isolated forms are written: thus 2, 15, 5, 7, 2 is an abbreviated method of writing $(a_1 a_2)^2 (a_2 a_3)^{15} (a_3 a_4)^5 (a_4 a_5)^7 (a_5 a_6)^2$. In every case the indices can be read backwards or forwards.

*General Classes.**—Typical covariant

$$C_{\delta+1} \equiv (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}.$$

$[\lambda_1, \lambda_2, \dots, \lambda_\delta] = 0$ implies that $C_{\delta+1}$ is reducible.

$$(A) \lambda_{r(1)} + \lambda_{r(2)} + \dots + \lambda_{r(\kappa)} = 2^\kappa - 1; \quad r^{(1)} < r^{(2)} < \dots < r^{(\kappa)}.$$

$$[\lambda_1, \lambda_2, \dots, \lambda_\delta] = 0, \text{ if}$$

$$(i.) [\lambda_{r(1)}, \lambda_{r(2)}, \dots, \lambda_{r(\kappa)}] = 0.$$

(ii.) The sum of the indices is odd in any of the groups into which the indices are divided by the removal of $\lambda_{r(1)}, \lambda_{r(2)}, \dots, \lambda_{r(\kappa)}$.

$$(B) \lambda_{r(1)} + \lambda_{r(2)} + \dots + \lambda_{r(\kappa)} = 2^\kappa.$$

$$(a) [\dots, \mu_1, \dots, \mu_2, \dots, (\mu_\epsilon + 1), \lambda_{r+1}, \dots, \lambda_\delta] = 0, \text{ if}$$

$$[\mu_1, \mu_2, \dots, \mu_\epsilon] = 0,$$

and also $\lambda_{r+1} + \lambda_{r+2} + \dots + \lambda_\delta$ is odd.

$$(\beta) \text{ Sum of two indices} = 4.$$

$$[\lambda_1, \dots, \lambda_{r-2}, 2, \lambda_r, 2, \lambda_{r+2}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_r \text{ is odd.}$$

$$(\gamma) \text{ Sum of three indices} = 8.$$

$$[\lambda_1, \dots, \lambda_{r-2}, 2, \lambda_r, 3, 3, \lambda_{r+1}, \dots, \lambda_\delta] = 0, \text{ if } \lambda_r \text{ is odd.}$$

Specially $[2, \lambda_2, 3, 3, \dots] = 0$, whatever λ_2 may be.

$$(C) \lambda_{r(1)} + \lambda_{r(2)} + \dots + \lambda_{r(\kappa)} = 2^\kappa + 1.$$

$$(a) \text{ Sum of two indices} = 5.$$

$$[2, \lambda_2, 3, \lambda_4, \dots, \lambda_\delta] = 0, \text{ if } \lambda_2 \text{ is odd.}$$

N.B.— $[3, \lambda_2, 2, \lambda_4, \dots, \lambda_\delta]$ is not necessarily zero, if λ_2 is odd.

$$(D) [\lambda_1, \lambda_2, \dots, \lambda_\delta] = 0, \text{ if } \delta = 2\epsilon - 1, \text{ and}$$

$$\lambda_1 = \lambda_{2\epsilon-1}, \lambda_2 = \lambda_{2\epsilon-2}, \dots, \lambda_{\epsilon-1} = \lambda_{\epsilon+1},$$

so that the indices read the same backwards and forwards.

* I have not thought it necessary to point out again that no term can have the sum of any κ of its indices less than $2^\kappa - 1$.

Special Reducible Covariants (not included in the preceding general classes).

(E) Degree 5.—
 1, 6, 4, 4,
 1, 4, 6, 6.

(F) Degree 6.— $C_6 = (a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} (a_3 a_4)^{\lambda_3} (a_4 a_5)^{\lambda_4} (a_5 a_6)^{\lambda_5}$.

(i.) $\lambda_1 = 1$.—

	reversible	
1, 14,	3, 8, 5	
1, 13,	4, 8, 5	
1, 6,	7, 13, 4	
1, 5,	8, 13, 4	
	reversible	

(The last four indices may be read either backwards or forwards.*)

1, λ_2 , 4, 4, λ_5 λ_2 and λ_5 both even.

(ii.) $\lambda_2 = 1$.—

14, 1, 5, 8, 3
 14, 1, 3, 8, 5

(iii.) $\lambda_3 = 1$.—None.

(iv.) $\lambda_1 = 2$.—

2, 15, 5, 7, 2
 2, 14, 2, 8, 5
 2, 13, 9, 3, 4
 2, 13, 6, 7, 3
 2, 13, 5, 9, 2
 2, 10, 3, 11, 5
 2, 6, 6, 3, 14
 2, 3, 6, 15, 5

(G) $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$.—One index is $2^{\delta-1}-1$ (see also §§ 5 and 16).

(i.) $\delta = 5$ (one index 15).

(a) $\lambda_2 = 15$ or $\lambda_4 = 15$.—

3, 15, 3, 7 3.

* Each sequence of five indices can also be read backwards or forwards, so that the set above includes $[5, 8, 13, 4, 1] = 0, \dots$

(β) $\lambda_3 = 15$.—

$[\lambda_1, \lambda_2, 15, \lambda_4, \lambda_5] = 0$, if $\lambda_1 + \lambda_2$ is even except in the case where λ_1 is either 1 or 2, or λ_5 is either 1 or 2. (But even in this case there is one reducible under class (D), viz.,

$$[2, 6, 15, 6, 2] = 0.)$$

(ii.) $\delta = 6$ (one index 31), $\lambda_1 = 1$.

(a) $\lambda_3 = 31$ or $\lambda_5 = 31$.—

reversible					
1,	13,	31,	11,	5,	2
1,	13,	31,	11,	4,	3
1,	13,	31,	11,	3,	4
1,	11,	31,	12,	4,	4
1,	11,	31,	8,	8,	4
1,	9,	31,	4,	5,	13
1,	7,	31,	6,	15,	3
1,	6,	31,	10,	13,	2
1,	6,	31,	8,	13,	4
1,	5,	31,	16,	6,	4
1,	3,	31,	6,	15,	7
reversible					

(The last five indices can be read either backwards or forwards.*)

(β) $\lambda_4 = 31$.—

reversible					
1,	14,	7,	31,	6,	4
1,	13,	10,	31,	6,	2
1,	5,	11,	31,	13,	2
reversible					

(The last five indices can be read either backwards or forwards.*)

* Each sequence of six indices can also be read backwards or forwards, so that the sets above include
and
 $[6, 31, 8, 13, 4, 1] = 0, \dots$
 $[5, 11, 31, 13, 2, 1] = 0, \dots$

(iii.) $\delta = 6$ (one index 31), $\lambda_2 = 1$.

(a) $\lambda_4 = 31$ or $\lambda_5 = 31$.—(Other cases by inspection; v. § 16.)

reversible

10, 1, 11, 31, 5, 5
4, 1, 13, 31, 11, 3
4, 1, 5, 31, 9, 13
2, 1, 13, 31, 11, 5
2, 1, 6, 31, 10, 13

(The last four indices
can be read either
backwards or for-
wards.*)

reversible

(iv.) $\delta = 6$ (one index 31), $\lambda_3 = \frac{1}{2}1$.—None.

(v.) $\delta = 6$ (one index 31), $\lambda_1 = 2$.

(a) $\lambda_2 = 31$.—

(a) 2, 31, 15, 3, 5, 7
2, 31, 15, 3, 4, 8
2, 31, 14, 3, 8, 5
2, 31, 6, 6, 15, 3
2, 31, 6, 4, 13, 7
2, 31, 5, 16, 7, 2
2, 31, 5, 7, 16, 2
2, 31, 5, 5, 7, 13

(b) $[2, 31, \lambda_3, 15, \lambda_5, \lambda_6] = 0$, if λ_3 is odd and λ_6 is neither 1 nor 2.

(c) $[2, 31, 5, \lambda_4, 3, \lambda_6] = 0$, if λ_4 is odd.

(d) $[2, 31, 5, \lambda_4, 2, \lambda_6] = 0$, if λ_4 is odd.

(β) $\lambda = 31$.—

(a) 2, 14, 31, 7, 3, 6
2, 14, 31, 3, 8, 5
2, 11, 31, 13, 4, 2
2, 11, 31, 3, 11, 5
2, 7, 31, 7, 13, 3
2, 6, 31, 16, 5, 3
2, 6, 31, 6, 16, 2
2, 3, 31, 14, 10, 3
2, 2, 31, 6, 15, 7

* Each sequence of six indices can also be read backwards or forwards, so that the above set includes $[13, 31, 11, 5, 1, 2] = 0, \dots$

$$(b) [2, \lambda_2, 31, \lambda_4, 3, \lambda_6] = 0, \text{ if } \lambda_2 = \lambda_4.$$

$$(c) [2, \lambda_2, 31, \lambda_4, 2, \lambda_6] = 0, \text{ if } \lambda_2 = \lambda_4.$$

$$(\gamma) \lambda_4 = 31. —$$

$$(a) \quad \begin{array}{cccccc} 2, & 8, & 4, & 31, & 4, & 14 \\ 2, & 3, & 11, & 31, & 13, & 3 \\ 2, & 3, & 6, & 31, & 6, & 15 \\ 2, & 3, & 6, & 31, & 5, & 16 \end{array}$$

$$(b) [2, 15, \lambda_3, 31, \lambda_5, \lambda_6] = 0, \text{ if } \lambda_3 \text{ is odd and } \lambda_6 \text{ is neither 1 nor 2.}$$

$$(c) [2, \lambda_2, 5, 31, 3, \lambda_6] = 0, \text{ if } \lambda_2 \text{ is odd.}$$

$$(d) [2, \lambda_2, 5, 31, 2, \lambda_6] = 0, \text{ if } \lambda_2 \text{ is odd.}$$

$$(\delta) \lambda_5 = 31. —$$

$$\begin{array}{cccccc} 2, & 15, & 6, & 6, & 31, & 3 \\ 2, & 8, & 14, & 4, & 31, & 4 \\ 2, & 7, & 13, & 4, & 31, & 6 \\ 2, & 7, & 4, & 15, & 31, & 4 \\ 2, & 7, & 4, & 4, & 31, & 15 \\ 2, & 5, & 13, & 5, & 31, & 7 \\ 2, & 4, & 2, & 11, & 31, & 13 \\ 2, & 2, & 15, & 6, & 31, & 7 \\ 2, & 2, & 4, & 11, & 31, & 13 \end{array}$$

16. No zero term occurs twice in the preceding list; if any zero term falls under two headings, it is introduced under one only: *e.g.*, $[2, 15, 5, 7, 2]$ does not appear in (G), (i.), (a), since it appears in (F), (iv.).

Moreover, in accordance with the observation of § 5, a certain number of reducibles have been omitted from the list (G), but all such are easily determined by inspection. The following are examples of such obvious omissions:—

$$(i.) [1, 16, 31, 7, 4, 4] = 0, \text{ since}$$

$$[1, 15, 7, 4, 4] = 0 \quad \text{and} \quad [1, 16, 6, 4, 4] = 0,$$

both the latter coefficients being zero because $[1, 6, 4, 4] = 0$.

$$(ii.) [2, 31, 15, 2, 8, 5] = 0, \text{ since}$$

$$[1, 15, 2, 8, 5] = 0 \quad \text{and} \quad [2, 14, 2, 8, 5] = 0.$$

(iii.) $[2, 16, 5, 3, 31, 6] = 0$, since

$$[2, 16, 5, 3, 5] = 0 \quad \text{and} \quad [2, 16, 5, 2, 6] = 0,$$

both the latter coefficients being zero, since $[2, 5, 3, 5] = 0$ by (C), and $[2, 5, 2, 6] = 0$ by (B), (β).

(iv.) $[1, 5, 8, 4, 31, 14] = 0$, since

$$[1, 5, 8, 3, 14] = 0 \quad \text{and} \quad [1, 5, 8, 4, 13] = 0 \text{ by (F), (i).}$$

(v.) $[2, 16, 3, 6, 31, 5] = 0$, for $[2, 3, 6, 15, 5] = 0$ [see (F), (iv.)], implies necessarily $[2, 3, 5, 5] + [2, 3, 6, 4] = 0$, and therefore $[2, 16, 3, 5, 5] + [2, 16, 3, 6, 4] = 0$.

It may be assumed in general that, if any such coefficient is zero, it is either included in the list or comes under the observation of § 5, taken in conjunction with the general classes and the special zeros catalogued.

The catalogue will therefore give, besides general classes, definite information as to the reducibility of any covariant of one of the following kinds:—

(1) Degree 3, 4, or 5.

(2) Degree 6, if any index is 1 or 15, or if the first or last index is 2.

(3) Degree 7, if any index is 31, and also

(i.) any other index is 1,

or (ii.) the first or last index is 2.

[Note added April 10th, 1907.—

All the covariants considered above are of the minimum weight for irreducibility: that is the covariants involving the $(\delta+1)$ symbols $a_1, a_2, \dots, a_{\delta+1}$ are of weight $2^\delta - 1$. It is easily seen that the conditions for the reducibility of the covariant $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$ of weight 2^δ are

$$[\lambda_1 - 1, \lambda_2, \dots, \lambda_\delta] = [\lambda_1, \lambda_2 - 1, \dots, \lambda_\delta] = \dots = [\lambda_1, \lambda_2, \dots, \lambda_\delta - 1] = 0.$$

These δ conditions are certainly satisfied if $[\lambda_{r^{(1)}}, \lambda_{r^{(2)}}, \dots, \lambda_{r^{(\epsilon)}}] = 0$, where $0 < r^{(1)} < r^{(2)} < \dots < r^{(\epsilon)} \leq \delta$, and this last condition is also found to be necessary in all the cases for which the tables have been actually calculated. The following general result is therefore suggested:—

The covariant, $(a_1 a_2)^{\lambda_1} (a_2 a_3)^{\lambda_2} \dots (a_\delta a_{\delta+1})^{\lambda_\delta}$, of weight greater than $2^\delta - 1$, is reducible if, and only if, the covariant

$$(b_1 b_2)^{\lambda_{r^{(1)}}} (b_2 b_3)^{\lambda_{r^{(2)}}} \dots (b_\epsilon b_{\epsilon+1})^{\lambda_{r^{(\epsilon)}}},$$

of weight $2^\epsilon - 1$, is reducible, where $0 < r^{(1)} < r^{(2)} < \dots < r^{(\epsilon)} \leq \delta$.]

ON THE SINGULARITIES OF FUNCTIONS DEFINED BY
TAYLOR'S SERIES (*Remarks in addition to a former Paper*)

By G. H. HARDY.

[Received and Read January 10th, 1907.]

1. The following note is a result of some remarks made to me by Prof. Bromwich. Its object is to indicate some extensions and further applications of the method which I used in my paper "A Method &c." (*Proceedings*, Vol. 3, pp. 381-9). This method is an adaptation of an idea due originally to Hadamard (*Journal de Math.*, 1893), who used it to establish certain general conclusions since made more precise by Le Roy.* In my paper I introduced certain loop integrals which enable us to obtain information in some respects more general and in others more precise than that given by the line integrals of the two writers just quoted.

2. Consider the series

$$f(x) = a_0 b_0 + a_1 b_1 x + a_2 b_2 x^2 + \dots$$

where a_n is defined as in my former paper, and

$$F(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

is a function of x , possibly many-valued, but having a branch (the *principal branch*) which is regular all over the plane with the exception of $x = 1$ and $x = \infty$, and one-valued inside the domain T formed by slitting the plane along the line $(1, \infty)$. Thus $F(x)$ might be

$$(1-x)^{-\delta}, \quad x^{-1} \log(1-x), \quad F(a, b, c, x).$$

Then, using the notation and arguments of my former paper, but writing for shortness

$$\Phi(u) = (\log u)^{a-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u),$$

we obtain the equation

$$(1) \quad f(x) = \frac{1}{2i \sin(a+\beta)\pi} \int_C \Phi(u) F(xu) du;$$

unless $a+\beta$ is an integer k , in which case

$$(1a) \quad f(x) = \frac{(-)^k}{2\pi i} \int_C \Phi(u) F(xu) \log(u-1) du.$$

* *Annales de la Faculté des Sciences de Toulouse*, 1900.

In particular, if $\alpha = \beta = \gamma = 1$, so that $\Phi(u) = \phi(u)$ is regular in a domain including $(0, 1)$ in its interior,

$$(1b) \quad f(x) = \frac{1}{2\pi i} \int_C \phi(u) F(xu) \log(u-1) du.$$

This expression gives the analytic continuation of $f(x)$ all over the region T bounded by a slit along $(1, \infty)$. The branch of $f(x)$ thus defined is called the *principal branch*.

By taking as our fundamental contour C , not a loop including the line $(0, 1)$, but a loop including some other standard path from 0 to 1, we can replace T by another region T' bounded by a different slit from 1 to ∞ . We infer that the only finite singularity of the principal branch of $f(x)$ is $x = 1$: or, as we may say, $x = 1$ is the *only finite principal singularity* of $f(x)$.

The argument is easily extended to meet the case in which $F(x)$ has any number of singular points. We define a uniform *principal branch* of $f(x)$ by appropriate slits from its singular points to ∞ , and we infer that a corresponding principal branch of $f(x)$ exists, and that *the only principal singularities of $f(x)$ are those of $F(x)$* .

We have now to consider the *subsidiary* singular points of $f(x)$, i.e. those singular points which are not singular for the principal branch. Let us suppose first that $F(x)$ has a simple pole at $x = 1$. This is equivalent to taking

$$F(xu) = 1/(1-xu).$$

Le Roy considers this case in some detail. Applying to his fundamental line-integral Hermite's methods of dealing with *intégrales à coupures*, he proceeds to calculate the increment of $F(x)$ when x moves m times round $x = 1$, and gives the result

$$2m\pi i \frac{1}{x} \Phi\left(\frac{1}{x}\right).$$

This result is true only if $\Phi(u)$ is one-valued for circuits round $u = 1$, i.e., if $\alpha + \beta$ is an integer. It then follows at once by applying the transformation of my former paper to the integral (1a): we deduce that, if $\bar{f}(x)$ is the principal branch of $f(x)$,

$$\bar{f}(x) + \frac{(-)^k}{x} \Phi\left(\frac{1}{x}\right) \log\left(\frac{1}{x} - 1\right)$$

is regular near $x = 1$. In the more general case the increment is

$$\frac{2\pi i}{x} \left\{ \Phi_1\left(\frac{1}{x}\right) + \dots + \Phi_m\left(\frac{1}{x}\right) \right\}$$

where the suffices denote different values of Φ . The difference is most easily grasped by considering such a case as

$$f(x) = \frac{1}{2} \left\{ \log \left(\frac{1}{1-x} \right) \right\}^2 = \sum_0^{\infty} \left(1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \frac{x^{n+1}}{n+1}.$$

This point does not, however, affect the general conclusions which can be drawn. For, in any case, it follows that the only subsidiary singularities of $f(x)$ are given by

$$x = 0, \quad x = 1/\xi$$

where ξ is a singularity of Φ . Hence the singularities of $f(x)$ are $1, \infty$ (*principal*) and $0, 1/\xi$ (*subsidiary*), ξ being now any singularity of ϕ .

When F has a variety of simple poles $x = X$, the singularities of f are X, ∞ and $0, X/\xi$. The same conclusion holds when F has multiple poles or essential singularities near which it is one-valued. The treatment of such cases (as Le Roy shows) introduces no new difficulties of principle. When, however, $F(x)$ is many-valued (say at $x = 1$), the "increment" cannot be so expressed in so simple a form. In fact, as I shall show shortly, we can define another function $g(x)$ such that $f(x) + g(x)$ is one-valued near $x = 1$, but the complete determination of the singular points of $g(x)$ is, in general, a problem just as difficult as the original problem. And similarly if (as we may do) we calculate directly from Le Roy's line-integral the increment of $f(x)$ corresponding to a circuit round $x = 1$.

Le Roy, curiously enough, passes over this difficulty in silence, merely remarking that the result proved in the simpler case is true in general. I do not myself see how to effect the extension directly; but we can deduce the required result from Hadamard's "multiplication theorem," that the only singular points of $\sum a_n b_n x^n$ are $0, \alpha\beta$ where α is a singular point of $\sum a_n x^n$ and β one of $\sum b_n x^n$ (0 , of course, can only be a subsidiary singularity). We thus obtain Le Roy's theorem that the only singular points of $f(x)$ are

$$X, \infty; \quad 0, X/\xi.*$$

* It should be noted that the point $x = 1$ may be for the other branches of $f(x)$ a singularity of a kind quite different from what it is for the principal branch. For example, if

$$\alpha = \beta = \gamma = 1$$

and

$$\Phi(u) = \exp \{ 1/\sqrt{(2-u)+1} \},$$

the principal branch of $f(x)$ has an ordinary logarithmic singularity at $x = 1$. The point $x = \frac{1}{2}$ is a subsidiary singularity. If we encircle first 1 and then $\frac{1}{2}$ (as in the figure), we return to 1 with a value of $f(x)$ which has an exponentially essential singularity there. A discussion of the difficulties which are involved in the extension of Hadamard's theorem to subsidiary singularities will be found in a recent paper by O. Faber (*Jahresbericht der Deutschen Math. Ver.*, 1907, Bd. xvi., p. 285).



3. The preceding discussion does not contain anything really novel. I have included it because (1) the use of loop integrals instead of line integrals enables us to reach somewhat more general conclusions (the reduction of a loop to a line integral being only possible when the constants $\alpha, \beta, \dots$ satisfy certain quite irrelevant inequalities); (2) there are several points which Le Roy passes over rather hastily; and (3) some preliminary discussion is needed in order to make intelligible the later sections of this note and the longer paper which follows.

4. I now proceed to consider the application of the method of my former paper to the case in which $F(x)$ is many-valued. The typical case is that in which $F(x) = (1-x)^{-\delta}$ where δ is not an integer.

We introduce the contour C' of my former paper, and either the contour C_1 shown in Fig. 1 or the contour C_2 shown in Fig. 2.

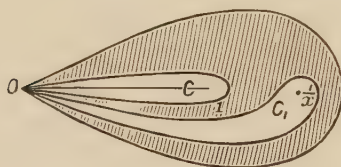


FIG. 1.

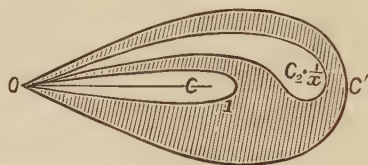


FIG. 2.

In either case the subject of integration is one-valued and regular within the area bounded by the the three loops (and shaded). Also

$\int_{C'}$ is regular near $x = 1$. Hence (Fig. 1)

$$\int_C = \int_{C'} - \int_{C_1}$$

and

$$f(x) + \frac{1}{2i \sin(\alpha + \beta)\pi} \int_{C_1} \Phi(x) \frac{du}{(1-xu)^\delta}$$

is regular near $x = 1$. When $\delta = 1$, we can calculate the last integral in finite terms; and so we arrive at the results of my former paper.

It may be observed that the initial and final values near the origin of the factors $(\log u)^{\alpha-1}$, $(u-1)^{\beta-1}$ are not the same as in the original integral. It is easy to see that

$$f(x) + g(x)$$

is regular where

$$g(x) = \frac{e^{(\delta-\alpha-\beta)\pi i}}{2i \sin(\alpha + \beta)\pi} \int_{C_1} \left(\log \frac{1}{u}\right)^{\alpha-1} (1-u)^{\beta-1} \frac{u^{\gamma-1} \phi(u) du}{(xu-1)^\delta}.$$

Here the initial and final values of $(1-u)^{\beta-1}$ are each unity, and

$$\begin{aligned} \left(\log \frac{1}{u}\right)^{\alpha-1} &= e^{(\alpha-1)\log \log 1/u} \\ &= \exp(\alpha-1) \left[\log \sqrt{\left\{ \left(\log \frac{1}{r}\right)^2 + \theta^2 \right\}} + i \tan^{-1} \left\{ \theta / \left(\log \frac{1}{r}\right) \right\} \right], \end{aligned}$$

the inverse tangent being initially and finally very small ; also $(xu-1)^\delta$ is $\exp \{ \delta \log (xu-1) \}$, $\log (xu-1)$ being real at the point of C_1 for which $xu > 1$.

Similarly, from Fig 2, we obtain

$$g(x) = \frac{e^{-(\delta-\alpha-\beta)\pi i}}{2i \sin (\alpha+\beta)\pi} \int_{C_2} \left(\log \frac{1}{u}\right)^{\alpha-1} (1-u)^{\beta-1} \frac{u^{\gamma-1} \phi(u) du}{(xu-1)^\delta}.$$

It is convenient to take the first or second form of $g(x)$ according as $R(x/i) \leq 0$: we can then suppose C_1 (or C_2) to be a loop closely surrounding the line $(0, 1/x)$.

If $F(x) = \psi(x)/(1-x)^\delta$, ψ being regular near $x = 1$, we have only to include an additional term $\psi(xu)$ under the sign of integration in either form of $g(x)$.

5. Suppose, *e.g.*, that $\alpha = 1$. We may write the above equations in the form

$$g(x) = K_1 x^{-\gamma} \int_{\Gamma_1} \left(1-\frac{u}{x}\right)^{\beta-1} \phi\left(\frac{u}{x}\right) \frac{u^{\gamma-1} \psi(u) du}{(u-1)^\delta}$$

(or a similar equation with K_2, Γ_2), the contour now being a loop surrounding $(0, 1)$, and the choice being determined by the relative situation of the points $0, 1, x$.

Then, if $|x| > 1$, we can expand $g(x)$ in the form

$$Kx^{-\gamma} \sum \frac{c_n d_n}{x^n}$$

where c_n is the coefficient of ξ^n in the expansion of

$$(1-\xi)^{\beta-1} \phi(\xi)$$

and

$$d_n = \int \frac{u^{\gamma+n-1} \psi(u) du}{(u-1)^\delta}.$$

We thus define the nature of the singularity of $f(x)$, but not as before in finite terms, the result being that

$$f(x) + x^{-\gamma} \chi\left(\frac{1}{x}\right)$$

is regular near $x = 1$. For example, if ϕ and ψ are each identically unity, we obtain a relation of this kind connecting two hypergeometric

series. Relations of this kind are of common occurrence in the theory of many-valued functions defined by Taylor's series.

But one must conclude that relations as simple as those found in my former paper for series such as

$$\sum \frac{x^n}{(\gamma+n)^\alpha}, \quad \sum \frac{x^n}{1+(n+1)^2}, \quad \dots$$

do not exist for series such as

$$\sum \frac{\delta(\delta+1) \dots (\delta+n-1)}{1 \cdot 2 \dots n} \frac{x^n}{(\gamma+n)^\alpha}$$

(or even series as simple as the ordinary hypergeometric series). All that can be done is to investigate relations such as those indicated in this paragraph, which will in general involve a second transcendent of at least equal complexity; or to obtain *asymptotic* formulæ valid near $x = 1$. This is much easier: for example, we can prove that

$$\begin{aligned} \sum \frac{\Gamma(\delta_1+n) \Gamma(\delta_2+n) \dots \Gamma(\delta_k+n)}{\Gamma(\epsilon_1+n) \Gamma(\epsilon_2+n) \dots \Gamma(\epsilon_k+n)} \frac{x^n}{(\gamma_1+n)^{\alpha_1} (\gamma_2+n)^{\alpha_2} \dots (\gamma_l+n)^{\alpha_l}} \\ = \frac{\Gamma(1+\sum \delta - \sum \epsilon - \sum \alpha)}{(1-x)^{1+\sum \delta - \sum \epsilon - \sum \alpha}} (1+\eta_x), \end{aligned}$$

where $\lim \eta_x = 0$ for any manner of approach of x to $x = 1$. In so far as modes of approach from *within* the circle of convergence are concerned this follows as a mere corollary from Pringsheim's generalisations of Appell's theorem (*Acta Math.*, t. xxviii., p. 1). To show that it is valid all round the singular point, some method depending on a formula giving the analytic continuation of the function is of course essential. A modification of the method of my former paper leads readily to the result, which I need only mention at present.

6. The method of my former paper is capable of interesting extensions in a different direction. Let us consider the more general contour integral

$$f(x) = \frac{1}{2i \sin(\alpha+\beta)\pi} \int_C \Phi(u) \frac{du}{1-x\psi(u)}$$

where $\psi(u)$ is regular in a domain including $(0, 1)$ in its interior. For sufficiently small values of x ,

$$f(x) = \sum a_n x^n$$

where

$$a_n = \frac{1}{2i \sin(\alpha+\beta)\pi} \int_C \Phi(u) \{\psi(u)\}^n du.$$

The contour integral gives the analytical continuation of the series over the domain bounded by a certain cut. This cut contains those values of x given by

$$x = 1/\psi(u) \quad (0 < u < 1).$$

By varying the path from 0 to 1 included in the loop we vary the cut. The singular points of $f(x)$ can only be found among points excluded from the domain of regularity of $f(x)$ by *all possible* positions of the cut—as, e.g., $x = 1$, $x = \infty$, when $\psi(u) \equiv u$.

Suppose, to take a definite case,

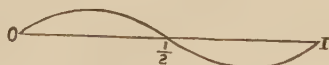
$$\psi(u) \equiv u(1-u).$$

When the path from 0 to 1 is along the real axis the cut begins at $+\infty$, goes along the real axis to $x = 4$, and returns to $+\infty$.

If $u = re^{i\theta}$, $1/u(1-u) = \xi + i\eta$, where

$$\xi = \frac{\cos \theta - r \cos 2\theta}{r(1 - 2r \cos \theta + r^2)}, \quad \eta = -\frac{\sin \theta - r \sin 2\theta}{r(1 - 2r \cos \theta + r^2)}.$$

Thus $\eta = 0$ (i.) if u is real and (ii.) if $r \cos \theta = \frac{1}{2}$, i.e., when the path from 0 to 1 crosses the line $R(u) = \frac{1}{2}$. When this is so, $\xi = 4 \cos^2 \theta < 4$. Hence the cut always passes from ∞ to ∞ , passing between 0 and 4, i.e., *all possible positions of the cut exclude the point $x = 4$* . For example, if the path from 0 to 1 is the semicircle described on (0, 1) as diameter, the cut is the line $R(x) = 2$. Again, if we take the path from 0 to 1 to be as shown in the figure (the two curved portions being congruent) the



cut is a curve from 4 to ∞ described twice in opposite directions, and nowhere meeting the real axis except at 4. Thus $x = 4$ is the only principal singular point.

Suppose x is real and a little less than 4. The roots of

$$1 - xu(1-u) = 0$$

are $u = \frac{1}{2} \pm i\sqrt{\{(4-x)/4x\}}$;

as x approaches 4 these poles of the subject of integration approach $\frac{1}{2}$ from opposite sides, nipping the loop C between them. Introducing a contour C' which encloses the poles, we obtain

$$\int_C = \int_{C'} + \frac{2\pi}{\sqrt{\{x(4-x)\}}} \left\{ \Phi\left[\frac{1}{2} + i\sqrt{\{(4-x)/4x\}}\right] - \Phi\left[\frac{1}{2} - i\sqrt{\{(4-x)/4x\}}\right] \right\}.$$

We thus determine the behaviour of $f(x)$ near $x = 4$.

Suppose, *e.g.*, $\alpha = 1$, $\phi \equiv 1$. Then

$$a_n = \int_0^1 (1-u)^{\beta-1+n} u^{\gamma-1+n} du = \frac{\Gamma(\beta+n) \Gamma(\gamma+n)}{\Gamma(\beta+\gamma+2n)}.$$

Thus the only principal singular points of

$$f(x) = \sum \frac{\Gamma(\beta+n) \Gamma(\gamma+n)}{\Gamma(\beta+\gamma+2n)} x^n$$

are $x = 4, \infty$. Near $x = 4$, $f(x)$ behaves like

$$\frac{i\pi}{\sin \beta\pi} \frac{1}{\sqrt{\{x(4-x)\}}} \{(-\tfrac{1}{2}+iX)^{\beta-1} (\tfrac{1}{2}+iX)^{\gamma-1} - (-\tfrac{1}{2}-iX)^{\beta-1} (\tfrac{1}{2}-iX)^{\gamma-1}\},$$

where $X = \sqrt{\{(4-x)/4x\}}$. We may verify the result by supposing $\beta = \gamma = 1$, when we must use the degenerate integral

$$\frac{1}{2\pi i} \int \frac{\log(u-1)}{1-xu(1-u)} du.$$

We find for the irregular part

$$-\frac{2}{\sqrt{\{x(4-x)\}}} \tan^{-1} \sqrt{\left(\frac{4-x}{x}\right)}.$$

As a matter of fact

$$f(x) = 1 + \frac{2}{3} \left(\frac{x}{4}\right) + \frac{2 \cdot 4}{3 \cdot 5} \left(\frac{x}{4}\right)^2 + \dots = \frac{2}{\sqrt{\{x(4-x)\}}} \tan^{-1} \sqrt{\left(\frac{x}{4-x}\right)}.$$

In this (and in the more general case above) 4, ∞ , 0 are the only singularities, the last being subsidiary.

Another interesting set of series is given by supposing

$$\psi(u) = u \log \left(\frac{1}{u}\right).$$

If $\beta = 1$, $\phi \equiv 1$, this gives rise to the series

$$\sum \frac{\Gamma(n+\alpha)}{(n+\gamma)^{n+\alpha}} x^n.$$

The nature of the cross cuts may be seen to be much the same as in the last example: the one finite principal singularity is in this case $x = e$. As x approaches e the contour C is nipped by two poles which both approach the point $u = 1/e$. The expression of the irregular part involves two transcendental functions, viz., two roots of

$$1 + xu \log u = 0,$$

or $u^u = \xi$, where $\xi = e^{-1/x}$. These functions have been (to a limited extent) studied by Eisenstein and Seidel.

Further examples are given by

$$\psi(u) = u^m(1-u)^n, \quad u^m \left(\log \frac{1}{u} \right)^n, \quad u(1+u), \quad \frac{u}{1+u}, \quad \dots$$

7. Every admissible form of $\psi(u)$ (§ 6, beginning) gives rise to a family of series as extensive as that considered in my former paper. The preceding examples will sufficiently show the course of the argument in each case. I have purposely chosen an example— $\psi(u) = u(1-u)$ —in which the details of the analysis differ from those necessary in the simplest case, viz., $\psi(u) = u$. The whole method is obviously capable of numerous generalisations which it is difficult and hardly worth while to attempt to classify. It is better to allow the generalisations to be suggested by the particular classes of series with which they enable us to deal.

ON THE LIMITS OF REAL VARIANTS

By J. MERCER.

[Received November 26th, 1906.—Read December 13th, 1906.]

Introduction.

There are two well known theorems due to Cauchy which are concerned with the limits of variants* (*i.e.*, functions of a variable positive integer n) for infinite indices. The first† is to the effect that, if X_n is any variant such that

$$X_{n+1} - X_n$$

tends to a definite limit λ when n is increased indefinitely, then

$$n^{-1} X_n$$

tends to the same limit. The second‡ shows that, if X_n is a *positive* variant such that

$$X_{n+1}/X_n$$

tends to a definite limit λ when n is increased indefinitely, then

$$X_n^{1/n}$$

tends to that limit.

Now the converse of neither of these theorems is necessarily true. To take a particular example, suppose that $X_n = a^n \theta(n)$ where $\theta(n) = n$ for odd values of n and $= \frac{1}{2}n$ for even values.

Then

$$1 \leq \theta(n) \leq n$$

for all values of n . Consequently

$$1 \leq \{\theta(n)\}^{1/n} \leq n^{1/n}.$$

Since $\text{Lt}_{n=\infty} n^{1/n} = 1$, it is clear that

$$\text{Lt}_{n=\infty} \{a^n \theta(n)\}^{1/n} = a.$$

But $\text{Lt}_{n=\infty} X_{n+1}/X_n$ does not exist, since

$$\text{Lt}_{n=\infty} X_{2n+1}/X_{2n} = 2a \quad \text{and} \quad \text{Lt}_{n=\infty} X_{2n}/X_{2n-1} = \frac{1}{2}a.$$

* Cf. Meray, *Nouveau Précis d'Analyse infinitésimale*, p. 1.

† “Cours d'Analyse de l'Ecole royale polytechnique” (1821), *Œuvres Complètes*, tome. Série, t. III., p. 54–58, p. 62–63.

‡ Cauchy, *loc. cit.*, pp. 51–63.

Examples such as this suggest the question: Can we find some single variant analogous to the $X_{n+1} - X_n$ or X_{n+1}/X_n of Cauchy's theorems such that the existence of its limit for an infinite index will ensure the existence of both

$$\lim_{n=\infty} (X_{n+1} - X_n), \quad \lim_{n=\infty} n^{-1} X_n \quad \text{or} \quad \lim_{n=\infty} X_{n+1}/X_n, \quad \lim_{n=\infty} X_n^{1/n}$$

as the case may be? This will be considered in Section I.* The succeeding section will be devoted to the extension of the theorems to the case of variants of more than one index. Section III. will be devoted to an extension of Pincherle's theorem on the radius of convergence of a power series which arises naturally from the analysis of the first section.

I.

1. We are to shew that there do exist single functions answering the above mentioned requirements. In fact, analogous to Cauchy's first theorem, we have the following†:—

THEOREM I.—Let X_n ‡ be any variant whose index is n and which is such that

$$X_{n+1} - X_n + \mu n^{-1} X_n$$

tends to a definite limit λ when n tends to infinity; then (i.), if λ is finite,

$$X_{n+1} - X_n \quad \text{and} \quad n^{-1} X_n$$

tend simultaneously to the limit $\lambda/(\mu+1)$, provided that $\mu+1 > 0$; (ii.) if λ is infinite ($\pm\infty$), $n^{-1} X_n$ tends to λ when $\mu+1 > 0$, whilst $X_{n+1} - X_n$ certainly does so only if $0 \geq \mu > -1$.

It will be observed that this theorem when proved will give us a single limit whose existence is necessary and sufficient for the *two* limits $\lim_{n=\infty} X_{n+1} - X_n$ and $\lim_{n=\infty} n^{-1} X_n$ so long as they are finite.

We now proceed to prove it. Suppose, in the first place, that λ is finite. By the definition of limit we can, if ϵ is any arbitrarily assigned positive quantity, however small, find an integer m such that, if $n \geq m$, we have

$$\lambda - \sigma < X_{n+1} - X_n + \mu n^{-1} X_n < \lambda + \sigma \quad (\text{i.})$$

where $\sigma = \frac{1}{2}(\mu+1)\epsilon$, σ being positive since $\mu+1 > 0$.

Writing $n = m$ in this equality and dividing along by $(1 - \mu/m)$ (which can be taken to be positive, since we may always suppose m

* Where it will be answered in the affirmative.

† Of course, the theorem may be stated in various forms: e.g., we may write

$$X_n = S_1 + S_2 + \dots + S_n.$$

‡ It is assumed that X_n is finite for finite values of n .

so chosen as to be greater than μ), we obtain

$$\frac{\lambda - \sigma}{1 - \frac{\mu}{m}} < \frac{X_{m+1}}{1 - \frac{\mu}{m}} - X_m < \frac{\lambda + \sigma}{1 - \frac{\mu}{m}}.$$

Writing now $n = m + 1$ in (i.) and dividing along by

$$\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m+1}\right),$$

we have
$$\frac{\lambda - \sigma}{\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m+1}\right)} < \frac{X_{m+2}}{\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m+1}\right)} - \frac{X_{m+1}}{1 - \frac{\mu}{m}}$$

$$< \frac{\lambda + \sigma}{\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m+1}\right)}.$$

Proceeding in this manner, we obtain p double inequalities, of which the last is

$$\frac{\lambda - \sigma}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)} < \frac{X_{m+p}}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)} - \frac{X_{m+p-1}}{\prod_{t=0}^{p-2} \left(1 - \frac{\mu}{m+t}\right)} < \frac{\lambda + \sigma}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)}.$$

If we add all these inequalities, we obtain

$$(\lambda - \sigma) S_{p-1} < \frac{X_{m+p}}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)} - X_m < (\lambda + \sigma) S_{p-1} \quad (\text{ii.})$$

where S_{p-1} is the sum of the series

$$\frac{1}{1 - \frac{\mu}{m}} + \frac{1}{\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m+1}\right)} + \dots + \frac{1}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)}.$$

Now, by a well known theorem of elementary algebra,* the sum of this series is ($\mu + 1 \neq 0$)

$$\frac{1}{\mu + 1} \left\{ \frac{m+p}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)} - m \right\}. \quad (\text{iii.})$$

Consequently, multiplying along the equality (ii.) by

$$\frac{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)}{m+p},$$

* Cf. *A Treatise on Algebra*, C. Smith, Fourth Edition (1893), p. 410.

after having added X_m to each term of it; we obtain

$$\frac{\lambda - \sigma}{\mu + 1} + \left\{ \frac{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t} \right)}{m+p} \right\} \left\{ X_m - \frac{m(\lambda - \sigma)}{\mu + 1} \right\}$$

$$< \frac{X_{m+p}}{m+p} < \frac{\lambda + \sigma}{\mu + 1} + \left\{ \frac{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t} \right)}{m+p} \right\} \left\{ X_m - \frac{m(\lambda + \sigma)}{\mu + 1} \right\}. \quad (\text{iv.})$$

Now, provided that $\mu + 1 > 0$, the expression

$$\frac{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t} \right)^*}{m+p}$$

tends to zero† when p is increased indefinitely: we can therefore choose a value of p —say, P —such that $p \geq P$; the second members of the extreme terms of this last inequality are less than $\frac{1}{2}\epsilon$ in absolute magnitude. We have then, on writing $m+p = n$, $m+P = N$,

$$\frac{\lambda}{\mu + 1} - \epsilon < \frac{X_n}{n} < \frac{\lambda}{\mu + 1} + \epsilon,$$

provided that $n \geq N$.

This proves that $n^{-1}X_n$ tends to the limit $\lambda/(\mu+1)$ as n increases indefinitely.

From this it follows at once, by well known theorems on limits, that

$$\begin{aligned} \text{Lt}_{n=\infty} (X_{n+1} - X_n) &= \text{Lt}_{n=\infty} \left(X_{n+1} - X_n + \mu \frac{X_n}{n} \right) - \mu \text{Lt}_{n=\infty} \frac{X_n}{n} \\ &= \lambda - \frac{\lambda\mu}{\mu + 1} = \frac{\lambda}{\mu + 1}, \end{aligned}$$

which completes the proof of the theorem for the case where λ is finite.

2. Next, suppose that $\lambda = +\infty$.

Let H be any positive number as great as we please. Then, corresponding to (i.) above, we have the single inequality

$$X_{n+1} - X_n + \mu n^{-1} X_n > H(\mu + 1) + \sigma \quad (n \geq m),$$

σ being a positive number.

Corresponding to (ii.), we have

$$\frac{X_{m+p}}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t} \right)} - X_m > [H(\mu + 1) + \sigma] S_{p-1};$$

* Hereafter to be denoted by η_p .

† Cf. C. Smith, *op. cit.*, pp. 423-4.

whilst, instead of (iv.), we have

$$\frac{X_{m+p}}{m+p} > H + \frac{\sigma}{\mu+1} + S_p$$

where S_p tends to zero when p is increased indefinitely. Consequently, choosing P so great that for $p \geq P$ we have $|S_p| < \sigma/(\mu+1)$, we deduce, on writing $m+p = n$, $m+P = N$,

$$n^{-1}X_n > H \quad (n \geq N),$$

which proves that

$$\text{Lt}_{n=\infty} n^{-1}X_n = +\infty.$$

It follows, then, at once that

$$\text{Lt}_{n=\infty} (X_{n+1} - X_n) = \text{Lt}_{n=\infty} [X_{n+1} - X_n + \mu n^{-1}X_n] - \mu \text{Lt}_{n=\infty} n^{-1}X_n = +\infty,$$

if $\mu \leq 0$. The remaining case, viz. $\lambda = -\infty$, may at once be deduced (by writing $X_n = -X'_n$) from the preceding.

The theorem is then completely established.

3. As corollaries of Theorem I. we have :

COR. I.—If we write $\mu = 0$ in the above, we at once obtain Cauchy's theorem.

COR. II.—Suppose that X_n is such that $n^{-2}X_n$ tends to zero when n becomes indefinitely great.

The theorem may be then stated in the form :

If X_n be any variant such that $n^{-2}X_n$ tends to zero when n increases indefinitely, whilst at the same time

$$n^{-\mu} \{ n^\mu X_{n+1} - (n-1)^\mu X_n \}$$

tends to a definite limit λ , then, $\mu+1 > 0$, we have

$$\text{Lt}_{n=\infty} (X_{n+1} - X_n) = \text{Lt}_{n=\infty} \frac{X_n}{n} = \frac{\lambda}{\mu+1}.$$

As a simple application of this corollary we may evaluate

$$\text{Lt}_{n=\infty} \frac{1^\mu + 2^\mu + \dots + n^\mu}{n^{\mu+1}} \quad (\mu+1 > 0).$$

Write

$$X_{n+1} = \frac{1^\mu + 2^\mu + \dots + n^\mu}{n^\mu}.$$

Obviously, if $\mu > 0$,

$$X_{n+1} < n;$$

if $\mu < 0$,

$$X_{n+1} < n^{1-\mu}.$$

Consequently, $\mu+1 > 0$, the $\text{Lt}_{n=\infty} n^{-2} X_n = 0$.

Again, $n^\mu X_{n+1} - (n-1)^\mu X_n = n^\mu$.

We have therefore in this case $\lambda = 1$; therefore

$$\text{Lt}_{n=\infty} \frac{X_n}{n} = \text{Lt}_{n=\infty} \frac{1^\mu + 2^\mu + \dots + n^\mu}{n^{\mu+1}} = \frac{1}{\mu+1}.*$$

4. Before proceeding further, let us examine the limitation (viz., $\mu+1 > 0$) which has been forced upon us in the course of the proof when λ is finite.

It will be observed that the proof turns upon the fact that

$$\text{Lt}_{p=\infty} \frac{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)}{m+p}$$

tends to zero when p is increased indefinitely, provided $\mu+1 > 0$. If $\mu+1 \leq 0$, our argument breaks down, and, if the theorem is true for a wider range of values of μ , we should have to find some new line of reasoning.

It will be seen, however, by the following particular example, that it is impossible to widen the range of values of μ in the general case.

Let

$$X_{n+1} = (1-\mu) \left(1 - \frac{\mu}{2}\right) \dots \left(1 - \frac{\mu}{n}\right).$$

Then we have always $X_{n+1} - \left(1 - \frac{\mu}{n}\right) X_n = 0$.

But

$$\frac{X_{n+1}}{n+1} = \frac{(1-\mu) \left(1 - \frac{\mu}{2}\right) \dots \left(1 - \frac{\mu}{n}\right)}{(1+1) \left(1 + \frac{1}{2}\right) \dots \left(1 + \frac{1}{n}\right)},$$

which tends to the limits 0, 1, ∞ according as $\mu+1 >, =$, or < 0 .

This shows at once that $\mu+1 > 0$ is the greatest range of values for which the theorem is generally true.

5. We have next to inquire whether in the case $\lambda = \pm \infty$ we could hope to prove generally that $\text{Lt}_{n=\infty} (X_{n+1} - X_n) = \lambda$ when $\mu > 0$. As in the preceding, we shall prove that this is impossible by a particular example.

* In the language of the integral calculus $\int_0^1 x^\mu dx = \frac{1}{\mu+1}$ ($\mu+1 > 0$), it is easy to see how to evaluate directly such integrals as $\int_0^1 \int_0^x x^\mu y^\nu dx dy$ ($\nu+1 > 0$, $\mu+\nu+2 > 0$).

Let a variant be defined by the following :—

$$X_{2n+1} = \frac{1}{2}n(n+3), \quad X_{2n} = \frac{1}{2}n(n+1).$$

Obviously $X_{2n+1} - X_{2n} = n, \quad X_{2n} - X_{2n-1} = 1.$

But $\lim_{n \rightarrow \infty} \{X_{n+1} - X_n + \mu n^{-1} X_n\} = +\infty \quad (\mu > 0),$

whilst the two equations just given above show that $X_{n+1} - X_n$ oscillates between 1 and $+\infty$.

It is easy to see, however, that, if we regard the numbers of the sequence $X_{n+1} - X_n$ as represented by points on a line, even if $\mu > 0$, λ is a limiting point of the set.

6. Before leaving this theorem we may remark that the condition

$$\lim_{n \rightarrow \infty} (X_{n+1} - X_n + \mu n^{-1} X_n) = \lambda \quad (\mu + 1 > 0)$$

is necessary *and* sufficient for the existence of the limits

$$\lim_{n \rightarrow \infty} (X_{n+1} - X_n) \quad \text{and} \quad \lim_{n \rightarrow \infty} n^{-1} X_n$$

only if λ is finite. If λ is infinite, the above condition is necessary, if $\mu > 0$, but is not sufficient; whilst, if $\mu < 0$ but $\mu + 1 > 0$, the condition is sufficient, but not necessary. This last point may be made clear by the following example :—

Let μ be a negative number (> -2), r a positive number $= -\frac{4}{\mu} - 1$, and let a variant be defined by the equations

$$X_{2n+1} - X_{2n} = rn, \quad X_{2n} - X_{2n-1} = n;$$

so that $X_{2n+1} = \frac{(1+r)n(n+1)}{2}, \quad X_{2n} = \frac{rn(n-1)}{2} + \frac{n(n+1)}{2}.$

Then, obviously, $\lim_{n \rightarrow \infty} (X_{n+1} - X_n) = \infty, \quad \lim_{n \rightarrow \infty} n^{-1} X_n = \infty;$

but $\lim_{n \rightarrow \infty} \left[X_{2n+1} - X_{2n} + \mu \frac{X_{2n}}{2n} \right] = +\infty,$

$$\lim_{n \rightarrow \infty} \left[X_{2n} - X_{2n-1} + \mu \frac{X_{2n-1}}{2n-1} \right] = \text{finite quantity}.$$

7. It is easy to see that, by a slight modification of the above argument, we can generalise this theorem.* Let $f(x)$ be a function of x defined for all real values of x greater than a fixed number and

* Cf. Cauchy, *op. cit.*, pp. 54-58.

which is finite for finite values of x . The theorem referred to is the following :—

THEOREM II.—If for increasing values of x the difference

$$f(x+1) - \left(1 - \frac{r}{x}\right) f(x)$$

tends to the limit λ , then the functions

$$f(x+1) - f(x); \quad x^{-1}f(x)$$

tend at the same time to the limit $\lambda/(r+1)$, provided that λ is finite and $r+1 > 0$: in the case when $\lambda = \pm \infty$ these each tend to λ if $0 \geq r > -1$; whilst, if $r > 0$, $\text{Lt}_{n=\infty} x^{-1}f(x) = \lambda$, but $f(x+1) - f(x)$ may not tend to this limit.

8. Corresponding to Cauchy's second theorem, we have the following :—

THEOREM III.—Let X_n be a variant, positive for great enough values of n , such that

$$\frac{X_{n+1}}{X_n^{(1-\mu/n)}}$$

tends to a definite limit λ when n increases indefinitely; then

$$X_{n+1}/X_n \quad \text{and} \quad X_n^{1/n}$$

each tends to the limit $\lambda^{1/(n+1)}$, if λ is finite and not zero and $\mu+1 > 0$, whilst, if $\lambda = 0$ or ∞ , each tends to the limit λ , provided that $0 \geq \mu > -1$. In the latter case $X_n^{1/n}$ certainly tends to the limit λ if $\mu > 0$, though X_{n+1}/X_n may not do so.

As before, we may remark that this will give a single limit whose existence is necessary and sufficient for that of two—in this case

$$\text{Lt}_{n=\infty} X_{n+1}/X_n \quad \text{and} \quad \text{Lt}_{n=\infty} X_n^{1/n}$$

—provided λ is neither zero nor infinite.

It is, of course, quite clear that Theorem III. may be deduced from Theorem I. by writing $\log X'_n = X_n$; but, following Cauchy's lead in his proof of the classical theorems, a separate proof which does not require the theory of logarithms may be supplied. Space does not, however, permit such proof being given here. The reader will have little difficulty in constructing it if he so desires.

Corresponding to Theorem II., it is easily seen that we have the following generalisation of Theorem III.:—

THEOREM IV.—If for increasing values of x the quotient

$$\frac{f(x+1)}{[f(x)]^{1-r/x}}$$

tends to the definite limit λ , then (1), if λ is finite and not zero, the functions

$$[f(x+1)/f(x)] \quad \text{and} \quad [f(x)]^{1/x}$$

tend at the same time to the limit $\lambda^{1/(r+1)}$ ($r+1 > 0$); (2), if $\lambda = \infty$ or 0, these each tend to λ if $0 \geq r > -1$; whilst, if $r > 0$, $\lim_{r=\infty} [f(x)]^{1/x} = \lambda$, though $f(x+1)/f(x)$ may not tend to this limit.

II.

9. In extending the theorems of I. to variants of more than one index we will confine ourselves to the consideration of the case when there are only two, as the statement of the theorems and their proof will be sufficiently obvious in the more complex cases.

Corresponding to Theorem I. of the previous section, we have:

THEOREM I.—Let $X_{m,n}$ be a variant of two positive indices m and n , such that $(X_{m+1,n} - X_{m,n})$ is limited* for m infinite and $(X_{m,n+1} - X_{m,n})$ for n infinite; moreover, suppose that for increasing values of m and n the variant

$$X_{m+1,n+1} - \left(1 - \frac{\mu}{m}\right) X_{m,n+1} - \left(1 - \frac{\nu}{n}\right) X_{m+1,n} + \left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\nu}{n}\right) X_{m,n}$$

tends to a definite limit λ . Then

(i.) If λ is finite,

The variants $X_{m+1,n+1} - X_{m,n+1} - X_{m+1,n} + X_{m,n}$,

$$\frac{X_{m+1,n} - X_{m,n}}{n}, \quad \frac{X_{m,n+1} - X_{m,n}}{m}, \quad \frac{X_{m,n}}{mn}$$

all tend to the limit†

$$\frac{\lambda}{(\mu+1)(\nu+1)},$$

* A variant $X_{m,n}$ is limited for m infinite if for any finite value of n we can find two finite numbers l_n and L_n such that $l_n < X_{m,n} < L_n$ for all values of m . It is easy to see that, if $X_{m+1,n} - X_{m,n}$ is so limited, then $m^{-1}X_{m,n}$ is also limited.

† To remove all possibility of doubt, it may be stated that here and throughout a variant of two indices $X_{m,n}$ will be said to tend to a finite number λ for increasing values of m and n , when or any assigned positive number ϵ we can find integers M and N so that $|X_{m,n} - \lambda| < \epsilon$, provided $m \geq M$ and $n \geq N$. Should λ be infinite, the definition is modified in the usual manner.

when m and n are increased indefinitely, provided that $\mu+1 > 0$, $\nu+1 > 0$.

(ii.) If $\lambda = \pm \infty$,

(a) The last of the above variants tends to λ , provided that

$$\mu+1 > 0, \quad \nu+1 > 0.$$

(b) The third variant tends to λ , provided that

$$0 > \nu > -1, \quad \mu+1 > 0.$$

(c) The second tends to λ , provided that $0 > \mu > -1$, $\nu+1 > 0$.

(d) The first variant tends to λ , provided that μ or $\nu = 0$, and the one $\neq 0$ lies between 0 and -1 .

We remark that this theorem, when proved, will give us a single variant the existence of whose (finite) limit for m and n indefinitely increased is the necessary and sufficient condition that the four variants mentioned above should have (finite) limits.

10. We proceed to prove the theorem, assuming in the first place that λ is finite.

Consider the last variant, viz., $m^{-1}n^{-1}X_{m,n}$. Then, if ϵ be any arbitrarily assigned positive number, however small, we can under the proposed conditions find two integers m and n , such that when $N \geq n$, $M \geq m$, we have

$$\begin{aligned} \lambda - \sigma < \left[X_{M+1, N+1} - \left(1 - \frac{\mu}{M}\right) X_{M, N+1} - \left(1 - \frac{\nu}{N}\right) X_{M+1, N} \right. \\ \left. + \left(1 - \frac{\mu}{M}\right) \left(1 - \frac{\nu}{N}\right) X_{M, N} \right] < \lambda + \sigma \quad (a) \end{aligned}$$

where

$$\sigma = \frac{1}{2}(\mu+1)(\nu+1)\epsilon.$$

Writing now $N = n$, $M = m$, and dividing through the inequality by $\left(1 - \frac{\mu}{m}\right)\left(1 - \frac{\nu}{n}\right)$ (which may be assumed positive, since we can always assume m and n chosen greater than μ and ν respectively), we have

$$\begin{aligned} \frac{\lambda - \sigma}{\left(1 - \frac{\mu}{m}\right)\left(1 - \frac{\nu}{n}\right)} &< \left\{ \frac{X_{m+1, n+1}}{\left(1 - \frac{\mu}{m}\right)\left(1 - \frac{\nu}{n}\right)} - \frac{X_{m, n+1}}{\left(1 - \frac{\nu}{n}\right)} - \frac{X_{m+1, n}}{\left(1 - \frac{\mu}{m}\right)} + X_{m, n} \right\} \\ &< \frac{\lambda + \sigma}{\left(1 - \frac{\mu}{m}\right)\left(1 - \frac{\nu}{n}\right)}. \end{aligned}$$

Again, writing $M = m+1$, $N = n$, and dividing through by

$$\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m+1}\right) \left(1 - \frac{\nu}{n}\right),$$

we obtain

$$\begin{aligned} & \frac{\lambda - \sigma}{\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m-1}\right) \left(1 - \frac{\nu}{n}\right)} \\ & < \left\{ \frac{X_{m+2, n+1}}{\prod_{t=0}^1 \left(1 - \frac{\mu}{m+t}\right) \left(1 - \frac{\nu}{n}\right)} - \frac{X_{m+1, n+1}}{\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\nu}{n}\right)} \dots \right\} \\ & < \frac{\lambda + \sigma}{\left(1 - \frac{\mu}{m}\right) \left(1 - \frac{\mu}{m+1}\right) \left(1 - \frac{\nu}{n}\right)}. \end{aligned}$$

Proceeding in this manner and giving M all possible values $< m+p$, and N all possible values $< n+q$, we have a number of inequalities of which the last is

$$\begin{aligned} & \frac{\lambda - \sigma}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right) \prod_{t'=0}^{q-1} \left(1 - \frac{\nu}{n+t'}\right)} < \frac{X_{m+p, n+q}}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right) \prod_{t'=0}^{q-1} \left(1 - \frac{\nu}{n+t'}\right)} \dots \\ & < \frac{\lambda + \sigma}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right) \prod_{t'=0}^{q-1} \left(1 - \frac{\nu}{n+t'}\right)}. \quad (\beta) \end{aligned}$$

Adding all these inequalities, we obtain

$$\begin{aligned} (\lambda - \sigma) S_{p-1, q-1} & < \left\{ \frac{X_{m+p, n+q}}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right) \prod_{t'=0}^{q-1} \left(1 - \frac{\nu}{n+t'}\right)} - \frac{X_{m, n+q}}{\prod_{t'=0}^{q-1} \left(1 - \frac{\nu}{n+t'}\right)} \right. \\ & \quad \left. - \frac{X_{m+p, n}}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)} + X_{m, n} \right\} \\ & < (\lambda + \sigma) S_{p-1, q-1} \end{aligned}$$

where

$$S_{p-1, q-1} = \sum_{s=0}^{p-1} \sum_{s'=0}^{q-1} \frac{1}{\prod_{t=0}^s \left(1 - \frac{\mu}{m+t}\right) \prod_{t'=0}^{s'} \left(1 - \frac{\nu}{n+t'}\right)}.$$

Now it is clear that the sum of this series is equal to

$$\sum_{s=0}^{p-1} \frac{1}{\prod_{t=0}^s \left(1 - \frac{\mu}{m+t}\right)} \sum_{s'=0}^{q-1} \frac{1}{\prod_{t'=0}^{s'} \left(1 - \frac{\nu}{n+t'}\right)}.$$

Consequently, if we denote by η_p the same quantity as in the previous section, and by ξ_q the corresponding quantity when n is written for m and q for p , we have after an easy reduction

$$\begin{aligned} & \frac{\lambda - \sigma}{(\mu + 1)(\nu + 1)} (1 - m\eta_p)(1 - n\xi_q) + \frac{X_{m, n+q}}{n+q} \eta_p + \frac{X_{m+p, n}}{m+p} \xi_q - X_{m, n} \eta_p \xi_q \\ & < \frac{X_{m+p, n+q}}{(m+p)(n+q)} < \frac{\lambda + \sigma}{(\mu + 1)(\nu + 1)} (1 - m\eta_p)(1 - n\xi_q) \\ & + \frac{X_{m, n+q}}{n+q} \eta_p + \frac{X_{m+p, n}}{m+p} \xi_q - X_{m, n} \eta_p \xi_q. \end{aligned}$$

Now it is obvious* that when p and q increase indefinitely the left- and right-hand sides of this double inequality tend to the limits $\frac{\lambda - \sigma}{(\mu + 1)(\nu + 1)}$ and $\frac{\lambda + \sigma}{(\mu + 1)(\nu + 1)}$ respectively, *i.e.*, to the limits $\frac{\lambda}{(\mu + 1)(\nu + 1)} - \frac{\epsilon}{2}$ and $\frac{\lambda}{(\mu + 1)(\nu + 1)} + \frac{\epsilon}{2}$. It is possible therefore to choose two positive integers P and Q such that

$$\frac{\lambda}{(\mu + 1)(\nu + 1)} - \epsilon < \frac{X_{m+p, n+q}}{(m+p)(n+q)} < \frac{\lambda}{(\mu + 1)(\nu + 1)} + \epsilon,$$

provided that $p \geq P$, $q \geq Q$. This is sufficient to show that

$$\text{Lt}_{m=\infty, n=\infty} \frac{X_{m, n}}{mn} = \frac{\lambda}{(\mu + 1)(\nu + 1)}.$$

11. Now let us consider the third variant. The number σ used in the inequality (a) is quite at our disposal, though, of course, m and n vary with it. The number ϵ being as before, choose $\sigma = \frac{1}{2}(\mu + 1)\epsilon$. Taking those of the inequalities (b) for which $q = 1$ and adding them, we obtain

$$\begin{aligned} & \frac{\lambda - \sigma}{\left(1 - \frac{\nu}{n}\right)} S_{p-1} \\ & < \left\{ \frac{X_{m+p, n+1}}{\left(1 - \frac{\nu}{n}\right) \prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)} - \frac{X_{m+p, n}}{\prod_{t=0}^{p-1} \left(1 - \frac{\mu}{m+t}\right)} - \frac{X_{m, n+1}}{\left(1 - \frac{\nu}{n}\right)} + X_{m, n} \right\} \\ & < \frac{\lambda + \sigma}{\left(1 - \frac{\nu}{n}\right)} S_{p-1}. \end{aligned}$$

* Since $\text{Lt}_{p=\infty} \eta_p = 0$, $\text{Lt}_{q=\infty} \xi_q = 0$, and in virtue of the conditions of limitedness imposed on the variant.

Using the notation of I., § 1, we obtain

$$\begin{aligned} \frac{(\lambda - \sigma)}{\mu + 1} (1 - m\eta_p) + [X_{m, n+1} - X_{m, n}] \eta_p + \frac{\nu}{n} X_{m, n} \eta_p - \nu \frac{X_{m+p, n}}{(m+p)n} \\ < \frac{X_{m+p, n+1} - X_{m+p, n}}{m+p} < \frac{\lambda + \sigma}{\mu + 1} (1 - m\eta_p) + \dots \end{aligned}$$

Now consider the terms on the left-hand side when p and n increase indefinitely. The first tends to the limit $(\lambda - \sigma)/(\mu + 1)$; the second and third vanish in virtue of the conditions of limitedness imposed; the fourth tends to the limit

$$\frac{\lambda \nu}{(\mu + 1)(\nu + 1)},$$

in virtue of the result proved in the preceding paragraph. Similarly for terms on the right.

Recalling that $\sigma/(\mu + 1) = \frac{1}{2}\epsilon$, it follows that we can choose two positive integers P and N such that, if $p \geq P$ and $n \geq N$, we have

$$\frac{\lambda}{(\mu + 1)(\nu + 1)} - \epsilon < \frac{X_{m+p, n+1} - X_{m+p, n}}{m+p} < \frac{\lambda}{(\mu + 1)(\nu + 1)} + \epsilon.$$

This is sufficient to prove that

$$\text{Lt}_{m=\infty, n=\infty} \frac{X_{m, n+1} - X_{m, n}}{m} = \frac{\lambda}{(\mu + 1)(\nu + 1)}.$$

In the same manner we may prove that the second variant tends to the same limit. It then follows at once by methods analogous to those used at the end of § 1 that the first variant tends to this limit. We have therefore proved the theorem for the case where λ is finite. The proof of the second part of the theorem we do not propose to give: the reader will have no difficulty in constructing it, in view of what has just been done here and in § 2 above.

12. We have the following corollaries of this theorem:—

COR. I.—Writing $\mu = \nu = 0$, we have the two-dimensional analogue of Cauchy's first theorem, viz.,

If for increasing values of m and n the variant

$$X_{m+1, n+1} - X_{m, n+1} - X_{n+1, m} + X_{n, n}$$

tends to a definite limit λ , then

$$\frac{X_{m,n}}{mn}, \quad \frac{X_{m+1,n} - X_{m,n}}{n}, \quad \frac{X_{m,n+1} - X_{m,n}}{m}$$

tend to the same limit.

The theorem may, of course, be stated in various ways, *e.g.*, writing

$$X_{m,n} = \sum_{r=1}^m \sum_{s=1}^n S_{rs},$$

we have it in a new form.

COR. II.—Let

$$\text{Lt}_{m=\infty, n=\infty} \frac{X_{m,n+1} - X_{m,n}}{m^2} = \text{Lt}_{m=\infty, n=\infty} \frac{X_{m+1,n} - X_{m,n}}{n^2} = \text{Lt}_{m=\infty, n=\infty} \frac{X_{m,n}}{n^r m^s} = 0,$$

the latter for all positive integer values of r and s , $r = s = 1$ excluded. We deduce at once—

If for increasing values of m and n the variant

$$m^{-\mu} n^{-\nu} \{ m^{\mu} n^{\nu} X_{m+1, n+1} - (n-1)^{\mu} n^{\nu} X_{m, n+1} - (n-1)^{\nu} m^{\mu} X_{m+1, n} + (m-1)^{\mu} (n-1)^{\nu} X_{m, n} \}$$

tends to the definite limit λ , then

$$\text{Lt}_{m=\infty, n=\infty} \frac{X_{m,n}}{mn} = \frac{\lambda}{(\mu+1)(\nu+1)} \quad (\mu+1 > 0, \nu+1 > 0).$$

As a simple example of this consider

$$\text{Lt}_{\substack{m=\infty, n=\infty, \\ r+s+2 > 0, \\ r+1 > 0}} \frac{1^s (1^r + 2^r + 3^r + \dots + m^r) + 2^s [1^r + 2^r + \dots + (2m-1)^r + (2m)^r] + \dots + n^s [1^r + 2^r + \dots + (nm-1)^r + (nm)^r]}{n^{r+s+2} m^{r+1}}.$$

If we write this equal to $\frac{X_{m+1, n+1}}{mn}$, it is easy to see that the above conditions are all fulfilled; moreover, we have

$$m^r [n^{r+s+1} X_{m+1, n+1} - (n-1)^{r+s+1} X_{m+1, n}] = n^s \{ 1^r + 2^r + \dots + (nm-1)^r + n^r m^r \}.$$

From which

$$m^{-r} n^{-(r+s+1)} \{ m^r n^{r+s+1} X_{m+1, n+1} \dots \} = \frac{1 + \left(1 - \frac{1}{nm}\right)^r + \left(1 - \frac{2}{nm}\right)^r + \dots + \left(1 - \frac{1}{m}\right)^r}{n}.$$

Obviously this tends to the limit 1 when m and n are increased indefinitely.

We see therefore that

$$\text{Lt}_{m=\infty, n=\infty} \frac{X_{m+1, n+1}}{(m+1)(n+1)} = \text{Lt}_{m=\infty, n=\infty} \frac{X_{m, n}}{mn} = \frac{\lambda}{(r+1)(r+s+2)},^*$$

provided, of course, $r+1$ and $r+s+2 > 0$.

13. Sufficient indications have now been given to show how the theorems of the preceding section extend to the case of variants of two indices. We will content ourselves therefore with enunciating the following theorems,[†] which are easily established:—

THEOREM II.—If $f(x, y)$ is a real function of two positive variables x and y (defined for all values of $x > X$ and of $y > Y$ when X and Y are fixed numbers) such that $f(x+1, y) - f(x, y)$ is limited for $x > X$, $y \leq$ any finite number $> Y$, with a corresponding condition for $f(x, y+1) - f(x, y)$, then, if when x and y are increased indefinitely the function

$$\begin{aligned} f(x+1, y+1) - \left(1 - \frac{r}{x}\right) f(x, y+1) - \left(1 - \frac{s}{y}\right) f(x+1, y) \\ + \left(1 - \frac{r}{x}\right) \left(1 - \frac{s}{y}\right) f(x, y) \end{aligned}$$

tends to a finite limit λ ,

$$\begin{aligned} \text{Lt}_{x=\infty, y=\infty} \frac{f(x, y)}{xy} &= \text{Lt}_{x=\infty, y=\infty} \frac{f(x+1, y) - f(x, y)}{y} = \text{Lt}_{x=\infty, y=\infty} \frac{f(x, y+1) - f(x, y)}{x} \\ &= \text{Lt}_{x=\infty, y=\infty} [f(x+1, y+1) - f(x, y+1) - f(x+1, y) + f(x, y)] \\ &= \frac{\lambda}{(r+1)(s+1)} \quad (\text{provided } r+1 > 0, s+1 > 0). \end{aligned}$$

THEOREM III.—Let $X_{m, n}$ be a positive variant of two indices m and n , such that $\left(\frac{X_{m+1, n}}{X_{m, n}}\right)^{\pm 1}$ is limited for m infinite, and $\left(\frac{X_{m, n+1}}{X_{m, n}}\right)^{\pm 1}$ for n infinite; then, if for increasing values of m and n the variant

$$\frac{X_{m+1, n+1} X_{m, n}^{(1-\mu/m)(1-\nu/n)}}{X_{m, n+1}^{(1-\mu'/m)} X_{m+1, n}^{(1-\nu'/n)}}$$

* The limit evaluated is, of course, $\int_0^1 dx \int_0^y x^s y^r dy$ for one particular method of dividing the field of integration into infinitesimal elements. An evaluation when the method of division is different (squares) is referred to in the footnote attached to the end of § 3.

† We only enunciate the theorems for the case in which λ is finite, and in the last two theorems different from zero. There is no difficulty in stating them for other cases (cf. Theorem of this section).

tends to a definite finite limit λ different from zero, the variants

$$\frac{X_{m+1, n+1} X_{m, n}}{X_{m, n+1} X_{m+1, n}}, \quad \left(\frac{X_{m+1, n}}{X_{m, n}}\right)^{1/n}, \quad \left(\frac{X_{m, n+1}}{X_{m, n}}\right)^{1/m}, \quad (X_{m, n})^{1/mn}$$

tend simultaneously to the limit

$$\lambda^{1/(\mu+1)(\nu+1)},$$

provided that $\mu+1 > 0$, $\nu+1 > 0$.

COR. I.—Writing $\mu = \nu = 0$, we have the two-dimensional analogue of Cauchy's second theorem.

THEOREM IV.—If $f(x, y)$ is a real positive function of two variables x and y (defined for all values of $x > X$ and of $y > Y$, where X, Y are fixed numbers), such that $[f(x+1, y)/f(x, y)]^{\pm 1}$ is limited for $x > X$ and $y \leq$ any finite number $> Y$, with a corresponding condition for $[f(x, y+1)/f(x, y)]^{\pm 1}$, then, if when x and y are increased indefinitely the function

$$\frac{f(x+1, y+1) [f(x, y)]^{(1-r/x)(1-s/y)}}{[f(x, y+1)]^{(1-r/x)} [f(x+1, y)]^{(1-s/y)}}$$

tends to a finite limit λ different from zero,

$$\begin{aligned} \text{Lt}_{x=\infty, y=\infty} \frac{f(x+1, y+1) f(x, y)}{f(x, y+1) f(x+1, y)} \\ = \text{Lt}_{x=\infty, y=\infty} \left(\frac{f(x, y+1)}{f(x, y)} \right)^{1/x} = \text{Lt}_{x=\infty, y=\infty} \left(\frac{f(x+1, y)}{f(x, y)} \right)^{1/y} \\ = \text{Lt}_{x=\infty, y=\infty} [f(x, y)]^{1/xy} = \lambda^{1/(r+1)(s+1)}, \end{aligned}$$

provided that $r+1 > 0$, $s+1 > 0$.

III.

14. The object of the present section is to prove an extension of Pincherle's well known theorem on the radius of convergence of a power series, and in some measure to bridge the gap between it and the Cauchy-Hadamard theorem on the same subject. Pincherle's theorem* may be stated as follows:—

Let a_r be the modulus of the coefficients of z^r in a power series $P(z)$.

* Pincherle, *Lezioni sulla teoria delle funzioni analitiche*, Bologna, 1899–1900 (lithographed). See also Vivanti-Gutzmer, *Theorie der eindeutigen analytischen Funktionen*, Leipzig, 1906, p. 64.

Then, if L and l be the greatest* and least† of the limits of the set of variants

$$\frac{\alpha_1}{\alpha_0}, \quad \frac{\alpha_2}{\alpha_1}, \quad \dots, \quad \frac{\alpha_{n+1}}{\alpha_n}, \quad \dots,$$

the radius of convergence (ρ) of the series satisfies the inequalities

$$\rho \geq L^{-1}, \text{ if } L \text{ be finite,}$$

$$\rho \leq l^{-1}, \text{ if } l \neq 0.$$

15. Using the same convention with regard to α_r , the theorem we propose to prove is—

If L and l are respectively the greatest and least limits of the set of variants

$$\frac{\alpha_2}{\alpha_1^{(1-r)}}, \quad \frac{\alpha_3}{\alpha_2^{(1-\frac{1}{2}r)}}, \quad \dots, \quad \frac{\alpha_{n+1}}{\alpha_n^{(1-r/n)}}, \quad \dots, \quad (r+1 > 0),$$

then the radius (ρ) of convergence of the series satisfies the inequalities

$$(1) \quad \rho \geq L^{-1/(r+1)}, \text{ if } L \text{ is finite;}$$

$$(2) \quad \rho \leq l^{-1/(r+1)}, \text{ if } l \neq 0.$$

Let the power series be

$$P(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots,$$

so that $|a_n| = a_n$, and let $\xi = |z|$.

To prove (1) it will be sufficient to show that, if ξ be any number $< 1/L^{1/(r+1)}$, the above series is convergent; supposing this to be the case, let K be a number such that

$$\xi^{-(r+1)} > K > L.$$

Then, by hypothesis, the above mentioned set contains only a finite number of elements greater than K . We are able then to choose m so that, if $n \geq m$,

$$\frac{\alpha_{n+1}}{\alpha_n^{(1-r/n)}} \leq K.$$

Writing $n = m$ in this inequality and supposing that m is chosen $> r$, we have, on extracting the $(1-r/m)$ -th root,

$$\alpha_m^{-1} \alpha_{m+1}^{1/(1-r/m)} \leq K^{1/(1-r/m)}.$$

* "La plus grande des limites," Borel, *Séries à termes positifs* (1902), p. 9.

† "La plus petite des limites," *ibid.*, p. 10.

Again, if we write $n = m + 1$, and extract the $1/\{(1-r/m)[1-r/(m+1)]\}$ -th root, we obtain

$$\alpha_{m+1}^{-[1/(1-r/m)]} \alpha_{m+2}^{1\{1-r/m\}[1-r/(m+1)]} \leq K^{1/\{1-r/m\}[1-r/(m+1)]}.$$

Next, writing $n = m + 2$, and so on, we obtain a number of inequalities of which, if we take p of them, the last is

$$\alpha_{m+p-1}^{-[1/\{\prod_{t=0}^{p-2} (1-\frac{r}{m+t})\}]} \alpha_{m+p}^{1/\{\prod_{t=0}^{p-1} (1-\frac{r}{m+t})\}} \leq K^{1/\{\prod_{t=0}^{p-1} (1-\frac{r}{m+t})\}}.$$

Multiplying these inequalities together, we obtain

$$\alpha_m^{-1} \alpha_{m+p}^{1/\{\prod_{t=0}^{p-1} (1-\frac{r}{m+t})\}} \leq K^{S_{p-1}}. * \quad (i.)$$

Consequently, if we multiply along (i.) by α_m and raise each side to the power η_p , we have

$$\alpha_{m+p}^{1/(m+p)} \leq \alpha_m^{\eta_p} K^{(1-m\eta_p)(r+1)}.$$

Now, when p is increased indefinitely the right-hand side tends to the limit $K^{1/(r+1)}$ ($r+1 > 0$). Moreover, $1 > K^{1/(r+1)} \xi$; so that we can choose a positive number θ satisfying the inequalities

$$1 > \theta > K^{1/(r+1)} \xi.$$

It follows, then, that we can choose a number P such that, if $p \geq P$, we have

$$\alpha_m^{\eta_p} K^{-[m(r+1)]\eta_p} \leq \theta/K^{1/(r+1)} \xi,$$

since the right side is > 1 .

We have therefore ($p \geq P$)

$$\alpha_{m+p}^{1/(m+p)} \xi \leq \theta.$$

In other words, if $m+P = N$, we have

$$\alpha_n \xi^n \leq \theta^n,$$

provided that $n \geq N$. Since θ is a proper fraction, the first part of the theorem is proved.

16. The proof of (2) is quite similar, and we will content ourselves with sketching it. It is sufficient to show that $P(z)$ diverges for values of z such that $\xi > l^{-1/(r+1)}$. For any such value of ξ we can choose a number k such that

$$l > k > \xi^{-(r+1)}.$$

* For the notation see § 1, Section I.

It is then possible to choose a positive integer m such that, if $n \geq m$, we have

$$\frac{\alpha_{n+1}}{\alpha_n^{(1-r/n)}} \geq k.$$

From this it easily follows that we can find a fixed number $\sigma > 1$, such that a corresponding positive integer N can be found in order to make

$$\alpha_n \xi^n \geq \sigma^n,$$

provided $n \geq N$.

This, of course, proves that the series diverges for all values of z whose modulus is ξ . The theorem is then completely established.

COR. I.—Writing $r = 0$, we have Pincherle's theorem.

COR. II.—If the set be such that there is only one limiting point, *i.e.*, $l = L$, then

$$\rho = L^{-1/(r+1)}.$$

ON PARTIAL DIFFERENTIAL COEFFICIENTS AND ON REPEATED LIMITS IN GENERAL

By E. W. HOBSON.

[Received December 7th, 1906.—Read November 8th, 1906.]

1. Let it be assumed that a function $f(x, y)$ is defined for values of x and y such that $x > a$, $y > b$, in a neighbourhood of the point (a, b) . The function $f(x, y)$ considered as a function of y only, with x constant and greater than a , has, in general, two functional limits

$$\overline{f(x, b+0)}, \quad \underline{f(x, b+0)},$$

the upper and lower limits at the point (x, b) ; these may be denoted* by $\overline{\lim}_{y=b} f(x, y)$, $\underline{\lim}_{y=b} f(x, y)$ respectively. In case these two limits are, for any particular value of x , identical, their common value may be denoted by $\lim_{y=b} f(x, y)$. If either of the limits $\overline{\lim}_{y=b} f(x, y)$, $\underline{\lim}_{y=b} f(x, y)$ is to be taken indifferently, we may denote them by $\overline{\lim}_{y=b} f(x, y)$. This may be regarded as a function of x , such that its value at the point x is multiple-valued and has $\overline{\lim}_{y=b} f(x, y)$, $\underline{\lim}_{y=b} f(x, y)$ for its limits of indeterminacy.

It may happen that $\overline{\lim}_{y=b} f(x, y)$, considered as a function of x , has a definite functional limit at the point $x = a$; this limit may be either finite, or infinite with fixed sign. In case such a limit exists, it may be denoted by $\lim_{x=a} \overline{\lim}_{y=b} f(x, y)$, and it may be termed the *repeated limit* of $f(x, y)$ at the point (a, b) , the order of the limits being that the limit for $y = b$ is taken first and then afterwards the limit for $x = a$.

The repeated limit $\lim_{y=b} \lim_{x=a} f(x, y)$, in which the limit with respect to x is first taken, and afterwards that with respect to y , may be defined in a precisely similar manner.

It is clear that the functional *values* on the straight lines $x = a$, $y = b$, in case they are defined, are irrelevant as regards the existence or the values of the repeated limits.

It is unnecessary for the existence of the repeated limit that $f(x, y)$ be defined for all values of (x, y) in a two-dimensional neighbourhood

* The notation here employed is that introduced by Pringsheim.

of (a, b) for which $x > a$, $y > b$. It is sufficient that the function be defined for a set of points in such a neighbourhood, of such a character that (x, b) is, for each value of x belonging to a set of points on the x -axis with the point (a, b) as limiting point, the limiting point of a set of points (x, y) on a straight line through (x, b) parallel to the y -axis, and, further, that a similar condition should hold for points (a, y) on the y -axis.

If for the values (x, y) , where $x > a$, $y > b$, for which $f(x, y)$ is defined, a number A exists such that, corresponding to an arbitrarily chosen positive number ϵ , a neighbourhood defined by $a < x < a + \delta$, $b < y < b + \delta$ can be determined for which the condition $|A - f(x, y)| < \epsilon$ is satisfied for every point (x, y) belonging to the domain for which $f(x, y)$ is defined, and lying within the neighbourhood, then the number A defines the double limit of $f(x, y)$ at (a, b) . When this double limit exists it is denoted by $\lim_{x=a, y=b} f(x, y)$.

In case the double limit exists as a definite number, it follows that the two repeated limits $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ both exist and are equal to the double limit.

The converse of this statement does not hold good; it is possible that the repeated limits may both exist and have one and the same value, and yet that the double limit may not exist.

2. Necessary and sufficient conditions will now be investigated that the two repeated limits $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ may both exist and have one and the same finite value.

It will be observed that the existence of $\lim_{x=a, y=b} f(x, y)$ does not necessarily involve the existence of $\lim_{y=b} f(x, y)$ as a definite number, since $\lim_{x=a} \overline{\lim}_{y=b} f(x, y)$, $\lim_{x=a} \underline{\lim}_{y=b} f(x, y)$ may both exist and have the same value, without it being necessarily the case that $\overline{\lim}_{y=b} f(x, y)$, $\underline{\lim}_{y=b} f(x, y)$ are identical for any value of x . It is, however, necessary that

$$\overline{\lim}_{y=b} f(x, y) - \underline{\lim}_{y=b} f(x, y)$$

should converge to the limit zero as x converges to the value a . The necessary and sufficient conditions required are contained in the following theorem:—

In order that the repeated limits $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ may both exist and have the same finite value, it is necessary and sufficient (1) that $\overline{\lim}_{y=b} f(x, y) - \underline{\lim}_{y=b} f(x, y)$ should have the limit zero for $x = a$,

and that $\overline{\lim}_{x=a} f(x, y) - \lim_{x=a} f(x, y)$ should have the limit zero for $y = b$; and (2) that, corresponding to any fixed positive number ϵ arbitrarily chosen, a positive number β can be determined such that for each value of y interior to the interval $(b, b + \beta)$ a positive number α_y , in general dependent on y , exists such that, for this value of y , $f(x, y)$ lies between $\overline{\lim}_{y=b} f(x, y) + \epsilon$ and $\lim_{y=b} f(x, y) - \epsilon$, for all values of x interior to the interval $(a, a + \alpha_y)$.

To prove that the conditions stated in the theorem are sufficient, let us assume that they are satisfied. A value of y may, in virtue of (1), be so chosen that the difference of the two limits $\overline{\lim}_{y=b} f(x, y)$ and $\lim_{y=b} f(x, y)$ is less than an arbitrarily chosen number η ; and this value of y may also be so chosen that it is interior to the interval $(b, b + \beta)$. For this fixed value of y , an interval $(a, a + \alpha'_y)$ for x may be so chosen that $f(x, y)$ lies between $\overline{\lim}_{x=a} f(x, y) + \epsilon$ and $\lim_{x=a} f(x, y) - \epsilon$, provided y has the fixed value and $a < x < a + \alpha'_y$: this follows from the definition of the upper and lower limits. Again, from the condition (1), a number α'' can be determined such that, if x be interior to the interval $(a, a + \alpha'')$, the difference between the two limits $\overline{\lim}_{y=b} f(x, y)$, $\lim_{y=b} f(x, y)$ is less than η . Now let $\bar{\alpha}_y$ be the smallest of the three numbers α_y , α'_y , α'' ; then, if x_1, x_2 be any two values of x within the interval $(a, a + \bar{\alpha}_y)$, and y have the fixed value, by applying the conditions of the theorem, we see that the inequalities

$$\begin{aligned} |f(x_1, y) - f(x_2, y)| &< \eta + 2\epsilon, \\ |f(x_1, y) - \overline{\lim}_{y=b} f(x_1, y)| &< \eta + \epsilon, \\ |f(x_2, y) - \overline{\lim}_{y=b} f(x_2, y)| &< \eta + \epsilon \end{aligned}$$

are all satisfied. It follows that

$$|\overline{\lim}_{y=b} f(x_1, y) - \overline{\lim}_{y=b} f(x_2, y)| < 3\eta + 4\epsilon$$

for every pair of points within the interval $(a, a + \bar{\alpha}_y)$. Hence, since ϵ, η are both arbitrarily small, $\overline{\lim}_{y=b} f(x, y)$ converges for $x = a$ to a definite value which is the limit of both $\overline{\lim}_{y=b} f(x, y)$ and of $\lim_{y=b} f(x, y)$ when $x = a$; and thus $\lim_{x=a} \lim_{y=b} f(x, y)$ exists.

Again, since $\lim_{x=a} \lim_{y=b} f(x, y)$ has a definite value, an interval $(a, a + \delta)$ can be determined such that, for any point interior to it,

$$|\lim_{x=a} \lim_{y=b} f(x, y) - \overline{\lim}_{y=b} f(x, y)| < \epsilon.$$

Now $\lim_{x=\alpha} \lim_{y=b} f(x, y) - \overline{\lim}_{x=\alpha} f(x, y)$ is the sum of the three differences

$$\lim_{x=\alpha} \lim_{y=b} f(x, y) - \overline{\lim}_{y=b} f(x, y),$$

$$\overline{\lim}_{y=b} f(x, y) - f(x, y),$$

$$f(x, y) - \overline{\lim}_{x=\alpha} f(x, y);$$

and for a fixed y , chosen as before, x may be chosen so that it not only lies within the interval $(\alpha, \alpha + \delta)$, but is also such that

$$|f(x, y) - \overline{\lim}_{y=b} f(x, y)|, \quad |f(x, y) - \overline{\lim}_{x=\alpha} f(x, y)|$$

are each less than $\eta + 2\epsilon$. It follows that

$$|\lim_{x=\alpha} \lim_{y=b} f(x, y) - \overline{\lim}_{x=\alpha} f(x, y)| < 5\epsilon + 2\eta,$$

and thus that $\overline{\lim}_{x=\alpha} f(x, y)$ converges, as y converges to b , to the limit $\lim_{x=\alpha} \lim_{y=b} f(x, y)$. It has thus been shewn that the two repeated limits both exist and have the same value.

Conversely, let us assume that the repeated limits both exist, and are finite and equal. We have, then, $|\lim_{x=\alpha} \lim_{y=b} f(x, y) - \overline{\lim}_{x=\alpha} f(x, y)| < \xi$, provided y lies between b and $b + \beta$, where β is some fixed number, ξ being an arbitrarily chosen positive number; from this it follows that

$$|\overline{\lim}_{x=\alpha} f(x, y) - \lim_{x=\alpha} f(x, y)|$$

is less than 2ξ , for $b < y < b + \beta$. Also

$$|\overline{\lim}_{y=b} f(x, y) - \lim_{x=\alpha} \lim_{y=b} f(x, y)| < \xi,$$

provided x lies within some fixed interval $(\alpha, \alpha + \delta')$; and from this it follows that $|\overline{\lim}_{y=b} f(x, y) - \overline{\lim}_{y=b} f(x, y)|$ is $< 2\xi$, for $\alpha < x < \alpha + \delta'$. Since ξ is arbitrarily small, we now see that the condition (1) of the theorem is satisfied. Further, we see that

$$|f(x, y) - \overline{\lim}_{x=\alpha} f(x, y)| < 2\xi + \xi'$$

where ξ' is any arbitrarily chosen positive number, provided x lies within some interval $(\alpha, \alpha + a'_y)$, where a'_y depends upon y , and may diminish indefinitely as y approaches the value b . It follows from the three inequalities that

$$|f(x, y) - \overline{\lim}_{y=b} f(x, y)| < 4\xi + \xi',$$

provided $b < y < b + \beta$, and provided also that x lies within some interval

$(a, a + \alpha_y)$ where α_y depends in general upon y . Since ξ and ξ' are both arbitrarily small, it follows that the condition (2) of the theorem is satisfied.

If the condition (2) in the above general theorem be replaced by the more stringent condition that, corresponding to any fixed positive number ϵ arbitrarily chosen, a positive number β can be determined, which is such that, for each value of y interior to the interval $(b, b + \beta)$, a positive number α_y dependent on y exists such that, for this value of y and for all smaller values, $f(x, y)$ lies between $\overline{\lim}_{y \rightarrow b} f(x, y) + \epsilon$ and $\underline{\lim}_{y \rightarrow b} f(x, y) - \epsilon$, then this condition and the condition (1) are the necessary and sufficient conditions that not only $\lim_{x \rightarrow a} \lim_{y \rightarrow b} f(x, y)$, $\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y)$ exist and are equal, but also that the double limit $\lim_{y \rightarrow b} \lim_{x \rightarrow a} f(x, y)$ exists, having a definite value the same as the repeated limits.

For, under the conditions stated, we have, provided y lies within the interval $(b, b + \beta_1)$ where $\beta_1 < \beta$,

$$|f(x, y) - \overline{\lim}_{y=b} f(x, y)| < \epsilon + \eta$$

where x has any value in the interval $(a, a + \xi)$, ξ being the lesser of the two numbers α_{β_1} and δ' , the number δ' being so chosen that

$$|\overline{\lim}_{y=b} f(x, y) - \underline{\lim}_{y=b} f(x, y)| < \eta$$

for $a < x < a + \delta'$. Also

$$|\overline{\lim}_{y=b} f(x, y) - \lim_{x=a} \lim_{y=b} f(x, y)| < \epsilon,$$

provided x lies within an interval chosen sufficiently small. Hence the condition

$$|f(x, y) - \lim_{x=a} \lim_{y=b} f(x, y)| < 2\epsilon + \eta$$

is satisfied, provided that $b < y < b + \beta_1$ and provided x lies within an interval of which the length may depend on ϵ and η . It follows, since ϵ, η are arbitrarily small, that $f(x, y)$ has a definite double limit at the point (a, b) . That the conditions are necessary follows at once from the definition of $\lim_{x=a, y=b} f(x, y)$.

3. The theorem obtained in § 2 may be simplified in the case in which $\lim_{y=b} f(x, y)$, $\lim_{x=a} f(x, y)$ both have definite values at all points on the straight lines $x = a$, $y = b$ which are in sufficiently small neighbourhoods of the point (a, b) . We may then state the theorem as follows:—

If* $\lim_{y=b} f(x, y)$, $\lim_{x=a} f(x, y)$ have definite finite values in the neighbourhood of the point (a, b) , then the necessary and sufficient condition that the two repeated limits $\lim_{x=a} \lim_{y=b} f(x, y)$, $\lim_{y=b} \lim_{x=a} f(x, y)$ may both exist and have the same finite value is that, corresponding to any fixed positive number ϵ arbitrarily chosen, a positive number β can be determined which is such that, for each value of y interior to the interval $(b, b+\beta)$, a positive number α_y in general dependent on y exists, such that, for this value of y , $|f(x, y) - \lim_{y=b} f(x, y)| < \epsilon$ for all values of x within the interval $(a, a+\alpha_y)$.

In case the condition $|f(x, y) - \lim_{y=b} f(x, y)| < \epsilon$ for all values of x within $(a, a+\alpha_y)$ be satisfied not only for the particular value of y , but for all smaller values, and this holds for every ϵ , then the double limit exists and is equal to the repeated limits. In this case, the point (a, b) is a point of uniform convergence of the function $f(x, y)$ to the limit $\lim_{y=b} f(x, y)$ with respect to the parameter x ; and thus for such a point there exists for each value of ϵ an interval $(a, a+\alpha)$ where α depends in general upon ϵ , such that, for each value of x within this interval, the condition $|f(x, y) - \lim_{y=b} f(x, y)| < \epsilon$ is satisfied, provided y be less than some fixed value which is the same for the whole x -interval $(a, a+\alpha)$.

* This theorem may be transformed at once into the corresponding theorem for double sequences. Let $x-a = 1/\nu$, $y-b = 1/\mu$, and assume that $f(x, y)$ is defined only for those values of x and y which correspond to positive integral values of ν and μ ; also, let

$$f(x, y) = a_{\mu\nu}.$$

The theorem relating to the existence and equality of the repeated limits $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$, $\lim_{\nu=\infty} \lim_{\mu=\infty} a_{\mu\nu}$ is then as follows:—

If $a_{\mu\nu}$ be such that $\lim_{\mu=\infty} a_{\mu\nu}$, $\lim_{\nu=\infty} a_{\mu\nu}$ both have definite values for all positive integral values of ν and of μ respectively, then the necessary and sufficient condition that the two repeated limits $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$, $\lim_{\nu=\infty} \lim_{\mu=\infty} a_{\mu\nu}$ both exist and are equal to one another, is that corresponding to any fixed positive number ϵ arbitrarily chosen, a number m_0 can be determined such that, if any integer $\mu > m_0$ be taken, an integer n_μ can be found such that $|a_{\mu\nu} - \lim_{\mu=\infty} a_{\mu\nu}| < \epsilon$ for all values of ν such that $\nu > n_\mu$.

This is equivalent to a theorem which has been given by Prof. Bromwich, *Proc. Lond. Math. Soc.*, Ser. 2, Vol. 1, p. 185; except that, in the statement given by Prof. Bromwich, the condition is added that $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$ must exist and have a finite value. This last condition is necessarily satisfied if the other condition be satisfied. In the alternative theorem given by Prof. Bromwich (*loc. cit.*, p. 184), the condition that $\lim_{\mu=\infty} \lim_{\nu=\infty} a_{\mu\nu}$ must have a definite value is not redundant, and is therefore rightly included in the statement of his theorem. The occurrence of the condition in question in the theorem on p. 185 arose from the fact that Prof. Bromwich deduced that theorem from the one on p. 184 in which the condition is required.

4. The necessary and sufficient conditions for the existence and equality of the repeated limits may be stated in a different form from that given in § 2.

The necessary and sufficient conditions that

$$\lim_{x=a} \lim_{y=b} f(x, y) = \lim_{y=b} \lim_{x=a} f(x, y),$$

their value being finite, are (1) that $\overline{\lim}_{x=a} f(x, y)$ converges to a definite value $\lim_{y=b} \lim_{x=a} f(x, y)$ when y converges to b , and that

$$\overline{\lim}_{y=b} f(x, y) - \lim_{y=b} f(x, y)$$

should converge to zero for $x = a$; and (2) that, corresponding to any arbitrarily chosen positive number ϵ , and to an arbitrarily chosen value $b + \beta_0$ of y , a value $y_1 < b + \beta_0$ can be found, and also a positive number α , such that the condition that $f(x, y_1)$ lies between

$$\overline{\lim}_{y=b} f(x, y) + \epsilon \quad \text{and} \quad \lim_{y=b} f(x, y) - \epsilon$$

is satisfied for every value of x within the interval $(a, a + \alpha)$.

In case $\lim_{y=b} f(x, y)$ everywhere exists in the neighbourhood of $x = a$, the condition (2) is that $|f(x, y_1) - \lim_{y=b} f(x, y)| < \epsilon$, for every value of x within the interval $(a, a + \alpha)$.

That the conditions contained in the theorem are necessary is seen from the theorem of § 2; it will be shewn that they are sufficient. Let us assume that the conditions are satisfied. We have then

$$\begin{aligned} & \overline{\lim}_{y=b} f(x, y) - \lim_{y=b} \lim_{x=a} f(x, y) \\ &= [\overline{\lim}_{y=b} f(x, y) - f(x, y)] + [f(x, y) - \overline{\lim}_{x=a} f(x, y)] + [\overline{\lim}_{x=a} f(x, y) - \lim_{y=b} \lim_{x=a} f(x, y)]. \end{aligned}$$

A positive number β_1 can now be chosen, such that, if $b < y < b + \beta_1$, the condition $|\overline{\lim}_{x=a} f(x, y) - \lim_{y=b} \lim_{x=a} f(x, y)| < \epsilon$ is satisfied; moreover, we may choose β_1 so that it is less than β_0 .

Next, a value y_1 of y exists, such that $f(x, y_1)$ lies between

$$\overline{\lim}_{y=b} f(x, y) + \epsilon \quad \text{and} \quad \lim_{y=b} f(x, y) - \epsilon,$$

provided x lies within the interval $(a, a + \alpha)$: the value of β_0 may be so chosen that $\overline{\lim}_{x=a} f(x, y) - \lim_{x=a} f(x, y) < \epsilon$, for every value of y which is less than $b + \beta_0$, and therefore for the value y_1 of y . Again, an interval

for x , possibly less than $(a, a+a)$, can be so chosen that

$$\overline{\lim}_{y=b} f(x, y) - \underline{\lim}_{y=b} f(x, y) < \epsilon,$$

provided x lie within the interval. It follows that an interval $(a, a+a')$ for x can be found, such that $|\overline{\lim}_{y=b} f(x, y) - f(x, y_1)| < 3\epsilon$. Further, the interval within which x lies may, if necessary, be so restricted that

$$|f(x, y_1) - \overline{\lim}_{x=a} f(x, y_1)| < 2\epsilon.$$

Hence, provided x lies within a definite interval, we see that

$$|\overline{\lim}_{y=b} f(x, y) - \lim_{y=b} \lim_{x=a} f(x, y)| < 6\epsilon;$$

and, since this condition holds for an arbitrary ϵ , it follows that $\overline{\lim}_{y=b} f(x, y)$ converges for $x = a$ to $\lim_{y=b} \lim_{x=a} f(x, y)$, and thus the sufficiency of the conditions is established.

5. The differential coefficient $\frac{\partial}{\partial x_0} \frac{\partial f}{\partial y_0}$, or $\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$ is the repeated limit

$$\lim_{h=0} \lim_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)}{hk},$$

it being assumed that $\frac{\partial f}{\partial y_0}$ exists and is finite.

We may denote this repeated limit by $\lim_{h=0} \lim_{k=0} F(h, k)$. In order that the partial differential coefficient may exist, the value of this limit must be independent of the signs of h and k .

It is not essential for the existence of the repeated limit, or of the partial differential coefficient $\frac{\partial^2 f}{\partial x_0 \partial y_0}$, that

$$\lim_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} \quad \text{or} \quad \frac{\partial}{\partial y_0} f(x_0+h, y_0)$$

should exist when $h \neq 0$. Thus the repeated limit may have a definite value when

$$\lim_{h=0} \frac{1}{h} \left[\overline{\lim}_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} - \lim_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} \right]$$

vanishes. Therefore $\frac{\partial^2 f}{\partial x_0 \partial y_0}$ may exist when $\frac{\partial f}{\partial y}$ exists at the point (x_0, y_0) , but is indefinite in the neighbourhood of that point.

The repeated limit may have a definite finite value, and yet

$$\lim_{k=0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k} \quad \text{or} \quad \frac{\partial f(x_0, y_0)}{\partial y_0}$$

may be indefinite,* in which case $\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$ does not exist in accordance with the usual definition, in which the existence of $\frac{\partial f(x_0, y_0)}{\partial y_0}$ is presupposed. It is, in fact, only when $\frac{\partial f}{\partial y_0}$ exists that the repeated limit

$\lim_{h=0} \lim_{k=0} F(h, k)$ can be written in the form

$$\lim_{h=0} \frac{1}{h} \left[\lim_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k} - \lim_{k=0} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k} \right],$$

which is then equal to

$$\lim_{h=0} \lim_{k=0} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0)}{hk}.$$

Also, when $\lim_{h=0} \lim_{k=0} F(h, k)$ is infinite with a definite sign, in order that

$\frac{\partial^2 f}{\partial x_0 \partial y_0}$ may exist, it is necessary that $\frac{\partial f}{\partial y_0}$ should exist. Unless this last

condition be satisfied, $\frac{\partial^2 f}{\partial x_0 \partial y_0}$ does not exist, in accordance with the

usual definition. When this condition is satisfied, the value of $\frac{\partial^2 f}{\partial x_0 \partial y_0}$ is infinite with a definite sign.

6. The condition for the validity of the theorem

$$\frac{\partial^2 f}{\partial x_0 \partial y_0} = \frac{\partial^2 f}{\partial y_0 \partial x_0}$$

is the same as the condition that the two repeated limits $\lim_{h=0} \lim_{k=0} F(h, k)$,

$\lim_{k=0} \lim_{h=0} F(h, k)$ should exist and have the same value. Necessary and

sufficient conditions for this could be obtained by applying the theorem of § 2, remembering that in applying the conditions each of the numbers h, k must be regarded as being capable of having either sign. It is, however, convenient, for application in particular cases, to have sufficient conditions relating to the partial differential coefficients in the neighbourhood of the point (x_0, y_0) . Such sufficient conditions are contained in the well known investigation of Schwarz relating to this theorem.

* A referee called my attention to this possibility. It is illustrated by an example given by Hardy in the *Messenger of Math.*, Vol. xxxii., p. 188. Let $f(x, y) = \phi(x) + \psi(y)$ where ϕ and ψ are not differentiable. Then $f(x_0+h, y_0+k) - f(x_0+h, y) - f(x_0, y_0+k) + f(x_0, y_0)$ vanishes identically, and therefore the repeated limit exists; but $\frac{\partial f}{\partial y_0}$ does not exist, and thus $\frac{\partial^2 f}{\partial x_0 \partial y_0}$ does not exist.

Sufficient conditions for the validity of the theorem will be here found, and are of a less stringent character than those given by Schwarz,* who assumed the additional condition that $\frac{\partial f(x, y_0)}{\partial y_0}$ exists and is finite for values of x in the neighbourhood of $x = x_0$, for the constant value y_0 .

The following theorem will be established:—

If (1) $\frac{\partial^2 f(x, y)}{\partial y \partial x}$ exist and be finite at all points in a two-dimensional neighbourhood of the point (x_0, y_0) , except that its existence at (x_0, y_0) is not assumed, and (2) the point (x_0, y_0) be a point of continuity of $\frac{\partial^2 f}{\partial y \partial x}$ with respect to (x, y) , the limit of this partial differential coefficient at (x_0, y_0) being a definite number A , and (3) $\frac{\partial f(x_0, y_0)}{\partial x_0}$ and $\frac{\partial f(x_0, y_0)}{\partial y_0}$ both exist, having definite values, then $\frac{\partial^2 f(x_0, y_0)}{\partial y_0 \partial x_0}$, $\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$ both exist, and have the same value A .

It will be observed that the condition (1) implies the existence of $\frac{\partial f(x, y)}{\partial x}$ at all points in a neighbourhood of (x_0, y_0) , except at that point itself, and that it is continuous with respect to y . From the condition (2), we have, corresponding to an arbitrarily chosen positive number ϵ ,

$$\frac{\partial^2 f(x_0 + h, y_0 + k)}{dy dx} = A + \alpha(h, k),$$

where $|\alpha| < \epsilon$, provided $|h|, |k|$ are each less than some fixed positive number η dependent on ϵ , and are not both zero.

Let $u(k')$ denote $\frac{\partial f(x_0 + h, y_0 + k')}{\partial x} - Ak'$, where k' lies in the interval $(0, k)$; we have then $\frac{du(k')}{dk'} = \alpha(h, k')$, and this is numerically less than ϵ .

It follows that $\frac{u(k) - u(0)}{k}$ is numerically less than ϵ ; for, by the mean value theorem† of the differential calculus, since $u(k')$ is continuous at $k' = 0$, and at $k' = k$, and possesses a definite differential coefficient at every interior point of the interval $(0, k)$, there exists a number $\bar{k}$ interior to the interval $(0, k)$ such that $\frac{u(k) - u(0)}{k} = \frac{du(\bar{k})}{d\bar{k}}$, and this is numerically

* *Gesammelte Abh.*, Vol. II., p. 275.

† The precise form of the mean-value theorem here employed is as follows:—If the function (x) be continuous in the interval $(x, x + h)$, and at every point in the interior of this interval $f'(x)$ exist, being either finite or infinite with fixed sign, then a point $x + \theta h$ exists, where $0 < \theta < 1$, and θ is neither 0 nor 1, such that $f(x + h) = f(x) + hf'(x + \theta h)$.

less than ϵ . We have now

$$\frac{\partial f(x_0+h, y_0+k)}{\partial x} - \frac{\partial f(x_0+h, y_0)}{\partial x} - Ak = k\alpha''(h, k),$$

where $\alpha''(h, k)$ is numerically less than ϵ . This holds for each value of h such that $0 < |h| < \eta$.

Let $v(h')$ denote $\frac{f(x_0+h', y_0+k) - f(x_0+h', y_0)}{k} - Ah'$, where h' lies

in the interval $(0, h)$; we have then $\frac{dv(h')}{dh'} = \alpha''(h', k)$, and this is numerically less than ϵ . As before, since $v(h')$ is, in virtue of (3), continuous at $h' = 0$ and also at $h' = h$, and possesses a definite differential coefficient at all interior points of the interval $(0, h)$, it follows that $\frac{v(h) - v(0)}{h}$ is numerically less than ϵ ; hence

$$\begin{aligned} hkF(h, k) &= f(x_0+h, y_0+k) - f(x_0+h, y_0) - f(x_0, y_0+k) + f(x_0, y_0) \\ &= Ahk + hk\alpha'''(h, k) \end{aligned}$$

where $|\alpha'''(h, k)| < \epsilon$.

We have now, corresponding to the arbitrarily chosen ϵ ,

$$|F(h, k) - A| < \epsilon,$$

provided $|h|, |k|$ are each less than some fixed positive number η dependent on ϵ . It follows that $F(h, k)$ is continuous at the point $h = 0, k = 0$ in the two-dimensional domain (h, k) and has A for its double limit. From this we conclude that the two limits $\lim_{h \rightarrow 0} \lim_{k \rightarrow 0} F(h, k)$, $\lim_{k \rightarrow 0} \lim_{h \rightarrow 0} F(h, k)$ exist, and are both equal to A . It follows that, when the condition (3) of the theorem is satisfied, the two partial differential coefficients

$$\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}, \quad \frac{\partial^2 f(x_0, y_0)}{\partial y_0 \partial x_0}$$

both exist and are equal to A .

7. The case in which the two partial differential coefficients are infinite with fixed sign is covered by the following theorem:—

If (1) $\frac{\partial^2 f(x, y)}{\partial y \partial x}$ exist and be finite at all points in a two-dimensional neighbourhood of the point (x_0, y_0) , except that its existence at (x_0, y_0) is not assumed, and (2) the function $\frac{\partial^2 f(x, y)}{\partial y \partial x}$ have the improper limit $+\infty$ or $-\infty$, with definite sign, at the point (x_0, y_0) , and (3) the differential

coefficients $\frac{\partial f(x_0, y_0)}{\partial x_0}$, $\frac{\partial f(x_0, y_0)}{\partial y_0}$ both exist and have definite values, then $\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$, $\frac{\partial^2 f(x_0, y_0)}{\partial y_0 \partial x_0}$ both exist, having the improper value $+\infty$ or $-\infty$ with definite sign.

To prove this theorem, let us assume that, if M be an arbitrarily chosen positive number, the condition $\frac{\partial^2 f(x_0+h, y_0+k)}{\partial y \partial x} > M$ is satisfied for all values of h and k which are not both zero, and are both numerically less than some fixed number η , dependent on M . Defining $u(k')$ as $\frac{\partial f(x_0+h, y_0+k')}{\partial x}$, we see, by means of the mean-value theorem, that

$$\frac{u(k)-u(0)}{k} > M \quad \text{or} \quad \frac{\partial f(x_0+h, y_0+k)}{\partial x} - \frac{\partial f(x_0+h, y_0)}{\partial x} > kM.$$

Next, defining $v(h')$ as $\frac{f(x_0+h', y_0+k)-f(x_0+h', y_0)}{k}$, we see, as before, that $\frac{v(h)-v(0)}{h} > M$; therefore $F(h, k) > M$, provided $|h|, |k|$ are both less than some number η dependent on M . It follows that $F(h, k)$ converges to the improper limit $+\infty$, with fixed sign, as h and k converge in any manner, each to the limit zero; thus both the limits

$$\lim_{h=0} \lim_{k=0} F(h, k), \quad \lim_{k=0} \lim_{h=0} F(h, k)$$

are $+\infty$. In order that $\frac{\partial^2 f(x_0, y_0)}{\partial x_0 \partial y_0}$, $\frac{\partial^2 f(x_0, y_0)}{\partial y_0 \partial x_0}$ may exist, in which case they both have the improper value $+\infty$, it is necessary to assume that $\frac{\partial f(x_0, y_0)}{\partial x_0}$, $\frac{\partial f(x_0, y_0)}{\partial y_0}$ both have definite values. The case in which the limits have both the improper value $-\infty$ may be treated in a precisely similar manner.

ON HYPER-EVEN NUMBERS AND ON FERMAT'S NUMBERS

By Lt.-Col. ALLAN CUNNINGHAM, R.E.

[Received January 5th, 1907.—Read January 10th, 1907.]

[The Author is indebted to Mr. H. J. Woodall, A.R.C.Sc., for help in reading the proof sheets, and for useful suggestions.]

1. *Introduction*.—This memoir is divided into three chapters; and the second chapter, which is the longest, is sub-divided into five sections.

Ch. I.—On Successive Haupt-Exponents, (ξ)	Art. 2-15
Ch. II.—On Hyper-Even Numbers, $(E_{r,n})$	Art. 16-44
Ch. II. A.—Residues of Hyper-Evens of First Order, (E_n)	Art. 18-26
Ch. II. B.—Residues of Hyper-Evens of Second and Higher Orders, $(E_n, \&c.)$	Art. 27-37
Ch. II. C.—Residues of Perfectly Even Numbers, (P_r)	Art. 38-44
Ch. II. D.—On Hyper-Exponential Numbers, $(H_{r,n})$	Art. 45
Ch. III.—Residuacity of Fermat's Numbers, (F_n)	Art. 46-54

1A. *Notation*.—The word “number” denotes in general a positive integer. m denotes a *modulus*; p denotes a *prime*. ω, Ω denote *odd* numbers; ϵ, E denote *even* numbers.

CHAPTER I.—ON SUCCESSIVE HAUPT-EXPONENTS, (ξ) .

2. *Haupt-Exponent*, (ξ) .—The *Haupt-exponent* (ξ) of a base (q) to any modulus (m) —(m prime to q)—is the exponent to which q appertains (mod m), *i.e.*, is defined by

$$q^\xi \equiv +1 \pmod{m}, \quad [\xi \text{ a min., but } > 0; m \text{ prime to } q]. \quad (1)$$

And, if $\tau = \phi(m)$, the *totient* of m , then

$$\tau = \nu \cdot \xi, \quad [\nu \text{ a max., } \xi \text{ a min.; } \nu \text{ and } \xi \nless 1], \quad (1a)$$

and ν is called the *max. residue-index* of q to mod m .

3. *Sub-modulus* (η) , *Sub-Haupt-exponent* (ξ) .—When the modulus (m) itself contains the base (q) , so that

$$\text{Modulus } m = q^a \cdot \eta, \quad [a \text{ a max.; } \eta \text{ prime to } q], \quad (2)$$

then the congruence $q^\xi \equiv 1 \pmod{m}$ is *impossible*, and q has no *Haupt-exponent* to mod m .

In this case, let ξ be the Haupt-exponent of q to mod η , so that

$$q^\xi \equiv +1 \pmod{\eta}, \quad [\xi \text{ a min., but } > 0; \eta \text{ prime to } q]. \quad (3)$$

Then η is called the *Sub-modulus* of m (for base q), and ξ is called the *Sub-Haupt-exponent* of q to mod m .

[When $a = 0$, then m is prime to q , and the Sub-modulus (η) and Sub-Haupt-exponent (ξ) are simply the modulus (m) and Haupt-exponent (ξ).]

4. *Residues R_n of q^n to mod $m = q^a \cdot \eta$.*—With the notation

$$R_n = \text{residue of } q^n \pmod{m = q^a \cdot \eta}; \quad [a \text{ a max., and } \nless 1; \eta \text{ prime to } q], \quad (4)$$

the residue-system (R_n)—when $n = 0, 1, 2, 3, \dots$ —consists in general of two parts.

1°. A *unique* (i.e., non-recurrent) part, containing say

$$h' = h + 1 \text{ members } (R_0, R_1, R_2, \dots, R_h). \quad (5a)$$

2°. A *recurring cycle* of say k members, $(R_{h+1}, R_{h+2}, \dots, R_{h+k})$, recurring *ad inf.* (5b)

5. *Period-lengths (h', k).*—The lengths (h', k) of the two parts may be shown to be

$$1^\circ. \text{ Unique part, } h' = h + 1 = a. \quad (6a)$$

$$2^\circ. \text{ Recurring Cycle, } k = \xi, \text{ (the sub-Haupt-exponent of } q \text{ to mod } m). \quad (6b)$$

[When $a = 0$, or m prime to q , then $h' = 0$, and $k = \xi$, the Haupt-exponent of q to mod m , and the residue-system (R_n) is the ordinary one.]

6. *Reduction of Residues of q^X .*—When X is a large exponent ($X > h + \xi$), then, writing $X = h + x + s\xi$,

$$q^X = q^{h+x+s\xi} \equiv q^{h+x} = q^{X-s\xi} \pmod{m = q^a \cdot \eta}. \quad (7)$$

Hence an exponent $X > h + \xi$ may always be reduced to the *smaller value* $(X - s\xi) < (h + \xi)$, but $> h$, by subtracting a sufficiently large multiple $s\xi$ of ξ , but so that the remainder left is still $> h$ (this condition is essential). This reduction is of great use in computing residues of high powers.

7. *Co-Haupt Exponential Moduli.*—If y is the A.P.F.* of $(q^\xi - 1)$ and

$$y = \mu \cdot y' y'' y''' \dots,$$

where $y', y'', \dots$ are the *specific† factors* of $(q^\xi - 1)$, then each of $y', y'', y''', \dots$,

* Here A.P.F. is used as an abbreviation for the *maximum* (irreducible) *algebraic prime factor*.

† A *specific factor* of $(q^\xi - 1)$ is one whose Haupt-exponent is ξ ; the A.P.F. of $(q^\xi - 1)$ may also possess factors (denoted by μ above) which have a Haupt-exponent $< \xi$. This is Reuschle's term.

and also any product thereof (including y itself), will all have the same *Haupt-exponent* (ξ). Hence

All moduli (m), having as sub-modulus (η) any of the above y' , y'' , ..., or any product thereof (including y itself), will all have the same sub-Haupt-exponent (ξ) (the q^a may differ in each); and no others can. (8)

8. *Co-Periodic Moduli* ($m = q^a \cdot \eta$).—From Art. 5–7 it follows that

All moduli (m), containing the same power of q , (i.e., q^a), will have the same *unique period-length* ($h' = a$); and no others can. (9a)

All moduli (m) having the same sub-Haupt-exponent (ξ) will have the same *cycle-length* ($k = \xi$); and no others can. (9b)

All moduli (m) having the same a , ξ will have the same *period-lengths in both periods* ($h' = a$, $k = \xi$); and no others can. (9c)

9. *Successive Sub-Moduli and Sub-Haupt Exponents*, (η_r , ξ_r).—Starting with m as the *original* modulus,

$$m = q^{a_0} \cdot \eta_0 \quad [a_0 \text{ a max., but may} = 0; \eta_0 \nless 1 \text{ and prime to } q], \quad (10)$$

the successive sub-moduli (η_r) and sub-Haupt-exponent (ξ_{r+1}) are formed by the rules of Art. 3 as follows:—

$$q^{\xi_1} \equiv 1 \pmod{\eta_0}, \quad q^{\xi_2} \equiv 1 \pmod{\eta_1}, \quad q^{\xi_3} \equiv 1 \pmod{\eta_2}, \quad \dots, \quad (11a)$$

$$\xi_1 = q^{a_1} \cdot \eta_1, \quad \xi_2 = q^{a_2} \cdot \eta_2, \quad \xi_3 = q^{a_3} \cdot \eta_3, \quad \dots \quad (11b)$$

Thus ξ_r , η_r , ξ_{r+1} , η_{r+1} are connected by the rules

$$\eta_r = \xi_r / q^{a_r}, \quad q^{\xi_{r+1}} \equiv 1 \pmod{\eta_r}, \quad \eta_{r+1} = \xi_{r+1} / q^{a_{r+1}}, \quad (12)$$

$$\text{and} \quad \left. \begin{array}{l} a_0, a_1, a_2, \dots \text{ are all max., but may} = 0 \\ \eta_0, \eta_1, \eta_2, \dots \text{ are all } \nless 1, \text{ and prime to } q \end{array} \right\}. \quad (13)$$

Here η_r may be called the r -th *sub-modulus* of the original modulus (m) to base q ; and ξ_r may be called the r -th *sub-Haupt-exponent* of q to that modulus (m).

10. *Train of Sub-Moduli and Sub-Haupt-Exponents* (η_r , ξ_r).—This train has the following properties:—

The whole train of η_r , ξ_r from m downwards is *definite*. (14a)

All moduli m having the same ξ_1 will have the same train of ξ_r , a_r , η_r . (14b)

The sub-moduli (η_r) decrease rapidly as r increases, ending (for all moduli m alike) with $\eta_r = 1$, which involves $\xi_{r+1} = 1$. (14c)

After which, $\xi_{r+x} = 1$, and $\eta_{r+x} = 1$, (for all values of x). (14d)

The modulus m itself is one of the train of ξ_r , (in fact $m = \xi_0$); any of the train of ξ_r might be considered as the original modulus. (14e)

Any of the a_r may = 0; but a long run of $a_r = 0$ is unusual, (especially when the base $q = 2$, see Art. 12). (14f)

The series may be continued *backwards* indefinitely, starting from any given ξ_r or η_r ; but, in the backward extension, each exponent α_x is quite arbitrary, and each η_r is incompletely defined by the preceding ξ_{r+1} , so that an infinite number of ξ_x and several η_x will arise at each backward step. (14g)

11. *Base 2*.—By far the most important base for subsequent applications is $q = 2$. Here

$$m = 2^{\alpha_0} \cdot \eta_0, [\alpha_0 \text{ a max., but may} = 0; \eta_0 \nless 1 \text{ and odd}]. \quad (15)$$

The residue-system of $R_n \equiv 2^n \pmod{m}$ has all the properties detailed in Art. 4 to 8; modified only by writing $q = 2$, and noting that η is to be *odd* (i.e., prime to the base 2) always.

The sub-modulus and sub-Haupt-exponent trains (η_r, ξ_r) are as defined in Art. 9; merely writing $q = 2$ therein, and noting that all the sub-moduli (η_r) are to be *odd* (being prime to the base 2). The train of η_r, ξ_r has all the properties detailed in Art. 10.

To these may be added

When $m = F_n = 2^N + 1$, a Fermat's number [$N = 2^n$], then

$$\alpha_0 = 0, \eta_0 = F_n; \xi_1 = 2^{n+1}, \alpha_1 = n+1, \eta_1 = 1; \quad (16a)$$

$$\alpha_2, \alpha_3, \alpha_4, \dots \text{ all} = 0; \eta_2, \eta_3, \eta_4, \dots \text{ all} = 1. \quad (16b)$$

All other *odd* moduli ($m = \eta_0, \alpha_0 = 0$) give—after a certain number of steps—

$$\eta_{r-1} = F_n \text{ (some Fermat's number)}; \xi_r = 2^{n+1}, \alpha_r = n+1, \eta_r = 1; \quad (17a)$$

$$\alpha_{r-1}, \alpha_{r+2}, \dots \text{ all} = 0; \eta_{r+1}, \eta_{r+2}, \dots \text{ all} = 1. \quad (17b)$$

From this also it is seen that for any *odd*, (but not very large), modulus (m), the number of steps required to give $\eta_r = 1$ is *small*; and that conversely, proceeding *backwards* from $\eta_r = 1$, through any chosen Fermat's number ($F_n = \eta_{r-1}$), the successive η_{r-x}, ξ_{r-x} rapidly become enormous.

Ex.—Taking $\eta_{r-1} = F_0 = 3$, and also taking each $\alpha_{r-x} = 0$, gives

$$\xi_{r-1} = \eta_{r-1} = 3, \quad \xi_{r-2} = \eta_{r-2} = 7, \quad \xi_{r-3} = \eta_{r-3} = 127, \quad \xi_{r-4} = \eta_{r-4} = 2^{17} - 1.$$

11A. *Example*.—Here follows an example of all the preceding articles, (but with base $q = 2$)—

$$(2^{100} - 1) \text{ has its A.P.F.} = (2^{50} + 1)/(2^{10} + 1) = 5 \cdot 101 \cdot 8101 \cdot 268501;$$

Excluding 5 [as being a factor of $(2^4 - 1)$], each of the *specific factors* 101, 8101, ..., and also any product thereof, and also $(2^{50} + 1)$, and $(2^{100} - 1)$ itself, have the same *Haupt-exponent* $\xi_1 = 100$ (Art. 7). Let η_0 denote any such number.

Then every *odd* modulus $m = \eta_0$ has the same *Haupt-exponent* $\xi_1 = 100$; and every modulus $m = 2^{\alpha_0} \cdot \eta_0$ (α_0 arbitrary) has the same *sub-Haupt-exponent* $\xi_1 = 100$ (Art. 7).

Hence every such modulus m has the same cycle-length ($k = 100$) in its 2^n residue-system (Art. 8); and every such modulus m , containing the same power 2^{α_0} , has the same *period-lengths* ($h' = \alpha_0, k = 100$) in both periods of its 2^n residue-system (Art. 8).

Also every such modulus m has the same *train* of sub-moduli (η_r) and *sub-Haupt-exponent* (ξ_r) , ending with $\eta_r = 1$ (Art. 10), as below—

$$\begin{array}{cccc} r = & 1, & 2, & 3, \\ \xi_r = & 2^{\alpha_r} \cdot \eta_r = 2^2 \cdot 25, & 2^2 \cdot 5, & 2 \cdot 1. \end{array}$$

12. *Prime-Chains*.—A series of primes connected by the relation

$$p_z = 2p_{z-1} + 1, \dots, p_3 = 2p_2 + 1, p_2 = 2p_1 + 1, \quad (18)$$

may be called a *Prime-chain*. In order that the chain may contain *more than three* consecutive primes, the members p must be of forms

$$\text{Every } p = 3\varpi - 1 = 10\varpi - 9 \equiv 2, 4, 5, 6 \pmod{7}, \quad (18a)$$

$$p_1 = 8\varpi + 1, 3, 7; p_2 = 8\varpi + 3, 7; p_3, p_4, \dots, p_z, \text{ all} = 8\varpi + 7. \quad (18b)$$

These prime-chains have the property

$$2^{p_z-1} \equiv 1 \pmod{p_z}, \dots, 2^{p_2} \equiv 1 \pmod{p_3}, \quad 2^{2p_1 \text{ or } p_1} \equiv 1 \pmod{p_2 = 8\varpi + 3, 7}, \quad (19)$$

so that each prime p_z is the sub-Haupt-exponent of 2 to the *next higher* prime p_{z+1} as modulus, and

Such a prime-chain is an η_r series (with base 2) to the highest prime p_z as modulus; and every $a_r = 0$ from p_z to p_2 ; $a_1 = 2$ or 1. (19a)

These prime-chains seem to give rise to the longest trains of *odd* Haupt-exponents ($\xi_r = \eta_r$) possible with prime moduli of moderate sizes.

Ex.—The longest prime-chains of this kind (known to the author) are the following of six terms each; thus confirming the result (14f) that long runs of $a_r = 0$ are unusual:—

$$\begin{array}{cccccc} 2879, & 1439, & 719, & 359, & 179, & 89; \\ 2029439, & 1014719, & 537359, & 253679, & 126839, & 63419; \\ 4068479, & 2034239, & 1017119, & 508559, & 254279, & 127139. \end{array}$$

The successive sub-moduli ($< p_1$) of such series run rapidly down to 1; thus $p_1 = 89$ has only 11, 5, 1 to complete its η_r series, [see Table I. at the end].

12A. *Other Prime-Chains*.—Prime-chains may also be formed with any power of 2 in the scale of relation, so that $p_{z+1} = 2^a \cdot p_z + 1$; but such chains seldom have more than three consecutive primes. These chains have the same property as (19) above, viz., that each prime p_z is the sub-Haupt-exponent (η) of 2 to the next higher prime p_{z+1} as modulus; but the a is usually not constant.

13. *L.C.M. of Sub-Haupt Exponent Train as Modulus*.—Let q be any *odd* number, not containing any repeated prime among its factors.

Let $\xi'_1, \xi'_2, \dots, \xi'_\rho$ be the sub-Haupt-exponent train of base 2 to mod q , where ρ is such that $\xi'_\rho = 2^{a'_\rho}$; $\eta'_\rho = 1$. Let

$$\varpi_\rho(q) = \text{L.C.M. of } q, \xi'_1, \xi'_2, \dots, \xi'_x, \dots, \xi'_\rho, [x < \rho],$$

$$\varpi_{\rho-x}(q) = \text{L.C.M. of } \xi'_x, \xi'_{x+1}, \dots, \xi'_\rho.$$

Hence $\varpi_2(q) = \text{L.C.M. of } \xi'_{\rho-2}, \xi'_{\rho-1}, \xi'_\rho$;

$$\varpi_1(q) = \text{L.C.M. of } \xi'_{\rho-1}, \xi'_\rho; \quad \varpi_0(q) = \xi'_\rho.$$

Now, take $m = \varpi_\rho(q)$ as the modulus; here m may be *even*.

And let $\xi_1, \xi_2, \dots, \xi_r$ be the sub-Haupt-exponent train of base 2 to mod m , where r is such that $\xi_r = 2^{a_r}$, $\eta_r = 1$.

Then, forming the sub-Haupt-exponents in succession, it will be found that

$$\xi_1 = \varpi_{\rho-1}(q), \xi_2 = \varpi_{\rho-2}(q), \dots, \xi_x = \varpi_{\rho-x}(q), \dots, \xi_{\rho-1} = \varpi_1(q), \dots, \xi_\rho = \varpi_0(q) = \xi'_\rho = 2^{a'_\rho}. \quad (20)$$

$$\text{Hence } \xi_r = 2^{a'_r} = \xi'_\rho, \text{ so that } r = \rho, \text{ and } a_r = a'_\rho. \quad (20a)$$

These results (20), (20a) are remarkably simple.

Ex.—Take $q = 47$; then $\xi'_1 = 23$, $\xi'_2 = 11$, $\xi'_3 = 2.5$, $\xi'_4 = 4$; $\rho = 4$;

$$m = \varpi_\rho(47) = 4.5.11.23.47;$$

and $\xi_1 = 4.5.11.23$, $\xi_2 = 4.5.11$, $\xi_3 = 4.5$, $\xi_4 = 4$; $r = 4$.

14. *Power Moduli*, (q^n).—When the modulus $m = q^n$, (q odd), the sub-Haupt-exponent train usually contains more than n terms.

Let $(\xi'_1, \xi'_2, \dots, \xi'_\rho)$, $(\xi_1, \xi_2, \dots, \xi_r)$ be the sub-Haupt-exponent trains of base 2 to mod q and q^n respectively, where ρ, r are such that

$$\xi'_\rho = 2^{a'_\rho}, \quad \eta'_\rho = 1; \quad \xi_r = 2^{a_r}, \quad \eta_r = 1.$$

14A. CASE i. ($q = F_\nu$ or f_ν).—When the modulus m is a power of a Fermat's number (F_ν) or of a Fermat's factor* (f_ν), the results take a simple form.

Here $q = F_\nu$ or f_ν ; giving (Art. 11) $\xi'_1 = 2^{\nu+1}$, $a'_1 = \nu+1$, $\rho = 1$, (21a)

and $m = q^n = F_\nu^n$ or f_ν^n ; giving

$$\xi_1 = 2^{\nu+1} \cdot q^{n-1}, \quad \xi_2 = 2^{\nu+1} \cdot q^{n-2}, \quad \dots, \quad \xi_{n-1} = 2^{\nu+1} \cdot q, \quad \xi_n = 2^{\nu+1}. \quad (21b)$$

Here $r = n$; and this is the only case in which $r = n$. (21c)

14B. CASE ii.— q an odd prime, or product of odd primes (none repeated), with the condition

The sub-moduli ($q, \eta'_1, \eta'_2, \dots, \eta'_{\rho-1}$) of q to be mutually prime. (22)

Let $\lambda(\xi'_x) = \text{L.C.M. of } \xi'_x, \xi'_{x-1}, \dots, \xi'_2, \xi'_1$, (beginning with ξ'_x),

$\varpi(\xi'_x) = \text{L.C.M. of } \xi'_\rho, \xi'_{\rho-1}, \dots, \xi'_x$, (ending with ξ'_x).

The condition (22) involves that the odd factors of all the different ξ'_r are mutually prime, so that $\lambda(\xi'_x)$, $\varpi(\xi'_x)$ are easily calculated, and

$$\lambda(\xi'_\rho) = \varpi(\xi'_1).$$

* *I.e.*, a factor of a Fermat's Number (F_ν).

Hence it will be found that $m = q^n$ gives (when $n > \rho$)

$$\xi_1 = \xi'_1 q^{n-1}, \xi_2 = \lambda(\xi'_2) q^{n-2}, \dots, \text{ up to } \xi_\rho = \lambda(\xi'_\rho) q^{n-\rho} \text{ [if } n > \rho], \quad (23a)$$

after which, $\xi_{\rho+x} = \lambda(\xi'_\rho) q^{n-\rho-x}$, where $\lambda(\xi'_\rho)$ is constant, [if $n > \rho + x$], (23b)

until $x = n - \rho$, giving $\xi_n = \lambda(\xi'_\rho) = \varpi(\xi'_1)$, [q has disappeared], (23c)

after which $\xi'_1, \xi'_2, \dots$, drop off step by step, so that

$$\xi_{n+1} = \varpi(\xi'_2), \xi_{n+2} = \varpi(\xi'_3), \dots, \xi_{n+\rho-2} = \varpi(\xi'_{\rho-1}), \quad (23d)$$

ending with $\xi_{n+\rho-1} = \varpi(\xi'_\rho) = \xi'_\rho = 2^{a_r} \eta_{n+\rho-1}$; and $r = n + \rho - 1$. (23e)

The slight modification needed in the above when $n =$ or $< \rho$ is easily made; for q disappears from ξ_x always at the n -th step, *i.e.*, from ξ_n , and the remaining steps ($\xi_{n+1}, \dots$) are the same as above, and finally $r = n + \rho - 1$ *always*.

15. *Table of Sub-Haupt-Exponent Trains, (ξ_r).*—Table I., at end of this paper, gives the *complete* trains of successive Haupt-exponents (ξ_r) of the bases $q = 2, 3, 5$ to argument $r = 1, 2, 3, \dots$ for the various moduli (m) stated below:

Mod.	Base 2.	Base 3.	Base 5.
$p =$	3 to 103	2 to 101	2 to 103
$p^r =$	9 to 125	4 to 125	4 to 125
$m =$	29 primes* > 103	10, 10 ² , 10 ³	

Each train is *complete* only up to the value ξ_r just > 1 ; the next entry, which would be $\xi_{r+1} = 1$ in every case, is omitted (to save space).

CHAPTER II.—ON HYPER-EVEN NUMBERS ($E_{r,n}$).

16. *Hyper-Even Numbers ($E_{r,n}$).*—It is proposed to attach this name to the number ($E_{r,n}$) $E_{r,n} = 2^{2^{2^{\dots^{2^n}}}}$ } r times (25)
figured in the margin, where the exponent 2 is repeated *exponentially* r times, and the uppermost exponent 2 is itself raised to the n -th power.

The number will be styled a *Hyper-even number* of order r and degree n (r, n may = 0). The symbol $E_{r,n}$ will be used by preference to the full symbol, as it saves the difficulty of printing (and reading) the heavy

* These are selected primes, and include three Mersenne's primes $M = (2^{13}-1), (2^{17}-1), (2^{19}-1)$.

exponentials; and note that hereby

$$E_{0,n} = 2^n, \quad E_{1,n} = 2^{E_{0,n}}, \quad E_{2,n} = 2^{E_{1,n}}, \quad (26a)$$

and, generally,

$$E_{r+1,n} = 2^{E_{r,n}}; \quad (26b)$$

so that each number $E_{r+1,n}$ has for its exponent the number $E_{r,n}$ of same degree (n) and next lower order (r).

To further relieve the notation, a *single suffix* symbol will often be used in the simpler cases (when one suffix is constant)

$$E_n = E_{1,n}, \quad \mathbf{E}_n = E_{2,n}, \quad P_r = E_{r,0}. \quad (26c)$$

A few of the actual values of $E_{r,n}$ are here recorded:

$n = 0, 1, 2, 3, 4, 5, 6, 7, \dots$	Abbreviation.
$E_{0,n} = 1, 2, 4, 8, 16, 32, 64, 128, \dots$	2^n
$E_{1,n} = 2, 4, 16, 256, 65536, 2^{32}, 2^{64}, 2^{128}, \dots$	E_n
$E_{2,n} = 4, 16, 65536, 2^{256}, 2^{65536}, 2^{65536^2}, 2^{65536^4}, \dots, \dots$	$\mathbf{E}_n$
$E_{3,n} = 16, 65536, 2^{65536}, \dots, \dots, \dots, \dots, \dots, \dots$	$\dots$
$\dots \dots$	
$E_{n,0} = 1, 2, 4, 16, 65536, 2^{65536}, \dots, \dots, \dots$	P_n

And it is seen at once that

$$E_{0,n} = 2^n \text{ is a simple geometrical progression of ratio } = 2. \quad (27a)$$

$$\text{In } E_{1,n} \text{ each term is the square of the preceding, } E_{n+1} = E_n^2. \quad (27b)$$

$$\text{In } E_{2,n} \text{ each exponent is the square of the preceding.} \quad (27c)$$

$$E_{n,0} = P_n \text{ is a special series, being really the zero column of all the series.} \quad (27d)$$

This last form (P_n) will be studied under the title of *perfectly even numbers* in Art. 38 *et seq.*

17. *Residues of Hyper-Even Numbers.*—The actual numbers ($E_{r,n}$) rise so rapidly with increase of r, n as to be too difficult* for direct treatment: but their residues ($R_{r,n}$) to various moduli can be readily dealt with. The residue-systems ($R_{r,n}$) of the several series of hyper-even numbers $E_{r,n}$ (r constant, $n = 0, 1, 2, \dots$) form series of considerable interest. The investigation of their properties forms the subject of Art. 18 to 37. Here

$$R_{r,n} \text{ denotes the residue of } E_{r,n} \text{ to mod } m, \quad (28)$$

and, to relieve the notation, a single suffix will be used in the simpler cases (when one suffix is constant), as in (26c)—

$$R_n, \mathbf{R}_n, \mathbf{R}_r \text{ denote the residues of } E_n, \mathbf{E}_n, P_r \text{ to mod } m. \quad (28a)$$

* Thus $E_{2,4} = E_{3,2} = E_{3,0} = 2^{65536} = 200353 \dots 156736$ (has 19729 figures),

$E_{4,2} = E_{3,4} = E_{6,0} = 2^{E_{3,0}} = \text{about } (2^{10^{19729}})^2$,

which has about 6.10^{19727} figures (a number inconceivably great).

These residue-systems $(R_{r,n})$ are, in the sequel, all formed to *odd* moduli (m) , and will be found to consist in general of two parts: one *unique* and one *recurrent* (forming a cycle).

[Note that the residue-systems of 2^n previously considered (Art. 4, 11) have *no unique period* when the modulus is *odd*.]

The modulus (m) , being *odd*, has its $a_0 = 0$; so that the sub-modulus (η_0) and sub-Haupt-exponent (ξ_1) are here *identical* with the modulus $(m = \eta_0)$ and Haupt-exponent (ξ_1) respectively (Art. 3).

CHAPTER II.A.—RESIDUES (R_n) OF HYPER-EVENS OF FIRST ORDER (E_n) .

18. *Notation*.—In this simple case, a simple notation suffices to save the difficulty of use of double exponentials.

$$2^n = N, \quad 2^h = H, \quad 2^x = X, \quad 2^k = K; \quad (29a)$$

$$E_n = 2^N, \quad E_h = 2^H, \quad E_x = 2^X, \quad E_k = 2^K; \quad (29b)$$

$$G_n = E_n - 1; \quad F_n = E_n + 1, \text{ (a Fermat's number); } f_n \text{ a factor of } F_n; \quad (29c)$$

$$R_n = \text{residue of } E_n \text{ to any modulus } m, [m \text{ odd}]. \quad (29d)$$

18a. *Notation Results*.—

$$E_{n+1} = E_n^2, \quad E_{n+2} = E_{n+1}^2 = E_n^4, \quad \dots, \quad E_{n+x} = E_n^X = E_x^N = E^{NX}; \quad (30a)$$

$$G_n = G_{n-1} \cdot F_{n-1} = G_{n-2} \cdot F_{n-2} \cdot F_{n-1} = \dots = F_0 \cdot F_1 \cdot F_2 \dots F_{n-1}. \quad (30b)$$

19. *Residue-System*.—The residues (R_n) of the successive numbers E_n to any *odd* modulus (m) —taking $n = 0, 1, 2, 3, \dots$ —form a series of some interest. It will be found to consist in general of two parts:

$$1^\circ. \text{ A } \textit{unique} \text{ portion of say } h' \text{ terms } (R_0, R_1, R_2, \dots, R_{h'}); [h' = h+1, h' \text{ may} = 0]. \quad (31a)$$

$$2^\circ. \text{ A } \textit{recurring} \text{ cycle of say } k \text{ terms } (R_{h+1}, R_{h+2}, \dots, R_{h+k}); [k \nless 1]. \quad (31b)$$

20. *Period-Lengths* (h', k) .—Let x be any number $\nless 1$, but $\nless k$; then the terms $E_{h+x}, E_{h+x+k}, \dots, E_{h+x+sk}$ are *all* to have the *same residue* (R_{h+x}) , by (31b), so that the lengths $(h' = h+1, k)$ of the two portions of the series are defined by the relation

$$E_{h+x+sk} \equiv E_{h+x}, \pmod{m}, \text{ for all values of } s, [x \nless k]. \quad (32)$$

Now $2^{h+x} = HX$, $2^{sk} = K^s = K\Sigma$ (for shortness),

$$2^{h+x+sk} = HXK^s = HXK\Sigma,$$

and the congruence (32) becomes

$$E_{h+x+sk} - E_{h+x} = 2^{HXK\Sigma} - 2^{HX} \equiv 0, \pmod{m};$$

therefore $2^{HX(K\Sigma-1)}-1 \equiv 0, \pmod{m}$, since m is *odd*.

Therefore $2^{HX(K\Sigma-1)} \equiv 1 \equiv 2^{\xi_1}, \pmod{m}$;

therefore

$$HX(K\Sigma-1) \equiv 0 \pmod{\xi_1 = 2^{a_1} \cdot \eta_1}; \text{ or } = 2^{\mu_1} \omega_1 \cdot (2^{a_1} \cdot \eta_1), [\eta_1 \text{ odd}]. \quad (33)$$

The *even* factors on either side of this equation must be equal; hence

$$2^{h+x} = HX \equiv 0 \pmod{2^{a_1}}, \text{ whence } h+x = \mu_1 + a_1 \text{ or } \nless a_1. \quad (34)$$

Here h should be the *least* integer satisfying this inequality for *all* values of $x \nless 1$; this involves
$$h' = h+1 = a_1. \quad (34a)$$

The *odd* parts on either side of the equation (33) must also be equal; hence

$$2^{s_1} - 1 = K\Sigma - 1 \equiv 0 \pmod{\eta_1}.$$

As this is to be true for *all* values of s , this involves

$$2^k - 1 \equiv 0 \pmod{\eta_1},$$

and k should be the *least* value satisfying this; so that

$$k = \xi_2, \text{ (the Haupt-exponent of 2 to mod } \eta_1). \quad (35)$$

Hence the lengths (h', k) of the unique portion and cycle are definitely given by (34a), (35).

20A. *Simple Cases*.—The lengths (h', k) can be stated in some cases, when the modulus is a prime ($m = p$), without a knowledge of ξ_1 .

$$p = 8\pi - 1 \text{ gives } (2/p)_2 = +1, \text{ and } h' = a_1 = 0. \quad (36a)$$

$$p = 8\pi + 3 \text{ gives } (2/p)_2 = -1, \text{ and } h' = a_1 = 1. \quad (36b)$$

$$p = 8\pi + 5 \text{ gives } (2/p)_2 = -1, \text{ and } h' = a_1 = 2. \quad (36c)$$

$$p = 8\pi + 1 = 2^\lambda \cdot \omega + 1 \text{ gives } (2/p)_2 = +1, \text{ and } h' = a_1 \nless \frac{1}{2}\lambda.$$

$$\text{Here } (2/p)_4, (2/p)_8, (2/p)_{16} = -1, \dots, \text{ give } h' = a_1 = \frac{1}{2}\lambda, \frac{1}{4}\lambda, \frac{1}{8}\lambda, \dots, \text{ respectively.} \quad (37a)$$

Also, without knowing the value of ξ_2 , it is seen at once that

$$k \nless \text{totient of } \eta_1, \text{ and } \nless 1. \quad (38)$$

21. *Co-periodic Moduli*.—Since the quantities a_1, η_1, ξ_2 are the same for all *odd* moduli (m) which yield the same Haupt-exponent (ξ_1) with base 2, (Art. 7, 8, 11), it follows that all such moduli have the same period-lengths (h, k) in their E_n residue-systems. This will be the case for all the *specific** *factors* of any number of form $(2^q - 1)$, and also for any product of the same, (Art. 7, 11).

* See footnote † of Art. 7.

E_x .— $(2^{100}-1)$ has its A.P.F. = $(2^{50}+1)/(2^{10}+1) = 5.101.8101.268501$.

Omitting 5, as being a factor of (2^2+1) , $\xi_1 = 100 = 2^2 \cdot 5^2$ for each of the factors 101, 8101, ...; and the period-lengths of their E_n residue-systems are alike, viz., $a = 2$, $k = 4.5$.

21A. *Concyclic Moduli*.—All odd moduli (m) whose Haupt-exponents (ξ_1) for base 2 are terms of the series $\xi_1 = \eta_1, 2\eta_1, 4\eta_1, \dots, 2^{a_1} \cdot \eta_1$, [η_1 odd, the same for all], will thus have the same ξ_2 , so that the cycles in their E_n residue-systems will be all of same length ($k = \xi_2$). This will be the case for all the *specific factors* of the whole series of numbers of form $(2^{\nu \xi_1} - 1)$ which have their $\xi_1 = 2^{a_1} \cdot \eta_1$, [η_1 the same for all], and also for any product of the same.

E_x .—Take $\nu = 1$, $\eta_1 = 7$, giving $\xi_2 = 3$; it will be seen (below) that all the factors of the A.P.F.* of each member $(2^{\xi_1} - 1)$ have the same length of cycle $k = 3$. [For shortness' sake, p is here used to denote a *large prime*.]

$\xi_1 = 7$	2.7	4.7	8.7	16.7	32.7
A.P.F. of $(2^{\xi_1} - 1)$	127	43	29.113	p	5153. p
$h', k = 0, 3$	1, 3	2, 3	3, 3	4, 3	5, 3.

22. *Cycle when $R_{x+y} = R_x$* .—From Art. 20 it is seen that the equality of two residues ($R_{x+y} = R_x$) shows at once the existence of a cycle of length $k = y$ (or a sub-factor of y).

22A. *Cycle when $R_{x+y} = -R_x$* .—Since $R_{x+y+1} = R_{x+y}^2$, and $R_{x+1} = R_x^2$, the existence of $R_{x+y} = -R_x$ involves $R_{x+y+1} = R_{x+1}$, and therefore (Art. 22) shows the existence of a cycle of length $k = y$. Also, (supposing y a *minimum*), R_{x+1} is in one cycle, and R_{x+y+1} is in the next cycle, and R_{x+y} is also in the former cycle. Hence, since $R_{x+y} \neq R_x$, it follows that R_x must be the *last term* of the non-periodic portion, and R_{x+y} the last term of the cycle. Hence

$$h = x, \quad \alpha_1 = h' = h + 1 = x + 1, \quad y = k; \quad R_x = R_h, \quad R_{x+y} = R_{h+k}. \quad (39)$$

The property $R_{h+k} = -R_h$ is of great generality. Thus

$$R_{x+y} \neq -R_x \text{ only when } \alpha_1 = 0, \text{ i.e., when } \xi_1 \text{ is odd.} \quad (40a)$$

$$R_{h+k} = -R_h \text{ always when } \alpha_1 > 0, \text{ i.e., when } \xi_1 \text{ is even.} \quad (40b)$$

$$\text{And, in this case, } R_h = E_h \text{ when } E_h < m. \quad (40c)$$

The results may be shown in greater detail when $m = p$ (an *odd prime*), viz.,

$$R_{x+y} \neq -R_x, \quad \text{when } p = 8\omega - 1, \text{ or } = 8\omega + 1, \text{ with } \xi_1 = \omega, \quad (41a)$$

$$R_{h+k} = -R_h = -2, \text{ when } p = 8\omega + 3, \text{ or } = 2^\lambda \cdot \eta_1 + 1, \text{ with } \xi_1 = 2\eta_1, \quad (41b)$$

$$R_{h+k} = -R_h = -4, \text{ when } p = 8\omega + 5, \text{ or } = 2^\lambda \cdot \eta_1 + 1, \text{ with } \xi_1 = 4\eta_1, \quad (41c)$$

$$R_{h+k} = -R_h = -E_h, \text{ when } p = 2^\lambda \cdot \eta_1 + 1, \text{ with } \xi = 2^{a_1} \cdot \eta_1, [a_1 = h + 1, E_h < p]. \quad (41d)$$

* See footnote * of Art. 7.

23. *Fermat's and Mersenne's Numbers and Factors*, (F_n, f_n, M_q, m_q) .—

$F_n = 2^N + 1$, $[N = 2^n]$, a Fermat's Number ; f_n a factor of F_n .

$M_q = 2^q - 1$, $[q \text{ prime}]$, a Mersenne's Number ; m_q a factor of M_q .

It will be seen at once (Art. 11) that these two kinds of numbers (F_n, M_q) —when in use as moduli (m) —give their E_n residue-systems in strong contrast.

i. *Fermat's Numbers* (F_n) and *factors* (f_n) give

$$h' = n + 1 \text{ (a max.)}, \quad k = 1 \text{ (a min.)}, \quad R_n = -1, \quad R_{n+1} = +1, \quad \text{always.} \quad (42a)$$

$$R_x = 2^x, \quad [X = 2^n], \text{ up to } x = n, \text{ when } m = F_n \text{ is prime.} \quad (42b)$$

ii. *Mersenne's Numbers* (M_q) and *factors* (m_q) give

$$h' = 0 \text{ (a min.)}, \quad k = \xi_2 \text{ the Haupt-exponent of } q \text{ (a max.)}, \quad (43a)$$

$$R_x = 2^t \text{ (some power of 2) always, when } m = M_q \text{ is prime.} \quad (43b)$$

24. *Property of* $\xi_1 = 2^{a_1} \cdot 3$.—The residues R_h, R_{h+1}, R_{h+2} are in this case very simply connected.

Here $\eta_1 = 3$ gives $k = \xi_2 = 2$;

$$\text{so that the cycle has only two terms } (R_{h+1}, R_{h+2}). \quad (44)$$

Hence $R_{h+3} = R_{h+1}$, whence $E_{h+3} - E_{h+1} \equiv 0 \pmod{m}$.

Now $E_{h+3} = E_{h+1}^4$, so that $E_{h+1}(E_{h+1}^3 - 1) \equiv 0 \pmod{m}$; therefore

$$E_{h+1}^3 - 1 = (E_{h+1} - 1) \cdot (E_{h+1}^2 + E_{h+1} + 1) \equiv 0 \pmod{m} \quad (45)$$

and $E_{h+1} - 1 \not\equiv 0$, (as this is a property only of a Fermat's number or factor); therefore

$$E_{h+1}^2 + E_{h+1} + 1 \equiv 0, \quad \text{whence} \quad E_{h+2} + E_{h+1} + 1 \equiv 0.$$

Hence the sum of the cycle terms

$$R_{h+1} + R_{h+2} \equiv -1 \pmod{m}. \quad (45a)$$

And, since $E_{h+1} = E_h^2$, equation (45) gives $E_h^6 - 1 = (E_h^3 - 1)(E_h^3 + 1) \equiv 0$ and $E_h^3 - 1 \not\equiv 0$, since h is the least quantity giving $E_{h+1}^3 - 1 \equiv 0$; therefore

$$E_h^3 + 1 = (E_h + 1)(E_h^2 - E_h + 1) \equiv 0.$$

Also $E_h + 1 \not\equiv 0$, (as this is a property only of a Fermat's number or factor); therefore

$$E_h^2 - E_h + 1 = 0, \quad \text{whence} \quad E_{h+1} - E_h + 1 \equiv 0.$$

Hence $R_h - R_{h+1} = 1$, [here R_h is the last non-recurring term]. (45b)

The results (45a, b) together give

$$R_h + R_{h+2} = 0. \quad (45c)$$

Note that the three relations (45*a*, *b*, *c*) between R_h , R_{h+1} , R_{h+2} are peculiar to (prime) moduli whose Haupt-exponent $\xi_1 = 2^{a_1} \cdot 3$.

Ex.—See cases of $p = 13, 97, 193, 241, 673$ in Tables II., III.

25. *Case of $R = (\pm p \pm 1)/\mu$.*—An interesting problem arises in finding the values of x which give, (when the modulus p is* prime),

$$R_x = (\pm p \pm 1)/2^L, \text{ where } L = 2^l. \quad (46)$$

Here $2^x \equiv R_x \equiv (\pm p \pm 1)/2^L, \pmod{p},$

$$2^{x+L} \equiv (\pm p \pm 1) \equiv \pm 1, \pmod{p}. \quad (47)$$

The two cases, wherein the residue is ± 1 , require separate treatment.

25A. CASE i.— $2^{x+L} \equiv +1, \pmod{p}. \quad (48)$

Here $2^x + 2^l = X + L = m \cdot \xi_1 = m \cdot 2^{a_1} \cdot \eta_1$, [η_1 odd, a_1 may = 0].

Therefore $2^{x-l} + 1 = m \cdot 2^{a_1-l} \cdot \eta_1.$

Here $m \cdot 2^{a_1-l}$ must be odd, and $l \nless a_1. \quad (48a)$

Hence $2^{x-l} \equiv -1, \pmod{\eta_1},$

whence $x-l = \frac{1}{2}k$, [k even, and $\eta_1 = \text{prime}$, or a power of such], $(48b)$

giving $x = \frac{1}{2}k + l, \quad R_x = (\pm p + 1)/2^L. \quad (48c)$

25B. CASE ii.— $2^{x+L} \equiv -1, \pmod{p}. \quad (49)$

Here $2^x + 2^l = X + L = m \cdot \frac{1}{2}\xi_1 = m \cdot 2^{a_1-1} \cdot \eta_1$, [ξ_1 even, η_1 odd].

Therefore $2^{x-l} + 1 = m \cdot 2^{a_1-l-1} \cdot \eta_1.$

Here $m \cdot 2^{a_1-l-1}$ must be odd, and $l \nless (a_1 - 1). \quad (49a)$

Hence $2^{x-l} \equiv -1, \pmod{\eta_1},$

and $x-l = \frac{1}{2}k$, [k even, and $\eta_1 = \text{prime}$, or a power of such], $(49b)$

giving $x = \frac{1}{2}k + l, \quad R_x = (\pm p - 1)/2^L. \quad (49c)$

25c. *Conditions for above.*—The conditions necessary for above are

Cases i. and ii.— $k = \epsilon$, and $\eta_1 = \text{odd prime}$, or power of such. $(50a)$

Case ii.—Add $\xi_1 = \epsilon. \quad (50b)$

* The case of a composite modulus would be more complicated; so is not here treated.

25D. *Exceptional Cases* ($k = 1$).—If the modulus be a Fermat's number $F_n = E_n + 1$, (prime or composite); then $k = 1$ always (Art. 23), and $x = n$ gives

$$R_x = -1 = -(F_n - 1)/2^N; \quad x = n + 1 \text{ gives } R_x = +1 = (F_n - 1)/2^N. \quad (50c)$$

25E. *Results*.—Here follows a table showing the results, *i.e.* the values of x , R_x , reduced from the above (48c), (49c) for each of the principal forms of p ($= 8\omega + 1, 3, 5, 7$), with numerical examples of p , most of which can be readily verified from the Table II. at the end of this paper.

[Here Q denotes either an (odd) prime (q) or a power of such (q^e).]

Case.	Form of p .	Conditions for p .		Trial.		Results.		Examples of p .
		ξ_1	$a_1 \eta_1 k$	m	l	x	R_x	
Case i.	$8\omega-1$		$0, Q, \epsilon$	1, 0	$\frac{1}{2}k,$	$\frac{1}{2}(p+1)$	7, 23, 31	
	$8\omega-1$		$0, Q, \epsilon$	2, 1	$\frac{1}{2}k+1,$	$\frac{1}{4}(p+1)$	7, 23, 31	
	$16\omega-1$		$0, Q, \epsilon$	4, 2	$\frac{1}{2}k+2,$	$\frac{1}{16}(p+1)$	31	
	$8\omega+3$		$1, Q, \epsilon$	1, 1	$\frac{1}{2}k+1,$	$\frac{1}{4}(p+1)$	11, 19, 59, 67, 83, 251	
	$8\omega+1$	$(2/p)_8 = +1$	$0, Q, \epsilon$	1, 0	$\frac{1}{2}k,$	$-\frac{1}{2}(p-1)$	73, 89, 233	
	$8\omega+1$	$(2/p)_8 = +1$	$0, Q, \epsilon$	2, 1	$\frac{1}{2}k+1,$	$-\frac{1}{4}(p-1)$	73, 89, 233	
Case ii. [$\xi = \epsilon$]	$8\omega+3$		$1, Q, \epsilon$	1, 0	$\frac{1}{2}k,$	$\frac{1}{2}(p-1)$	11, 19, 59, 67, 83, 251	
	$4\omega+1$		$2, Q, \epsilon$	1, 1	$\frac{1}{2}k+1,$	$\frac{1}{4}(p-1)$	13, 37, 53, 101, 397	
	$8\omega+1$	$(2/p)_4 = -1$	$2, Q, \epsilon$	1, 1	$\frac{1}{2}k+1,$	$\frac{1}{4}(p-1)$	41, 137, 157	
	$16\omega+1$	$(2/p)_4 = -1$	$3, Q, \epsilon$	1, 2	$\frac{1}{2}k+2,$	$\frac{1}{16}(p-1)$	241, 401, 433	
	$32\omega+1$	$(2/p)_8 = -1$	$3, Q, \epsilon$	1, 2	$\frac{1}{2}k+2,$	$\frac{1}{16}(p-1)$	353	

25F. *Mersenne's Numbers*, (M_q).—These numbers, which are of form $M_q = 2^q - 1$, [q a prime], give the simplest illustration of above results: thus, whether prime or composite, they give when used as moduli, (Art. 23),

$\alpha = 0$, $\xi_1 = \eta_1 = q$ (a prime); $k = \text{Haupt-exponent of } 2 \text{ to mod } q$, (51)

and $R_x = \frac{1}{2}(M_q + 1)$, $\frac{1}{4}(M_q + 1)$, $\frac{1}{16}(M_q + 1)$, ..., up to $(M_q + 1)/2^L$, [$2^L < M_q + 1$]
 when $x = \frac{1}{2}k$, $\frac{1}{2}k + 1$, $\frac{1}{2}k + 2$, ..., $\frac{1}{2}k + l$ } (51a)

Ex.— $M_{11} = 2^{11} - 1 = 2047 = 23 \cdot 89$; $\alpha = 0$; $\xi_1 = q = 11$; $k = 10$;

and $R_x = 1024, 512, 128, 8$; [*i.e.*, as far as $l = 3$, $L = 8$, $2^L = 256$],

when $x = 5, 6, 7, 8$.

26. *Table of Residues* (R_n) of E_n .—Tables II., III. at the end of this paper give the *complete* residue-systems (R_n) of E_n up to the end of the first cycle, to argument n , *i.e.* from $n = 0$ to $n = h + k$, for the moduli (m) as follows:—

Table II. $m = p = 3$ to 101; $m = p^e = 9$ to 81.

Table III. $m = p > 100$, but < 1000 (20 selected primes); $m = 10$ and 100;

$m = 8191$ (a Mersenne's prime); $m = 6700417$, a factor of $F_5 = (2^{32} + 1)$.

The moduli (m) are printed in bold type in upper left corner; the small figures enclosed in parentheses on the right of the same line are the values of ξ_1, ξ_2 brought from Table I. :

Ex.— $p = 113 \dots (28, 3)$ means that $p = 113$ has $\xi_1 = 28, \xi_2 = 3$,

and it will be seen that in every case the number of terms (k) in the cycle is $k = \xi_2$.

The two parts of the residue system are printed differently :—

Unique period (R_0 to R_h) in usual type; *Cycle* (R_{h+1} to R_{h+k}), enclosed in parentheses, with the positive residues in old-face type (30), and the negative residues in usual type (4).

Ex.— $p = 113$ has two unique terms 2, 4; and a three-term cycle (16, 30, 4).

Also, to save space in printing, *negative* residues are printed ($\overline{44}$, in usual type) whenever the positive residue $R_n > \frac{1}{2}m$, excepting when $m = 10, 100, \dots$

CHAPTER II.B.—RESIDUES (R_n) OF HYPER-EVENS OF SECOND ORDER, (E_n).

27. *Notation.*—The following notation is introduced to relieve the heavy exponentials

$$E_n = 2^{E_n}, \quad [E_n = 2^N, \quad N = 2^n, \text{ as before}]; \quad (52)$$

$$R_n = \text{residue of } (E_n \text{ to mod } m), \quad [m \text{ odd}]; \quad (52a)$$

$$r_n = \text{residue of } E_n = 2^N \text{ to mod } \xi_1. \quad (52b)$$

27A. *Connexion of* E_n, E_n ; R_n, R_n .—Since $E_n = 2^{E_n}$;

$$\text{therefore } E_0 = E_1, \quad E_1 = E_2, \quad E_2 = E_4, \quad E_3 = E_8, \quad \dots, \quad E_n = E_N; \quad (53)$$

$$\text{therefore } R_0 = R_1, \quad R_1 = R_2, \quad R_2 = R_4, \quad R_3 = R_8, \quad \dots, \quad R_n = R_N. \quad (53a)$$

$$\text{Hence the residues } R_n \text{ will be all found among the system of } R_n. \quad (53b)$$

28. *Successive Residues* R_{n+1}, R_n ; r_{n+1}, r_n .—By the notation

$$R_{n+1} \equiv E_{n+1} = 2^{E_{n+1}} = 2^{(E_n^2)}, \quad (\text{mod } m), \quad (54)$$

$$r_{n+1} \equiv E_{n+1} \equiv E_n^2 \equiv r_n^2, \quad (\text{mod } \xi_1). \quad (54a)$$

$$\text{Hence} \quad R_{n+1} \equiv 2^{r_{n+1}} \equiv 2^{(r_n^2)}, \quad (\text{mod } m). \quad (54b)$$

Hence—in computing the residues R_n —the *exponents* (E_n) of 2 in the E_n series may be replaced by the residues r_n of E_n to mod ξ_1 ; and each reduced exponent (as r_{n+1}) is the *square of the preceding reduced exponent* (r_n) to mod ξ_1 . This is perhaps the most convenient way of computing the R_n series.

29. *Residue-Systems* ($\mathbf{R}_n$).—The properties of this residue-system ($\mathbf{R}_n$) are much the same as those of the system of R_n already studied (Art. 19, *et seq.*). It will suffice to state these without fresh proof. The $\mathbf{R}_n$ system for any *odd* modulus (m) consists in general of *two* parts.

1°. A *non-periodic* part of $h' = h + 1$ terms ($\mathbf{R}_0, \mathbf{R}_1, \dots, \mathbf{R}_h$) [h' may = 0]. (55a)

2°. A *cycle* of k terms ($\mathbf{R}_{h+1}, \mathbf{R}_{h+2}, \dots, \mathbf{R}_{h+k}$) [$k \nless 1$]. (55b)

Also, *mutatis mutandis*, the *results* as to co-periodic and concyclic moduli, (Art. 21, 21A); as to cycles defined by $\mathbf{R}_{x+y} = \pm \mathbf{R}_x$, (Art. 22, 22A), and as to Fermat's and Mersenne's numbers and factors as moduli (Art. 23) apply for the most part here.

It is manifest also that *all* the members of the cycle $\mathbf{R}_{h+1}$ to $\mathbf{R}_{h+k}$ must be contained in the cycle of R_n ; but the converse is not true, for some members of the cycle of R_n usually fall in the unique part of the $\mathbf{R}_n$ -system.

[*Ex.*—Compare Tables II., III. with Table IV., *passim*.]

30. *Period-Lengths*, (h'_2, k_2).—Treating this case as in Art. 20, it will be found—after a double application of that process—that

$$k_2 = \xi_3 \text{ (the Haupt-exponent of 2 to mod } \eta_2), \quad (56a)$$

$$h'_2 = \text{the minimum given by } 2^{h'} \nless a_1, \text{ and } h' \nless a_2. \quad (56b)$$

Here ξ_1, ξ_2, ξ_3 ; η_1, η_2 ; a_1, a_2 have the meanings obtained by writing $q = 2$ in the notation of (11a) to (13), Art. 9.

31. *Case of* $\mathbf{R}_{x+y} = -\mathbf{R}_x$.—Reasoning as in Art. 22A leads to

$$x = h_2 = h'_2 - 1, \text{ and } \nless a_2, y = k_2 = \xi_3. \quad (57)$$

Simple Cases ($m = p$).—Referring to Art. 22A, it will be seen that here,

$$\mathbf{R}_{x+y} \neq -\mathbf{R}_x, \text{ when } p = 8\pi + 3, 7, \quad (58a)$$

$$\mathbf{R}_{x+y} = -\mathbf{R}_x \text{ requires } p = 8\pi + 5; \text{ or } = 8\pi + 1 \text{ with } \xi_1 = 4\kappa, \\ \text{(along with other conditions).} \quad (58b)$$

Ex. (see Table IV.).— $\mathbf{R}_{x+1} = -\mathbf{R}_x$, when $p = 5, 17, 193, 241, \dots$;

$$\mathbf{R}_{x+y} = \mathbf{R}_x, \text{ when } p = 29, 113, 353, 433, \dots$$

32. *Table of Residues* $\mathbf{R}_n$ of $\mathbf{E}_n$.—Table IV. at the end of this paper gives the *complete* residue-systems $\mathbf{R}_n$ of $\mathbf{E}_n$ up to the end of the first

cycle, to argument n , i.e. from $n = 0$ to $n = h + k$ for the moduli (m) as follows:—

$m = p = 3$ to 101; $m = p^e = 9$ to 125; $m = p > 101$, but < 1000 (20 selected primes).

$m = 8191$ and 131071 (Mersenne's primes); $m = 65537$ (a Fermat's prime).

$m = 6700417$ a factor of $F_5 = (2^{32} + 1)$; $m = 10, 100, 1000$.

For details of the typography, see Art. 26. The small-type figures at the end of the modulus line are the values of ξ_3 , and it will be seen that in every case the number of terms (k_2) in the cycle is $k_2 = \xi_3$.

Residues ($R_{r,n}$) of Hyper-Evens of Higher Orders, ($E_{r,n}$).

33. *Notation.*—By aid of the double suffix notation of Art. 16, 17, *q.v.*, many of the above results for the simpler cases of E_n (or $E_{1,n}$) and E^n (or $E_{2,n}$) may be readily extended to the higher power orders ($E_{r,n}$).

34. *Residue-System ($R_{r,n}$).*—Considering the r -th order series ($E_{r,n}$), it is manifest that its residue-system ($R_{r,n}$) to any *odd* modulus (m) will have similar properties to those already proved for the simpler cases of $E_{1,n}$, $E_{2,n}$. It will suffice to state some of these without fresh formal proof. The $R_{r,n}$ -system for any *odd* modulus (m) will consist in general of two parts,

1°. A *non-periodic* part of $h'_r = h_r + 1$ terms ($R_{r,0}, R_{r,1}, R_{r,2}, \dots, R_{r,h}$) [h' may = 0]. (59a)

2°. A *cycle* of k_r terms ($R_{r,h+1}, R_{r,h+2}, \dots, R_{r,h+k}$) [$k_r \nless 1$]. (59b)

Also, *mutatis mutandis*, the results as to co-periodic and concyclic moduli (Art. 21, 21A); as to cycle lengths (k_r) defined by $R_{r,y} = \pm R_{r,x}$, (Art. 22, 22A), and as to Fermat's and Mersenne's numbers and factors, (Art. 23), apply to a great extent here also. It is manifest also that all the members of the cycle $R_{r,h+1}$ to $R_{r,h+k}$ must be contained *within the cycles* of all lower power orders ($E_{r-u,n}$); but the converse is not true.

35. *Period-Lengths, (h'_r, k_r).*—By an extension of the reasoning of Art. 20, 30, in the case of $E_{1,n}$, $E_{2,n}$, it will be seen that the lengths (h'_r, k_r) of the two parts of the $R_{r,n}$ series may be determined from the sub-modulus (η_r) and sub-Haupt-exponent (ξ_r) trains of Art. 9. Thus

1°. *Unique Part.*—The length $h'_r = h_r + 1$ is determined by a train of r inequalities; thus h'_r should be the *minimum* satisfying

$$E_{r-2,h'} \nless a_1, E_{r-3,h'} \nless a_2, \dots, E_{2,h} \nless a_{r-3}, E_{1,h'} \nless a_{r-2}, 2^{h'} \nless a_{r-1}, h' \nless a_r. \quad (60a)$$

The values of $E_{r,h'}$ rise so rapidly with increase of r , (whilst the a remain small), that the last two or three of these inequalities (viz., those on the right) will—if satisfied—usually involve the rest, (so that h'_r is readily found).

2°. *Cycle-Length, (k_r) .*—Here

$$k_r = \xi_{r+1} = \text{Haupt-exponent of 2 to mod } \eta_r. \quad (60b)$$

36. *Ultimate Cycle $(k_\rho = 1)$.*—It was shown (Art. 10) that for every odd modulus (m) there is a value of r giving $\eta_r = 1$. Now, let ρ be the least such value; then

$$\eta_\rho = \eta_{\rho+1} = \dots = \eta_{\rho+x} = 1, \quad \text{and} \quad \xi_{\rho+1} = \xi_{\rho+2} = \dots = \xi_{\rho+x+1} = 1, \quad (\text{for all values of } x). \quad (61a)$$

$$\text{Hence also} \quad k_\rho = k_{\rho+1} = \dots = k_{\rho+x} = 1, \quad (\text{for all values of } x). \quad (61b)$$

Thus the residue-system of the $E_{\rho,x}$ series and of all higher order series has a cycle of only one term, *i.e.* has a *constant term*, and this term has the same (numerical) value in all those series, *i.e.*,

$$R_{\rho+x, h'_x} = R_{\rho, h'} \quad (\text{for all values of } x), \quad (62)$$

although the length of the unique portion (h'_x) may differ therein.

37. *Disappearance of Unique Period $(h'_r = 0)$.*—Let ρ be the least value giving $\eta_\rho = 1$, $\xi_{\rho+1} = 1$, as in Art. 36. Then the non-recurring part of the $R_{\rho,n}$ system is of length h'_ρ defined by writing $r = \rho$ in (60a). In passing to series of higher order ($r = \rho + x$), the cycle remains of only one term (Art. 36), whilst the length of the unique portion (h'_x) decreases on the whole (though there may be fluctuations), until finally it vanishes, $h'_r = 0$, when $1 \nless a_{r-1}$, $2 \nless a_{r-2}$, $2^2 \nless a_{r-3}$, $2^4 \nless a_{r-4}$, ..., $P_x \nless a_{r-x-1}$, (63)

so that the residue-system ($R_{r,n}$) is hereby *reduced to the constant term* with the above stated value

$$R_{r,n} = R_{r,0} \quad (\text{for all values of } n), \quad (64)$$

and the same is true of all series of higher order ($r+x$).

CHAPTER II. C.—RESIDUES (R_r) OF PERFECTLY EVEN NUMBERS (P_r).

38. *Definition and Notation.*—The term *perfectly even number*, of order r , is here used to denote the number (P_r) defined by

$$P_0 = 1, \quad P_1 = 2^{P_0}, \quad P_2 = 2^{P_1}, \quad \dots, \quad P_{r+1} = 2^{P_r}, \quad (65)$$

so that $P_0 = 1$, $P_1 = 2$, $P_2 = 4$, $P_3 = 16$, $P_4 = 65536$, $P_5 = 2^{65536}$, ...

Also $\mathbf{R}_r$ denotes the residue of P_r to any *odd* modulus (m).

Thus these numbers (P_r) form the series of *zero degree* ($n = 0$) among the series $E_{r,n}$ defined in Art. 16, *i.e.*,

$$P_r = E_{r,0}, \quad \text{and} \quad \mathbf{R}_r = R_{r,0}, \quad (66)$$

but the present (simpler) notation is introduced to simplify the symbolic work.

39. *Residue-System, ($\mathbf{R}_r$).*—From Art. 34–37, it is manifest that the residue-system ($\mathbf{R}_r$) consists *always* of two parts,

$$1^\circ. \text{ A non-periodic part of say } h' = h + 1 \text{ terms } (\mathbf{R}_0, \mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_h), [h' > 0]. \quad (67a)$$

$$2^\circ. \text{ A constant term (or cycle of one term) } \mathbf{R}_{h'}, \text{ repeating ad inf.} \quad (67b)$$

The results of Art. 34 require some modification for the present case. Thus all moduli (m) alike give a *cycle of one term*: and all moduli, which have the same Haupt-exponent ξ_1 (see Art. 7, 11), have the non-periodic part of their P_r residue-system of same length h' , (cf. Art. 34). And the equation $\mathbf{R}_y = \pm \mathbf{R}_x$, ($y > x$), shows that $\mathbf{R}_y$ is the constant term.

In this last case of $\mathbf{R}^y = -\mathbf{R}_x$, it is clear also (since the cycle contains only one term) that

$$h' = h + 1 = \text{least value of } y, R_y = R_{h'} \text{ is the constant term.} \quad (68)$$

40. *Reduction of Residues.*—The numbers P_n rise so rapidly (when $n > 4$) that it is important to show a practical way of computing the residues $\mathbf{R}_n$ when n is not small. This is done by a systematic reduction of the successive large exponents (P_{n-x}) in the series

$$P_{n-x+1} = 2^{P_{n-x}}, P_{n-x+2} = 2^{P_{n-x+1}}, \dots, P_{n-1} = 2^{P_{n-2}}, P_n = 2^{P_{n-1}}, \quad (69)$$

by help of the sub-Haupt-exponent train $\xi_x, \xi_{x-1}, \dots, \xi_2, \xi_1$ (Art. 9), used as reducing moduli by the process of Art. 6 repeatedly applied.

$$\text{Let } R'_{n-x} \text{ denote the least residue of } P_{n-x} \text{ to mod } \xi_x, (x \succ n). \quad (70)$$

$$\text{Then} \quad R'_{n-x+1} \equiv P_{n-x+1} \equiv 2^{P_{n-x}} \pmod{\xi_{x-1}}, \quad (71)$$

and, by Art. 6, R'_{n-x} may be substituted for P_{n-x} in the above, (since ξ_x is the sub-Haupt-exponent of ξ_{x-1}); hence

$$R'_{n-x+1} \equiv 2^{R'_{n-x}} \pmod{\xi_{x-1}}. \quad (71a)$$

$$\text{Similarly } R'_{n-x+2} \equiv 2^{R'_{n-x+1}} \pmod{\xi_{x-2}}, \quad (71b)$$

$$\vdots$$

$$R'_{n-x+y} \equiv 2^{R'_{n-x+y-1}} \pmod{\xi_{x-y}}, \quad (71c)$$

$$\vdots$$

$$R'_{n-2} \equiv 2^{R'_{n-3}} \pmod{\xi_2}, \quad R'_{n-1} \equiv 2^{R'_{n-2}} \pmod{\xi_1}. \quad (71d)$$

$$\text{Lastly, } R'_n \equiv 2^{R'_{n-1}} \pmod{\xi_0 = m}, \text{ so that } R'_n = \text{the required } R_n. \quad (71e)$$

To apply the above, it is only necessary to take x , such that P_{n-x} may be so small that the starting residue R'_{n-x} may be *readily calculable*. Hereby $R'_{n-x} < \xi_x < \xi_{x-1}$, so that the next residue R'_{n-x+1} is also readily calculable. And similarly each reduced exponent is so small that each residue may be calculated in succession.

41. *Position (r) of Constant Term (R_r).*—This might be worked out direct from the definition $R_{r+x} = R_r$ by a process similar to that used in Art. 20, 30: but this is now unnecessary, as it may be taken from the results of Art. 36, 37, which practically embody it.

STEP i.—Let ρ be the *least* value giving $\eta_\rho = 1$, $\xi_{\rho+1} = 1$, (as in Art. 36). This determines the numerical value of this constant term as $= R_{\rho, n'}$ of the series $E_{\rho, n}$, and also fixes a limit $\rho \triangleright$ the required r .

STEP ii.—Determine the value of r ($\triangleleft \rho$) defining the $E_{r, n}$ series in which $h'_r = 0$ (as in Art. 37), by the conditions

$$1 \triangleleft \alpha_{r-1}, \quad 2 \triangleleft \alpha_{r-2}, \quad 2^2 \triangleleft \alpha_{r-3}, \quad 2^4 \triangleleft \alpha_{r-4}, \quad \dots, \quad P_x \triangleleft \alpha_{r-x-1}. \quad (72)$$

In practice a few of these conditions (on the left) suffice to involve the rest, as P_x increases rapidly with x , (when $x > 4$), whilst the α remain small, so that r is readily found. This is clearly the value of r required, giving (in the P_r series),

$$R_{r+x} = R_r, \text{ for all values of } x. \quad (73)$$

42. *Value of Constant Term (R_r).*—The actual value of the constant term R_r may now be found by the general process of Art. 40, substituting the value of r , (found from Art. 41) for the general symbol n (of Art. 40). This substitution (of r for n) carries the following important consequences.

1°. The reduced residues R'_{r-x} are the *actual constant residues* in the P_n series to the several moduli ξ_x . (74a)

2°. The subscripts $(r-x)$ are the *values of the general symbol* r defining the position of the constant term to those moduli (ξ_x) , (74b)

as will be more clear from the following scheme :—

$$\begin{array}{l} \text{Modulus: } \xi_x, \quad \xi_{x-1}, \quad \dots, \quad \xi_{x-y}, \quad \dots, \quad \xi_2, \quad \xi_1, \quad \xi_0 = m. \\ \text{Value of } r: r-x, r-x+1, \dots, r-x+y, \dots, r-2, r-1, r. \\ \text{Constant Residues: } R'_{r-x}, R'_{r-x+1}, \dots, R'_{r-x+y}, \dots, R'_{r-2}, R'_{r-1}, R'_r = \mathbf{R}_r. \\ \text{Number } P: P_{r-x}, P_{r-x+1}, \dots, P_{r-x+y}, \dots, P_{r-2}, P_{r-1}, P_r. \end{array}$$

It will be seen that the subscripts (r) of R' and P increase by 1 at each step as the sub-Haupt-exponents rise from ξ_x to ξ_0 , ending with $\xi_0 = m$.

The process is applied as follows :—

STEP i.—Find r by the process of Art. 41.

STEP ii.—Take x such that P_{r-x} is so small that the residue R'_{r-x} of P_{r-x} to mod ξ_x is readily calculated.

STEP iii., &c.—Compute the reduced residues $R'_{r-x+1}, \dots, R'_{r-1}, R'_r$ to the moduli

$$\xi_{x-1}, \dots, \xi_2, \xi_1, \xi_0 (= m)$$

in succession by the process of Art. 40. The last residue R'_r is the required $\mathbf{R}_r$, the constant residue of $P_r, P_{r+1}, \dots$ to mod m .

Example.—Take $m = 2879$. This has been chosen as a somewhat difficult example, involving proceeding to the 10th order (P_{10}).

STEP i. To find r .—The train of sub-Haupt-exponents (ξ_x) is (see Table I.)

$$\begin{array}{cccccccccc} x = & 0, & 1, & 2, & 3, & 4, & 5, & 6, & 7, & 8, & 9, \\ \xi_x = & 2879, & 1439, & 719, & 359, & 179, & 2.89, & 11, & 2.5, & 2^2, & 1. \end{array}$$

Since $\xi_x = 2^{a_x} \cdot \eta_x$, the limiting $\eta_\rho = 1$, gives $\rho = 8, a_\rho = 2$.

On trying $r = \rho + 1, 2, \dots$ or $r = 9, 10, \dots$, $r = 9$ is found too small. But $r = 10$ gives (since $\xi_x = 2^{a_x} \cdot \eta_x$) $a_{r-1} = a_9 = 0, a_{r-2} = a_8 = 2, a_{r-3} = a_7 = 1, \dots$; and these values satisfy all the conditions (72); therefore $r = 10$.

STEP ii.—Taking $x = 7$ gives $r - x = 3, \xi_x = \xi_7 = 10; P_{r-x} = P_3 = 16$.

Hence

$$R'_{r-x} = R'_3 \equiv P_3 \equiv 6 \pmod{\xi_7 = 10}.$$

[It is here easy to confirm (by actual trial) that $R'_3 = 6$ is really the constant term to mod 10, and must therefore lead to the constant terms with all the successive moduli. Except for the advantage of *easy confirmation* of the constant term at the start, the start might have been equally well made by taking $x = 6$, which gives $P_{r-x} = P_4 = 65536$, an easy number to start with.]

STEP iii.—The remaining steps are now simple :—

$$\begin{aligned} R'_{r-x+1} &= R'_4 \equiv 2^{R'_3} \equiv 2^6 \equiv 9 \pmod{\xi_6 = 11}, \\ R'_5 &\equiv 2^{R'_4} \equiv 2^9 \equiv 156 \pmod{\xi_5 = 2.89}, \\ R'_6 &\equiv 2^{R'_5} \equiv 2^{156} \equiv 82 \pmod{\xi_4 = 179}, \\ R'_7 &\equiv 2^{R'_6} \equiv 2^{82} \equiv 164 \pmod{\xi_3 = 359}, \\ R'_8 &\equiv 2^{R'_7} \equiv 2^{164} \equiv 507 \pmod{\xi_2 = 719}, \\ R'_9 &\equiv 2^{R'_8} \equiv 2^{507} \equiv 95 \pmod{\xi_1 = 1439}, \\ \mathbf{R}_r = R'_{10} &= 2^{R'_9} = 2^{95} = 1025 \pmod{\xi_0 = m = 2879}. \end{aligned}$$

43. *Fermat's, Joubin's, and Mersenne's Numbers, and Factors.*—When the modulus (m) is one of these numbers, or one of their factors, the

position (defined by r) and value ($\mathbf{R}_r$) of the constant term of their P_r Residue-system is readily found. Both r and $\mathbf{R}_r$ usually turn out to be small numbers, even for large values of such moduli.

43A. *Fermat's Numbers*, (F_n).—Here $F_n = E_n + 1$,

and (Art. 11), $\xi_1 = 2^{n+1}$, $\alpha_1 = n + 1$, $\eta_1 = 1$; $a_2, a_3, a_4, \dots$, all $= 0$. (75)

Hence r is found by making $\alpha_{r-x-1} = \alpha_1$ in (72), whence $x = r - 2$, and the only condition to be satisfied is

$$P_{r-2} \nless \alpha_1 = (n+1). \quad (76)$$

After the first four numbers (F_0 to F_3), the value of r , thus found, rises very slowly, as shown below

$$\begin{array}{l} n = 0, 1, 2, 3; \text{ 4 to 15; 16 to 65535; 65536 to } (2^{65535} - 1), \\ r = 2, 3, 4, 4; \quad 5, \quad 6, \quad 7, \end{array}$$

and $\mathbf{R}_r = +1$, *always*, (but $\mathbf{R}_{r-1} \neq -1$, except as below). (77)

43B. *Joubin's Numbers*, (J_n).—These are defined by $J_n = P_n + 1$; they are interesting from the supposition which has been made* that they are (possibly) *all prime*. It is easy to see that with these numbers, (being a special class of Fermat's numbers),

$$\mathbf{R}_n = -1, \text{ and } \mathbf{R}_{n+1} = +1, \text{ always; so that } r = n + 1. \quad (78)$$

43C. *Mersenne's Numbers*, (M_q).—Here $M_q = (2^q - 1)$, [q prime], and $\xi_1 = q$.

Let $\mathbf{R}_r, R_{r'}$ be the *constant terms* in the residue-systems of the P_n series for the moduli M_q, q respectively, the subscripts r, r' defining their positions, (Art. 41). Then—by the last paragraph of Art. 42—it follows (since $q = \xi_1$) that

$$r = r' + 1, \text{ and } \mathbf{R}_r \equiv 2^{R_{r'}} \pmod{M_q}. \quad (79)$$

Here q is usually so small that the values of $r', R_{r'}$ are readily found

* The earliest reference the author has been able to find to these numbers is in a "Note" at the end of M. Joubin's *Mémoire sur les Facteurs numériques*, Havre, 1831; M. Joubin there suggests that these were possibly the numbers really meant by Fermat in his celebrated theorem as to primes. Lucas mentions these numbers, as supposed to be all prime, in his *Récréations mathématiques*, t. II., 1883, p. 235, and again in his *Théorie des Nombres*, t. I., 1891, p. 355, but without quoting authority.

directly, (as in Table V.), after which the values of r , $\mathbf{R}_r$ are given by (79); or, in any case, r' , $R_{r'}$, r , $\mathbf{R}_r$ can be readily found by the processes of Art. 41, 42. The following abstract shows the values of r , $\mathbf{R}_r$ to argument q for every modulus M_q up to $q = 101$.

$$\begin{aligned} q &= 2, 3, 5, 7, \quad 11, 13, 17, 19, 23, 29, 31, \quad 37, 41, 43, \\ r &= 2, 3, 4, 4, \quad 5, \quad 4, \quad 5, \quad 5, \quad 6, \quad 5, \quad 5, \quad 5, \quad 5, \quad 5, \\ \mathbf{R}_r &= 1, 2, 2, 4, 512, \quad 8, \quad 2, 32, 64, 2^{25}, \quad 4, 512, 2^{18}, 16, \\ q &= 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, \\ r &= 7, \quad 5, \quad 6, \quad 5, \quad 6, \quad 5, \quad 5, \quad 5, \quad 6, \quad 6, \quad 5, \quad 6, \\ \mathbf{R}_r &= 2^{17}, 2^{28}, 2^{48}, 2^{22}, 2^{17}, \quad 8, 2^{55}, 2^{45}, 2^{30}, 2^{67}, 2^{61}, 2^{78}. \end{aligned}$$

Interesting cases of the above arise when the exponent q of the Mersenne's number (M_q) used as modulus is itself either a Fermat's number (F_ν), or a Mersenne's number (M_κ), as shown below.

43D. *Case of $q = F_\nu$.*—Here $F_\nu = E_\nu + 1$, a Fermat's number, and $m = M_q = 2^q - 1$, where $q = F_\nu$ may be prime or composite.

Here, by Art. 43A, $R_{r'} = +1, \pmod{F_\nu}$, always.

Hence, by (79), $r = r' + 1$, and $\mathbf{R}_r \equiv 2 \pmod{M_q, q = F_\nu}$, always. (80)

43E. *Mersenne's Chain.*—An interesting series of moduli is an exponential chain of Mersenne's numbers formed as follows:—

$$M_{0,\nu} = F_\nu, \quad M_{1,\nu} = 2^{M_{0,\nu}} - 1, \quad M_{2,\nu} = 2^{M_{1,\nu}} - 1, \quad \dots, \quad M_{x+1,\nu} = 2^{M_{x,\nu}} - 1; \quad (81)$$

where the leading modulus $M_{0,\nu}$ is a Fermat's number (F_ν), and the rest are Mersenne's numbers, formed so that each one is the exponent in the succeeding one. Here, if $R_{r'}$ be the constant term in the residue-system of the P_n series for the leading modulus (F_ν), the subscript r' marking its position, it follows from Art. 42, 43A-D, that the following scheme shows the values of r , $\mathbf{R}_r$ for the moduli $M_{0,\nu}, \dots, M_{x,\nu}$ for any value of ν whatever, (noting that $R_{r'} = 1$),

$$\begin{aligned} m &= F_\nu, \quad M_{1,\nu}, \quad M_{2,\nu}, \quad M_{3,\nu}, \quad M_{4,\nu}, \quad \dots, \quad M_{x,\nu}, \\ r &= r', \quad r' + 1, \quad r' + 2, \quad r' + 3, \quad r' + 4, \quad \dots, \quad r' + x, \end{aligned} \quad (82a)$$

$$\mathbf{R}_r = 1, \quad 2, \quad 4, \quad 16, \quad 65536, \quad \dots, \quad P_x, \quad (82b)$$

and it is now seen that—

The P_n series itself forms the series of the constant terms in its own residue-system to every such chain of moduli. (83)

Note that herein F_ν may be prime or composite, but that the rest of the members of the chain (so far as used above) should be all *prime*.

The abstract below shows the values of r , $\mathbf{R}_r$ for the first four such

chains, given by $F_\nu = 3, 5, 17, 257$, ($\nu = 0, 1, 2, 3$), carried in each case just *beyond the limit* of known primes, (so that the $r, \mathbf{R}_r$ of the last term in each case is *doubtful*).

$M_{x,\nu} = 3, 7, 127, (2^{127}-1), M_{4,0}$	$5, 31, (2^{31}-1), M_{3,1}$	$17, 131071, M_{2,2}$	$257, (2^{257}-1)$
$r = 2, 3, 4, 5, 6$	$3, 4, 5, 6$	$4, 5, 6$	$4, 5$
$\mathbf{R}_r = 1, 2, 4, 16, 65536$	$1, 2, 4, 16$	$1, 2, 4$	$1, 2$

43F. *Fermat's, Joubin's, and Mersenne's Factors*.—From Art. 7, 9, 11, 23 it is seen that factors of F_n, J_n, M_q have in general the same Haupt-exponent (ξ_1) as F_n, J_n, M_q themselves; and have therefore also the same train of ξ_r, a_r, η_r with F_n, J_n, M_q (Art. 10). The value of r which fixes the position of the constant term $\mathbf{R}_r$ in the residue-system of the P_n series will therefore be the same for the factors of F_n, J_n, M_q as for the F_n, J_n, M_q themselves.

Hence the constant term $\mathbf{R}_r = +1$ for all factors (f_n) of F_n and J_n (Art. 43A, B); but the $\mathbf{R}_r$ of factors of M_q usually differ (when reduced to "least residue" form) from each other, and also from the R_r of M_q itself.

Example.—When $q = 11$, $M_q = 2^{11} - 1 = 2047 = 23 \cdot 89$.

Here $r = 5$ for each of the moduli 23, 89, 2047.

This gives $\mathbf{R}_r \equiv 2^9 \pmod{2047}$, and also to mod 23 and 89; but, when reduced to *least* residue form, this gives $R_r \equiv 2^9 \equiv 6 \pmod{23}$, and $\equiv 2^9 \equiv 67 \pmod{89}$.

44. *Table of Residues of P_r* .—Table V., at end of this paper, gives the *complete* Residue-system $\mathbf{R}_r$ of P_r , up to (and including) the constant term, to argument r , for the moduli (m) as follows:—

$m = p = 3$ to 101; $m = p^* = 9$ to 125;
 $m = p > 101$, but < 1000 (23 selected primes);
 $m = 8191, 131071, 524287$ (Mersenne's primes);
 $m = 65537$ (a Fermat's prime); $m = 6700417$ a factor of $(2^{32} + 1)$;
 $m = 10, 100, 1000, 10000, 100000, 1000000$.

The constant term (the last term in each case) is—for sake of distinction—enclosed in parentheses, in old-face type when *positive*, and in usual type when *negative*: thus $p = 23$ has its constant term $\mathbf{R} = (6)$.

To save space in printing, the group of three or four terms common to many of the moduli is (not printed but) indicated by a sign, (* or †) as follows:—

2nd, 3rd, ..., rows.—The * indicates the omission of $r = 0, 1, 2$ for every modulus (p, p^*, m).

$\mathbf{R}_r = 1, 2, 4$

3rd, 5th, ..., rows.—The † indicates the omission of $r = 0, 1, 2, 3$ for every modulus (p or m).

$\mathbf{R}_r = 1, 2, 4, 16$

Special short tables, giving only the constant term ($\mathbf{R}_r$) and the subscript (r) (defining its position) will be found in Art. 43c, 43E.

CHAPTER II.D.—ON HYPER-EXPONENTIAL NUMBERS, $(H_{r,n})$.

45. *Hyper-Exponential Numbers, $(H_{r,n})$.*—It is proposed to attach this name to the number $(H_{r,n})$ figured in the margin, the exponent q being repeated (exponentially) r times.

$$H_{r,n} = q^{\overbrace{q^{\overbrace{\dots q^n}^{r \text{ times}}}}^{r \text{ times}}} \quad (84)$$

The leading properties of their residue-systems $(R_{r,n})$, for any modulus (m) prime to q , are quite similar to those detailed for hyper-even numbers, in that they consist of a unique part and a cycle, the lengths of which, (say h', k), depend in the same way on the sub-moduli (η_r) and sub-Haupt-exponents (ξ_r) of the base q defined in Art. 3, 9, 10, so that

$$k_1 \text{ of } H_{1,n} = \xi_2, \quad k_2 \text{ of } H_{2,n} = \xi_3, \quad \text{and so on,} \quad (85a)$$

$$h_1, h'_2, \dots, \text{ depend on inequalities similar to } (34a), (56b), (60a). \quad (85a)$$

In the case of odd base-numbers (q odd), the hyper-exponentials $(H_{1,n})$ of first order have the following important property, (cf. Art. 22A).

$$\text{If } R_y = -R_x, \text{ then } p = 4\varpi + 1, \text{ and } k = 2(y-x), \quad (86)$$

and the cycle begins with R_h , and consists of two half-cycles alike in all respects except that the equal terms are of opposite sign, i.e.,

$$R_y = -R_x, \text{ if } y-x = \frac{1}{2}k, \quad x \leq h+1. \quad (86a)$$

All these properties (of $H_{1,n}$) are sufficiently illustrated in Tables VI., VII. at the end of this paper.

45A. *Tables of Residues $(R_{1,n})$ of $H_{1,n}$.*—Tables VI., VII., at the end of this paper, give the complete residue-systems $(R_{1,n})$ of $H_{1,n}$, with base $q = 3, 5$, up to the end of the first cycle, to argument n , i.e. from $n = 0$ to $n = h+k$, for the moduli (m) as follows:—

Table VI.: $q = 3$, $H_{1,n} = 3^N$, where $N = 3^n$; $m = p = 2$ to 97, $p^* = 25, 49$; 4 to 128;
 $m = 10, 100.$

Table VII.: $q = 5$, $H_{1,n} = 5^N$, where $N = 5^n$; $m = p = 2$ to 101, $p^* = 9$ to 81; 4 to 128.

For details of the typography, see Art. 26.

CHAPTER III.—RESIDUACITY OF FERMAT'S NUMBERS, (F_n) .

46. *2-ic Relation of p to F_n .*—Let $F_n = E_n + 1$, a Fermat's number. Then the 2-ic relation of p to F_n , i.e. the value of $(p/F_n)_2$, is given by

$$\left(\frac{p}{F_n}\right)_2 = \left(\frac{F_n}{p}\right)_2 = \left(\frac{E_n+1}{p}\right)_2 = \left(\frac{R_n+1}{p}\right)_2 \quad (87)$$

and therefore depends only on the residue (R_n) of E_n to mod p .

This shows that the whole system of Fermat's numbers (F_n), from $n = 0$ upwards, can, in respect to their 2-ic relations to a given prime (p), be arranged similarly to the residue-system (R_n); [see Art. 19].

1°. A *unique* set of $(h+1)$ numbers (F_0 to F_h), whose 2-ic relations (p/F_n) are unique; (88a)

2°. A cycle of k numbers in a cycle (F_{h+1} , F_{h+2} , ..., to F_{h+k}), such that the 2-ic relation recurs; for, taking $x \nless k$,

$$\left(\frac{p}{F_{h+x+mk}}\right)_2 = \left(\frac{R_{h+x+mk}+1}{p}\right)_2 = \left(\frac{R_{h+x}+1}{p}\right)_2 = \left(\frac{p}{F_{h+x}}\right)_2. \quad (88b)$$

[In the above results it does not matter whether p or F_n is prime or composite.]

Examples.—The tables (Nos. II., III.) herewith, giving the values of R_n , enable the 2-ic relation of any prime (p) therein to the whole system of Fermat's numbers to be readily inferred: and, by compounding these results the relation can also be inferred for many composite moduli (p_1 , p_2 , ...).

$$(p/F_n)_2 = -1, \text{ when } p = 3, 5, 7; 6, 10, 12, 14, 20, \dots \quad (89a)$$

$$(p/F_n)_2 = -1 \text{ [if } n = \epsilon\text{]}; = +1 \text{ [if } n = \omega\text{]}, \text{ when } p = 11, 31, \dots \quad (89b)$$

$$(p/F_n)_2 = +1, \text{ when } p = 2, 13, 15, 21, \dots; 241, \dots \quad (89c)$$

$$(p/F_n)_2 = +1, \text{ when } p = F_m \text{ [if } m \text{ and } n > 1\text{]}. \quad (89d)$$

The condition $p < F_n$ is also required in many of the above results.

47. *2-ic Forms F_n .*—Every Fermat's number $F_n = E_n + 1$, (when > 5), is expressible algebraically in the 2-ic forms: [see *Notation*, Art. 18].

$$F_n = a^2 + b^2 \quad \left| \quad = c^2 + 2d^2 \quad \right| \quad = e^2 - 2f^2 \quad (90)$$

$$= 1^2 + E_{n-1}^2 \quad \left| \quad = G_{n-1}^2 + 2E_{n-2}^2 \quad \right| \quad = F_{n-1}^2 - 2E_{n-2}^2; \quad (90a)$$

therefore

$$a = +1, b = E_{n-1} \quad \left| \quad c^* = -G_{n-1}, d = E_{n-2} \quad \right| \quad e = +F_{n-1}, f = E_{n-2}. \quad (90b)$$

48. *4-ic Relations of p to F_n .*—The 4-ic relation of any prime (p) to any Fermat's prime (F_n), *i.e.* the value of $(p/F_n)_4$, is known to depend only on the value of the ratio $b:a \pmod{p}$, *i.e.* on the residue of $b = E_{n-1} \pmod{p}$, [since $a = 1$]. This shows that the whole set of Fermat's numbers ($F_n \nless 5$), from $r = 1$ upwards, can—in respect to their 4-ic relations to

* The odd number $G_{n-1} = (E_{n-1} - 1)$, being of form $(4\gamma - 1)$, the number c is to be reckoned negative (according to the usual convention of signs), so that $c = -G_{n-1} = -(E_{n-1} - 1)$ (and, in using the tabulated residues from Tables II., III. in this connexion, note that they are always the residues of $+E_n, +E_{n-1}, \dots$).

a given prime (p) ,—be arranged similarly to the system of Art. 46 (but starting from F_1).

1°. A *unique* set of $(h+1)$ numbers $(F_1$ to $F_{h+1})$, whose 4-ic relations $(p/F_n)_4$ are unique; followed by— (91a)

2°. A cycle of k numbers in a cycle $(F_{h+2}, F_{h+3}, \dots, F_{h+k-1})$, such that the 4-ic relation recurs; for, taking $x \nabla k$,

$$(p/F_{h+1+x+mk})_4 = (p/F_{h+1+x})_4. \quad (91b)$$

49. *8-ic Relations of p to F_n .*—The 4-ic relation of any prime (p) to any Fermat's prime (F_n) , *i.e.* the value of $(p/F_n)_8$, is known to depend only on the mutual relations of the residues of $a, b, c, d \pmod{p}$; *i.e.* on the residues of $E_{n-1}, E_{n-2} \pmod{p}$. This shows that the whole set of Fermat's numbers $(F_n > 5)$, from $n = 2$ upwards, can—in respect to their 8-ic relations to a given prime (p) ,—be arranged similarly to the system of Art. 46, 48, (but starting from F_2).

1°. A *unique* set of $(h+1)$ numbers $(F_2$ to $F_{h+2})$, whose 8-ic relations $(p/F_n)_8$ are unique; followed by— (92a)

2°. A cycle of k numbers in a cycle $(F_{h+3}, F_{h+4}, \dots, F_{h+k+2})$, such that the 8-ic relation recurs; for, taking $x \nabla k$,

$$(p/F_{h+2+x+mk})_8 = (p/F_{h+2+x})_8. \quad (92b)$$

50. *2-ic, 4-ic, and 8-ic Relations of p to F_n .*—From Art. 46 to 49 it appears that the 2-ic relations of any odd prime (p) to any Fermat's number (F_n) , and also the 4-ic and 8-ic relations of any odd prime p to any Fermat's prime $(F_n > 5)$, *recur* (when $n > h$) in cycles of k terms (h, k being as in Art. 19); so that, writing n in form

$$n = k.t + \nu, \text{ whence } n \equiv \nu \pmod{k}, \quad (93)$$

the question of

$$(p/F_n)_2 = \mp 1, (p/F_n)_4 = \mp 1, (p/F_n)_8 = \mp 1, \text{ depends (when } n > h) \text{ on the value of } \nu; \quad (94)$$

and can therefore be determined for all values of $n (> h)$ by recording it for each prime (p) for one cycle of $n = kt + \nu$, [$\nu = 0$ to $k-1$].

The table on next page shows the values of $n = kt + \nu$ ($n > 1$) which give

$$(p/F_n)_2 = \mp 1, (p/F_n)_4^* = \mp 1, \text{ for all odd primes } p < 50 \text{ [here } h \nabla 1].$$

$$(p/F_n)_8^\dagger = \mp 1 \text{ for all odd primes } p \nabla 43, \text{ except } p = 19, 37.$$

* The values of $(p/F_n)_4$ are reduced up to $p = 29$ from general formulæ up to that limit in Père Th. Pépin's *Mémoire sur la réciprocité relative aux résidus des puissances*, Rome, 1878. Those for $p > 29$ are reduced from general formulæ in the MS. papers of the late C. E. Bickmore, deduced by him from Pépin's general formulæ.

† The values of $(p/F_n)_8$ are reduced from general formulæ in the MS. papers of the late C. E. Bickmore, deduced by him from the law of 8-ic reciprocity given by Herr F. Goldscheider in his memoir *Das Reciprocitätsgesetz der achten Potenzreste*, Berlin, 1889.

p	$(p/F_n)_2 = -1$	$(p/F_n)_2 = +1$	$(p/F_n)_4 = -1$	$(p/F_n)_4 = +1$	$(p/F_n)_8 = -1$	$(p/F_n)_8 = +1$
3, 5, 7	$n > 1$	ω	$4t+1$	$4t-1$	$4t-1$	ϵ
11	ϵ	ω	ω	ϵ	ϵ	ϵ
13	$\cdot$	$n > 1$	ω	ϵ	ϵ	ϵ
17	$\cdot$	> 2	ω	> 2	> 3	3
19	$6t+0, 3$	$6t\pm 1, 2$	$6t\pm 2$	$6t\pm 1$	$6t\pm 1$	$6t\pm 1$
23	$10t+1, 2, 4, 6, 7, 9$	$10t+0, 3, 5, 8$	$10t+0, 5$	$10t+3, 8$	$10t+3$	$10t+8$
29	$3t\pm 1$	$3t$	$3t$	$3t$	$3t$	$3t$
31	ϵ	ω	ω	ϵ	ϵ	ϵ
37	$6t+0, 2, 3, 5$	$6t+1, 4$	$6t-4$	$6t+1$	$6t+1$	$6t+1$
41	> 1	> 1	> 1	> 1	> 1	> 1
43	$3t+0, 1$	$3t+2$	$3t+2$	$3t+2$	$3t+2$	$3t+2$
47	$11t+1, 3, 4, 5, 6, 7$	$11t+0, 2, 8, 9, 10$	$11t+0, 9, 10$	$11t+2, 8$	$11t+2, 8$	$11t+2, 8$

51. *Haupt-Exponents of F_{n+x} , G_{n+x} , ... to mod F_n , G_n , ...*—There are ten cases—shown in the scheme below—in which a base q ($=$ a certain F_v or G_v) has an *unusually small* Haupt-exponent (ξ') to the modulus m ($=$ some F_n , f_n or G_n) specified; and the theorems are true whether the modulus (m) be *prime or composite*.

$$\left. \begin{array}{l} \text{Case 1. } 2. \quad 3. \quad 4. \quad 5. \quad 6. \quad 7. \quad 8. \quad 9. \quad 10. \\ q = F_n, F_{n+x}, G_n, G_n, F_{n+x}, F_{n+x}; F_n, F_n, G_n, G_n \\ m = G_n, G_n, F_n, f_n, F_n, f_n; F_{n+1}, f_{n+1}, F_{n+1}, f_{n+1} \\ \xi' = N, N, 2N, 2N, 2N, 2N; 8N, 8N, 8N, 8N \end{array} \right\} \quad (95)$$

Noting that, as f_n denotes a factor of F_n , therefore

$$q^{3\xi'} \equiv -1, \text{ and } q^{\xi'} \equiv +1 \pmod{F_n} \text{ involve the same to mod } f_n,$$

so that the theorems, if proved for F_n (as modulus), need not be separately proved for f_n (as modulus).

$$\text{Now } 2^N \equiv +1 \pmod{G_n}, \text{ and } 2^N \equiv -1, \text{ and } 2^{2N} \equiv +1 \pmod{F_n \text{ and } f_n}.$$

Hence

$$F_n = E_n + 1 \equiv 2 \pmod{G_n}; \text{ therefore } F_n^N \equiv +1 \pmod{G_n}. \quad (96a)$$

$$F_{n+x} = E_{n+x} + 1 \equiv 2 \pmod{G_n}; \text{ therefore } F_{n+x}^N \equiv +1 \pmod{G_n}. \quad (96b)$$

$$G_n = E_n - 1 \equiv 2 \pmod{F_n}; \text{ therefore } G_n^N \equiv -1 \pmod{F_n \text{ and } f_n}. \quad (96c)$$

$$F_{n+x} = E_{n+x} + 1 \equiv 2 \pmod{F_n}; \text{ therefore } F_{n+x}^N \equiv -1 \pmod{F_n \text{ and } f_n}. \quad (96d)$$

This proves the *first six* theorems as to ξ' ; note that in contrast with above

$$G_{n+x} \equiv 0 \pmod{G_n; \text{ and mod } F_n \text{ and } f_n}.$$

Again, since

$$F_{n+1} = E_{n+1} + 1 = E_n^2 + 1 = \{(F_n - 1)^2 + 1^2\} = \{(G_n + 1)^2 + 1^2\},$$

$$\text{and } F_n^4 + 4 = \{(F_n - 1)^2 + 1^2\} \{(F_n + 1)^2 + 1^2\} \equiv 0 \pmod{F_{n+1}},$$

$$\text{and } G_n^4 + 4 = \{(G_n - 1)^2 + 1^2\} \{(G_n + 1)^2 + 1^2\} \equiv 0 \pmod{F_{n+1}}.$$

$$\text{Therefore } F_n^4 \text{ and } G_n^4 \equiv -4 \equiv -2^2 \pmod{F_{n+1}}. \quad (96e)$$

$$\text{Therefore } F_n^{4N} \equiv G_n^{4N} \equiv +2^{2N} \equiv -1 \pmod{F_{n+1} \text{ and } f_{n+1}}. \quad (96f)$$

This proves the *last four* theorems as to ξ' .

51A. *Maximum Residue-indices* (ν) *in above*.—When the modulus (F_n or F_{n+1}) in the above theorems, (Nos. 3, 5, 7, 9 of Art. 51) is a *prime*, then the maximum residue-index (ν), [see the definition of (1a) of Art. 2] is given by (1a). Thus

$$\left(\frac{G_n}{F_n}\right)_\nu = +1, \text{ and } \left(\frac{F_{n+x}}{F_n}\right)_\nu = +1, \text{ give } \nu = \frac{2^N}{2N} = 2^{N-n-1}. \quad (97a)$$

$$\left(\frac{F_n}{F_{n+1}}\right)_\nu = +1, \text{ and } \left(\frac{G_n}{F_{n+1}}\right)_\nu = +1, \text{ give } \nu = \frac{2^{2N}}{8N} = 2^{2N-n-3}. \quad (97b)$$

Thus the maximum residue-index in the above cases is always a high number when $n > 4$; and as to a *lesser* residue-index (ν') common to the whole of the series ($\nu' < \nu$),

$$(G_n/F_n)_{16} = +1, \text{ and } (F_{n+x}/F_n)_{16} = +1, \text{ always, } [n \nless 3]; \quad (97c)$$

$$(G_n/F_{n+1})_8 = +1, \text{ and } (F_n/F_{n-1})_8 = +1, \text{ always, } [n \nless 2]. \quad (97d)$$

52. *4-ic and 8-ic Relations of* $F_n, \dots$ *to* F_{n+x} .—It is proposed here to discuss the 4-ic and 8-ic relations of F_n and of its factors (f_n) to the *prime* modulus F_{n+x} . It has been already shown (Art. 46) that $F_0 = 3$, and $F_1 = 5$ (given by $n = 0, 1$) are primitive roots of every prime F_{n+x} , and it has also been shown (Art. 51A) that

$$(F_n/F_{n+1})_8 = +1, \quad (G_n/F_{n+1})_8 = +1, \text{ always, } [n > 1, F_{n+1} \text{ prime}]. \quad (98)$$

These cases may therefore be excluded, so that it suffices to assume in what follows—

$$n > 1, \quad x > 1, \quad n+x > 2, \quad F_n > 5, \quad F_{n+x} > 17. \quad (99)$$

As explained in Art. 48, 49 the 4-ic and 8-ic relations of any prime base $q = F_n = 8\kappa+1$ to the prime modulus $p = F_{n+x} = 8\omega+1$ depend on the residues of the (a, b), (c, d) of the 2-ic partitions of F_{n+x} to the smaller modulus $q = F_n$.

$$\text{Now} \quad F_{n+x} = a_{n+x}^2 + b_{n+x}^2 = c_{n+x}^2 + 2d_{n+x}^2 \quad (100a)$$

$$= 1 + E_{n+x-1}^2 = G_{n+x-1}^2 + 2E_{n+x-2}^2. \quad (100b)$$

$$\text{Here } a_{n+x} = +1 \text{ always,} \quad (100c)$$

$$b_{n+x} = E_{n+x-1} \equiv +1, \pmod{F_n, \text{ and mod each } f_n}, [n > 1, x > 1] \quad (100d)$$

$$c_{n+x}^* = -(E_{n+x-1} - 1) \equiv 0, \pmod{F_n, \text{ and mod each } f_n}, [n > 1, x > 1] \quad (100e)$$

52A. *4-ic Relations of* $F_n, \dots$ *to* F_{n+x} .—The 4-ic relation of a *prime*

* See the footnote * to Art. 47 as to the sign of c.

base $q = F_n$ or $f_n = 16\kappa + 1$ to a prime modulus $p = F_{n+x} = 16\varpi + 1$ is given by*

$$\begin{aligned} \left(\frac{q}{F_{n+x}}\right)_4 &= \left(\frac{a_{n+x} + b_{n+x} \cdot i}{a_{n+x} - b_{n+x} \cdot i}\right)^{\frac{1}{4}(q-1)} \pmod{q = F_n \text{ or } f_n} \\ &= \text{residue of } \left(\frac{1+i}{1-i}\right)^{4\kappa} \equiv \left(\frac{-2i}{2}\right)^{4\kappa} = +1 \\ &\pmod{q = F_n \text{ or } f_n}, [n > 1, x > 1]; \end{aligned}$$

therefore

$$\left(\frac{F_n \text{ and } f_n}{F_{n+x}}\right)_4 = +1, \quad [\text{when } n > 1, \text{ and } x > 1; \text{ and } F_{n+x} \text{ and } F_n \text{ or } f_n \text{ are prime}], \quad (101)$$

and, by (98) this is also true when $x = 1$; hence, more generally—

$$\left(\frac{F_n \text{ and } f_n}{F_{n+x}}\right)_4 \equiv +1, \quad [\text{when } F_{n+x} \text{ and } F_n \text{ or } f_n \text{ are prime; and } n > 1, x \nless 1]. \quad (101a)$$

52B. 8-ic Relations.—The 8-ic relations of a prime base $q = F_n$ or $f_n = 8\kappa + 1$ to a prime modulus $p = F_{n+x} = 8\varpi + 1$ is given by†

$$\begin{aligned} \left(\frac{q}{F_{n+x}}\right)_8 &\equiv \left(\frac{a_{n+x} + b_{n+x} \cdot i}{a_{n+x} - b_{n+x} \cdot i}\right)^{\frac{1}{8}(q-1)} \cdot \left(\frac{c_{n+x} + d_{n+x} \sqrt{-2}}{c_{n+x} - d_{n+x} \sqrt{-2}}\right)^{\frac{1}{8}(q-1)}, \pmod{q = F_n \text{ or } f_n} \\ &= \text{residue of } \left(\frac{1+i}{1-i}\right)^2 \cdot (-1)^4 \equiv \left(\frac{2i}{2}\right)^2 = -1, \\ &\pmod{F_2 = 17, \text{ when } n = 2, x > 1} \end{aligned}$$

$$\begin{aligned} \text{and} \quad &= \text{residue of } \left(\frac{1+i}{1-i}\right)^{4j} \cdot (-1)^{4j} = \left(\frac{2i}{2}\right)^{4j} = +1, \\ &\pmod{F_n > 17, \text{ and } f_n, \text{ when } n > 2, x > 1}. \end{aligned}$$

$$\text{Hence} \quad \left(\frac{F_2 = 17}{F_{n+x}}\right)_8 = -1, \quad [n = 2, x > 1], \quad (102a)$$

$$\left(\frac{F_n \text{ and } f_n}{F_{n+x}}\right)_8 = +1, \quad [F_n > 17, n > 2, x > 1]. \quad (102b)$$

* This form of the 4-ic criterion (which is especially convenient in the present case) is given in Père Th. Pépin's *Mémoire sur la réciprocité relative aux résidus des puissances*, Rome, 1878, Art. 32.

† This form of the 8-ic criterion (which is especially convenient in the present case) was deduced by the late Mr. Chas. E. Bickmore from the law of 8-ic reciprocity given by Herr F. Goldscheider in his memoir *Das Reciprocitätsgesetz der achten Potenzreste*, Berlin, 1889; it has not, however, yet appeared in print.

53. 8-ic Relations of G_n to F_{n+x} .—It has been shown, [Art. 51A, (97c)]

$$(G_{n+x}/F_{n+x})_8 = +1 \text{ always, [when } n+x \nless 3, F_{n+x} \text{ prime].}$$

Also $G_{n+x} = G_n \cdot (F_n \cdot F_{n+1} \cdot F_{n+2} \dots F_{n+x-1}) = G_n \cdot \Pi(F)$ suppose; therefore

$$\left(\frac{G_{n+x}}{F_{n+x}}\right)_8 = \left(\frac{G_n}{F_{n+x}}\right)_8 \cdot \left(\frac{\Pi(F)}{F_{n+x}}\right)_8 = +1$$

and $[\Pi(F)/F_{n+x}]_8 = +1$, since every $(F_{n+z}/F_{n+x})_8 = +1$, by (102b), $[z < x, n \nless 3]$; therefore

$$\left(\frac{G_n}{F_{n+x}}\right)_8 = +1, [n \nless 3]. \quad (103)$$

Also, when $n = 3$, then

$$\left(\frac{G_3}{F_{n+x}}\right)_8 = \left(\frac{G_2 \cdot F_2}{F_{n+x}}\right)_8 = \left(\frac{G_2}{F_{n+x}}\right)_8 \cdot \left(\frac{F_2}{F_{n+x}}\right)_8 = +1.$$

But $(F_2/F_{n+x})_8 = -1$, [see (102a)]; therefore $(G_2/F_{n+x}) = -1^*$. (103a)

Thus* $(15/F_{n+x})_8 = -1$, and $(17/F_{n+x})_8 = -1$, $[n+x > 3]$. [103b]

53A. 4-ic Relations of G_n to F_{n+x} .—The above results for the 8-ic relations of G_n to F_{n+x} prove also that

$$(G_n/F_{n+x}) = +1, [n \nless 2, n+x \nless 3], \quad (104a)$$

except the last result (103b), which here becomes

$$(15/F_{n+x})_4 = +1, (17/F_{n+x})_4 = +1, [n+x \nless 3]. \quad (104b)$$

54. Formula for $(F_n/F_{n+x})_\nu$.—The 2-ic, 4-ic, and 8-ic criteria for F_n with respect to the modulus F_{n+x} may be included in the single form

$$(F_n/F_{n+x})_\nu = i^{(F_n-1)/\nu} [\nu = 2, 4, 8; x > 1; F_{n+x} \text{ prime}]. \quad (105)$$

Even the cases of $n = 0, 1$, i.e. of $F_0 = 3$, $F_1 = 5$, are included in the above. This simple form is known to be true at present only as far as $\nu = 8$; but it seems probable† that it applies also when $n = 16$ (and perhaps higher). For, the author has found reason to believe‡ that

If $\phi(x, y) \equiv 0 \pmod{q}$ is the criterion of $(q/p)_8 = \mp 1$, when $p = x^4 + y^4$, then it is also the criterion of $(q/p)_{16} \equiv \mp 1$, when $p = x^8 + y^8$. (106)

* This result could also—but with much more trouble—be deduced direct from the law of 8-ic reciprocity.

† It seems impossible to test this by numerical trial at present, as there are no known primes $F_{n+x} (> F_4)$ available.

‡ It has been found true for $q = \overline{3}, 5, \overline{7}$ in the very few cases in which numerical verification is possible with such primes: a similar connexion between $(2/p)_{16}$ and $(2/p)_{32}$ has also been verified to some extent.

TABLE I.—*continued.*

Base $q = 5$; $\xi_r = 5^{\alpha_r} \cdot \eta_r$, $5^{\xi_{r+1}} \equiv +1 \pmod{\eta_r}$; ξ a min., α a max., $\eta \neq 5i$.

p	2	3	7	11	13	17	19	23	29	31	37	41	43
r	1	1	1, 2	1	1	1, 2	1, 2, 3	1, 2	1, 2, 3	1, 2	1, 2, 3	1, 2	1, 2, 3
ξ_r	1	2	6, 2	5	4	16, 4	9, 6, 2	22, 5	14, 6, 2	3, 2	36, 6, 2	20, 4	42, 6, 2

p	47	53	59	61	67	71	73	79	83	89
r	1, 2, 3	1, 2	1, 2, 3, 4	1, 2	1, 2	1	1, 2, 3	1, 2	1, 2, 3	1, 2
ξ_r	46, 22, 5	52, 4	29, 14, 6, 2	30, 2	22, 5	5	72, 6, 2	39, 4	82, 20, 4	44, 5

p	97	101	103
r	1, 2, 3	1	1, 2, 3
ξ_r	96, 8, 2	25	102, 16, 4

p^*	9	27	49	81	121	4	8	16	32	64	128
r	1, 2	1, 2, 3	1, 2, 3	1, 2, 3, 4	1, 2	1	1	1	1, 2	1, 2	1, 2, 3
ξ_r	6, 2	18, 6, 2	42, 6, 2	54, 18, 6, 2	55, 5	1	2	4	8, 2	16, 4	32, 8, 2

TABLE II.—*Residues R_n of $E_n \pmod{p, p^*}$.*

$$[E_n = 2^N, \quad N = 2^n.]$$

p	3; (2, 1)	5; (4, 1)	7; (3, 2)	11; $\xi = (10, 4)$	13; (12, 2)	17; (8, 1)	19; $\xi = (18, 6)$
n	0, 1	0, 1, 2	0, 1	0, 1, 2, 3, 4	0, 1, 2, 3	0, 1, 2, 3	0, 1, 2, 3, 4, 5, 6
R	1, (1)	2, 1, (1)	(2, 3)	2, (4, 5, 3, 2)	2, 4, (3, 4)	2, 4, 1, (1)	2, (4, 3, 9, 5, 6, 2)

p	23; $\xi = (11, 10)$	29; (28, 3)	31; (5, 4)	37; $\xi = (36, 6)$
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	0, 1, 2, 3, 4	0, 1, 2, 3	0, 1, 2, 3, 4, 5, 6, 7
R	(2, 4, 7, 3, 9, 11, 6, 10, 8, 5)	2, 4, (13, 5, 4)	(2, 4, 15, 8)	2, 4, (16, 3, 9, 7, 12, 4)

p	41; $\xi = (20, 4)$	43; (14, 3)	47; $\xi = (23, 11)$
n	0, 1, 2, 3, 4, 5	0, 1, 2, 3	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10
R	2, 4, (16, 10, 18, 4)	2, (4, 16, 2)	(2, 4, 16, 21, 18, 5, 22, 14, 8, 17, 7)

p	53; $\xi = (52, 12)$	59	61; $\xi = (60, 4)$
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13		0, 1, 2, 3, 4, 5
R	2, 4, (16, 9, 25, 11, 15, 13, 10, 8, 17, 24, 7, 4)	infra	2, 4, (16, 12, 22, 4)

p	59; $\xi = (58, 28)$
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28
R	2, (4, 16, 20, 13, 8, 5, 25, 24, 14, 19, 7, 10, 18, 29, 15, 11, 3, 9, 22, 12, 26, 27, 21, 28, 17, 6, 23, 2)

p	61	67; $\xi = (66, 10)$	71; $\xi = (35, 12)$
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10		0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
R	supra	2, (4, 16, 12, 10, 33, 17, 21, 28, 20, 2)	(2, 4, 16, 28, 3, 9, 10, 29, 11, 21, 15, 12)

p	73; $\xi = (9, 6)$	79; $\xi = (39, 12)$
n	0, 1, 2, 3, 4, 5	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11
R	(2, 4, 16, 36, 18, 32)	(2, 4, 16, 19, 34, 29, 28, 6, 36, 32, 3, 9)

p	83; $\xi = (82, 20)$
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20
R	2, (4, 16, 7, 34, 6, 36, 32, 28, 37, 41, 21, 26, 12, 22, 14, 30, 13, 3, 9, 2)

TABLE II.—*continued.*

p	89;	$\xi = (11, 10)$	97;	$\xi = (48, 2)$
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9		0, 1, 2, 3, 4, 5	
R	(2, 4, 16, 11, 32, 44, 22, 39, 8, 25)		2, 4, 16, 35, (36, 35)	
p	101;	$\xi = (100, 20)$		
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21			
R	2, 4, (16, 47, 13, 33, 22, 21, 37, 45, 5, 25, 19, 43, 31, 49, 23, 24, 30, 9, 20, 4)			
p^*	9; (6, 2)	25; $\xi = (20, 4)$	27; $\xi = (18, 6)$	49; $\xi = (21, 6)$
n	0, 1, 2	0, 1, 2, 3, 4, 5	0, 1, 2, 3, 4, 5, 6	0, 1, 2, 3, 4, 5, 6
R	2, (4, 2)	2, 4, (9, 6, 11, 4)	2, (4, 11, 13, 7, 5, 2)	2, (4, 16, 11, 23, 10, 2)
p^*	81;	$\xi = (54, 18)$		
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18			
R	2, (4, 16, 13, 7, 32, 29, 31, 11, 40, 20, 5, 25, 23, 38, 14, 34, 22, 2)			

TABLE III.—*Residues R_n of E_n , (mod p).*

$$[E_n = 2^N, \quad N = 2^n.]$$

p	113; $\xi = (28, 3)$	127; (7, 3)	137; $\xi = (68, 8)$	151; (15, 4)
n	0, 1, 2, 3, 4	0, 1, 2	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	0, 1, 2, 3
R	2, 4, (16, 30, 4)	(2, 4, 16)	2, 4, (16, 18, 50, 34, 60, 38, 63, 4)	(2, 4, 16, 46)
p	157; $\xi = (52, 12)$	193; $\xi = (96, 2)$		
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13	0, 1, 2, 3, 4, 5, 6		
R	2, 4, (16, 58, 67, 64, 14, 39, 49, 46, 75, 27, 56, 4)	2, 4, 16, 63, 84, (85, 84)		
p	233; $\xi = (29, 28)$			
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27			
R	(2, 4, 16, 23, 63, 8, 64, 98, 51, 38, 46, 19, 105, 74, 116, 58, 102, 81, 37, 29, 91, 107, 32, 92, 76, 49, 71, 85)			
p	241; (24, 2)			
n	0, 1, 2, 3, 4			
R	2, 4, 16, (15, 16)			
p	251; $\xi = (50, 20)$			
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20			
R	2, (4, 16, 5, 25, 123, 69, 8, 64, 80, 125, 63, 47, 50, 10, 100, 40, 94, 51, 91, 2)			
p	257; (16, 1)	281; $\xi = (70, 12)$	337; $\xi = (21, 6)$	
n	0, 1, 2, 3, 4	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	0, 1, 2, 3, 4, 5	
R	2, 4, 16, 1, (1)	2, (4, 16, 25, 63, 35, 101, 85, 81, 98, 50, 29, 2)	(2, 4, 16, 81, 158, 26)	
p	353; $\xi = (88, 10)$			
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12			
R	2, 4, 16, (97, 122, 58, 166, 22, 131, 138, 140, 168, 16)			
p	397; $\xi = (44, 10)$			
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11			
R	2, 4, (16, 141, 31, 167, 99, 124, 107, 64, 126, 4)			
p	401; $\xi = (200, 20)$			
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22			
R	2, 4, 16, (115, 173, 146, 63, 41, 77, 86, 178, 58, 25, 177, 51, 195, 70, 88, 125, 14, 196, 80, 16)			

TABLE III.—continued.

p	433;	$\xi = (72, 6)$	577;	$\xi = (144, 6)$		
n	0, 1, 2, 3, 4, 5, 6, 7, 8		0, 1, 2, 3, 4, 5, 6, 7, 8, 9			
R	2, 4, 16, (177, 153, 27, 137, 150, 16)		1, 4, 16, 256, (242, 287, 142, 31, 193, 256)			
p	641;	$\xi = (64, 1)$	673;	$\xi = (48, 2)$	769;	$\xi = (364, 2)$
n	0, 1, 2, 3, 4, 5, 6		0, 1, 2, 3, 4, 5		0, 1, 2, 3, 4, 5, 6, 7, 8	
R	2, 4, 16, 256, 154, 1, (1)		2, 4, 16, 256, (255, 256)		2, 4, 16, 256, 171, 19, 361, (360, 361)	
p	8191;	$\xi = (13, 12)$	6700417;	$\xi = (64, 1)$		
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11		0, 1, 2, 3, 4, 5, 6			
R	(2, 4, 16, 256, 8, 64, 4096, 2048, 512, 32, 1024, 128)		2, 4, 16, 256, 65536, 1, (1)			
m	10; (4, 1)	100;	$\xi = (20, 4)$			
n	0, 1, 2	0, 1, 2, 3, 4, 5				
R	2, 4, (6)	2, 4, (16, 56, 36, 96)				

TABLE IV.—Residues R_n of E_n , (mod p).

$$[E_n = 2^{E_n}, E_n = 2^N, N = 2^n.]$$

p	3;	5; (1)	7; (1)	11; (1)	13; (1)	17; (1)	19; (2)	23; (4)	29; (2)		
n	0	0, 1	0, 1	0, 1, 2	0, 1	0, 1, 2	0, 1, 2	0, 1, 2, 3, 4	0, 1, 2		
R	(1)	1, (1)	3, (2)	4, 5, (2)	4, (3)	4, 1, (1)	4, (3, 5)	4, (7, 9, 8, 6)	4, (13, 4)		
p	31; (1)	37; (2)	41; (1)	43; (2)	47;	(10)					
n	0, 1, 2	0, 1, 2	0, 1, 2	0, 1	0, 1, 2, 3, 4, 5, 6, 7, 8, 9						
R	4, 15, (2)	4, (16, 9)	4, 16, (18)	(4, 16)	(4, 16, 18, 8, 5, 7, 17, 14, 21, 22)						
p	53;	(2)	59;	(3)	61; (1)	67;	(4)	71;	(2)		
n	0, 1, 2, 3		0, 1, 2, 3, 4		0, 1, 2	0, 1, 2, 3, 4		0, 1, 2, 3			
R	4, 16, (25, 10)		4, 16, (13, 24, 11)		4, 16, (22)	4, (16, 10, 28, 17)		4, 16, (3, 11)			
p	73;	(2)	79;	(2)	83;	(4)	89;	(4)	97;	(1)	
n	0, 1, 2		0, 1, 2, 3		0, 1, 2, 3, 4, 5		0, 1, 2, 3, 4		0, 1, 2		
R	4, (16, 18)		4, 16, (34, 36)		4, 16, (34, 28, 30, 26)		4, (16, 32, 8, 22)		4, 16, (36)		
p	101;	(4)									
n	0; 1, 2, 3, 4, 5										
R	4, 16, (13, 37, 23, 19)										
p	9; (1)	25; (1)	27; (2)	49; (2)	81;	(6)	125;	(4)			
n	0, 1	0, 1, 2	0, 1, 2	0, 1, 2	0, 1, 2, 3, 4, 5, 6		0, 1, 2, 3, 4, 5				
R	4, (2)	4, 9, (11)	4, (11, 7)	4, (16, 23)	4, (16, 7, 11, 34, 38, 20)		4, 9, (36, 61, 14, 39)				
p	113;	(2)	127; (2)	137;	(1)	151;	(1)	157;	(2)	193;	(1)
n	0, 1, 2		0, 1	0, 1, 2, 3		0, 1, 2		0, 1, 2, 3		0, 1, 2, 3	
R	4, (16, 4)		(4, 16)	4, 16, 50, (63)		4, 16, (2)		4, 16, (67, 49)		4, 16, 84, (84)	
p	233;	(3)	241;	(1)	251;	(4)	257;	(1)	281;	(2)	
n	0, 1, 2, 3, 4		0, 1, 2		0, 1, 2, 3, 4, 5		0, 1, 2		0, 1, 2, 3		
R	4, 16, (63, 51, 102)		4, 16, (16)		4, 16, (25, 64, 40, 47)		4, 16, (1)		4, 16, (63, 81)		
p	337;	(2)	353;	(4)	397;	(4)					
n	0, 1, 2		0, 1, 2, 3, 4, 5		0, 1, 2, 3, 4						
R	4, (16, 158)		4, 16, (122, 131, 166, 16)		4, (16, 31, 107, 99)						

TABLE IV.—*continued.*

p	401;	(4)	433;	(2)	577;	(2)	641;	(1)
n	0, 1, 2, 3, 4, 5		0, 1, 2, 3		0, 1, 2, 3		0, 1, 2, 3	
R	4, 16, (173, 77, 70, 25)		4, 16, (153, 16)		4, 16, (242, 193)		4, 16, 154, (1)	
p	673;	(1)	769;	(2)				
n	0, 1, 2		0, 1, 2, 3					
R	4, 16, (255)		4, 16, 171, (361)					
p	8191;	(2)	131071;	(1)	65537;	(1)	6700417;	(1)
n	0, 1, 2, 3		0, 1, 2, 3		0, 1, 2, 3		0, 1, 2, 3	
R	4, 16, (8, 512)		4, 16, 65535, (2)		4, 16, 1, (1)		4, 16, 65536, (1)	
m	10; (1)	100; (1)	1000;	(4)				
n	0, 1	0, 1, 2	0, 1, 2, 3, 4, 5					
R	4, (6)	4, 16, (36)	4, 16, (536, 936, 736, 336)					

TABLE V.—*Residues (R_r) of Perfectly Even Numbers, (P_r), (mod p and p^r).*

$$P_0 = 1, P_1 = 2, P_2 = 4, P_3 = 16, P_4 = 65536, \dots, P_{r+1} = 2^{P_r}.$$

The * means $\begin{cases} r = 0, 1, 2; \\ R_r = 1, 2, 4; \end{cases}$ The † means $\begin{cases} r = 0, 1, 2, 3; \\ R_r = 1, 2, 4, 16. \end{cases}$

p	3	5	7	11	13	17	19
r	0, 1, 2	0, 1, 2, 3	0, 1, 2, 3	0, 1, 2, 3, 4	0, 1, 2, 3	0, 1, 2, 3, 4	0, 1, 2, 3, 4
R	1, 1, (1)	1, 2, 1, (1)	1, 2, 3, (2)	1, 2, 4, 5, (2)	1, 2, 4, (3)	1, 2, 4, 1, (1)	1, 2, 4, 3, (5)

p	23	29	31	37	41	43	47	53
r	0, 1, 2, 3, 4, 5	*, 3, 4	*, 3, 4	*, 3, 4	*, 3, 4	*, 3, 4	*, 3, 4, 5, 6	*, 3, 4
R	1, 2, 4, 7, 9, (6)	13, (4)	15, (2)	16, (9)	16, (18)	16, (4)	16, 18, 5, (17)	16, (25)

p	59	61	67	71	73	79	83	89	97	101
r	*, 3, 4, 5	*, 3, 4	*, 3, 4, 5	†, 4	†, 4	†, 4	†, 4, 5	†, 4, 5	†, 4	†, 4, 5
R	16, 13, (11)	16, (22)	16, 10, (17)	(3)	(18)	(34)	34, (30)	32, (22)	(36)	13, (23)

p^r	9	25	27	49	81	121	125
r	*, 3	*, 3, 4	*, 3, 4	*, 3, 4	*, 3, 4, 5	*, 3, 4, 5	*, 3, 4, 5
R	(2)	9, (11)	11, (7)	16, (23)	16, 7, (34)	16, 75, (76)	16, 36, (111)

p	113	127	137	151	157	179	193	233	241	251
r	†, 4	†, 4	†, 4, 5	†, 4	†, 4	†, 4, 5, 6	†, 4, 5	†, 4, 5	†, 4	†, 4, 5
R	(4)	(4)	50, (63)	(2)	(67)	22, 53, (82)	84, (84)	63, (102)	(16)	25, (40)

p	257	281	337	353	359	397	401	433	577
r	†, 4	†, 4	†, 4	†, 4, 5	†, 4, 5, 6, 7	†, 4, 5	†, 4, 5	†, 4, 5	†, 4
R	(1)	(63)	(158)	122, (166)	161, 107, 66, (164)	31, (99)	173, (70)	(153)	(242)

p	641	673	719	769	8191	65537	131071
r	†, 4, 5	†, 4	†, 4, 5, 6, 7, 8	†, 4, 5	†, 4	†, 4, 5	†, 4, 5
R	154, (1)	(255)	107, 624, 257, 123, (212)	171, (361)	(8)	1, (1)	65535, (2)

p	524287	6700417
r	†, 4, 5	†, 4, 5
R	65536, (32)	65536, (1)

TABLE V.—continued.

m	10	100	1000	10000	100000
r	*, 3	†, 4	†, 4, 5	†, 4, 5, 6	†, 4, 5, 6, 7
R	(6)	(36)	536, (736)	5536, 6736, (8736)	65536, 56736, 28736, (48736)

m	1000000
r	†, 4, 5, 6, 7, 8
R	65536, 156736, 428736, 748736, (948736)

TABLE VI.—Residues (R_n) of H_n (mod p , p^x , or m).

$$H_n = 3^N, \quad N = 3^n.$$

p	2; (1, 1)	5; (4, 2)	7; (6, 1)	11; (5, 4)	13; (3, 1)	17; (16, 4)	19; (18, 1)
n	0	0, 1	0, 1	0, 1, 2, 3	0, 1	0, 1, 2, 3	0, 1, 2
R_n	(1)	(2, 2)	3, (1)	(3, 5, 4, 2)	3, (1)	(3, 7, 3, 7)	3, 8, (1)

p	23; $\xi = (11, 5)$	29; $\xi = (28, 6)$	31; $\xi = (30, 4)$	37; (18, 1)	41; (8, 2)
n	0, 1, 2, 3, 4	0, 1, 2, 3, 4, 5	0, 1, 2, 3, 4	0, 1, 2	0, 1
R_n	(3, 4, 5, 10, 12)	(3, 2, 8, 10, 14, 11)	3, (4, 2, 8, 15)	3, 10, (1)	(3, 14)

p	43; $\xi = (42, 6)$	47; $\xi = (23, 11)$	53; $\xi = (52, 6)$
n	0, 1, 2, 3, 4, 5, 6	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10	0, 1, 2, 3, 4, 5
R_n	3, (16, 11, 2, 8, 4, 21)	(3, 20, 10, 13, 12, 11, 15, 9, 23, 6, 19)	(3, 26, 20, 3, 26, 20)

p	59; $\xi = (29, 28)$
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27
R_n	(3, 27, 23, 13, 14, 29, 22, 28, 4, 5, 7, 11, 26, 6, 20, 24, 18, 9, 21, 2, 8, 19, 15, 12, 17, 16, 25, 10)

p	61; (10, 4)	67; $\xi = (22, 5)$
n	0, 1, 2, 3	0, 1, 2, 3, 4
R_n	(3, 27, 20, 9)	(3, 27, 15, 25, 14)

p	71; $\xi = (35, 12)$	73; (12, 2)	79; (78, 3)
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	0, 1, 2	0, 2, 3, 4
R_n	(3, 27, 16, 22, 2, 8, 15, 33, 11, 18, 10, 6)	3, (27, 27)	1, (27, 12, 10)

p	83; $\xi = (41, 8)$	89; $\xi = (88, 10)$	97; (48, 4)
n	0, 1, 2, 3, 4, 5, 6, 7	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11	0, 1, 2, 3, 4
R_n	(3, 27, 12, 15, 28, 40, 7, 11)	3, 27, (17, 18, 42, 40, 9, 17, 18, 42, 40, 9)	3, (27, 8, 27, 8)

p^x	4	8	16	32	64; (16, 4)	128; $\xi = (32, 8)$
n	0	0	0, 1	0, 1	0, 1, 2, 3	0, 1, 2, 3, 4, 5, 6, 7
R_n	(1)	(3)	(3, 5)	(3, 5)	(3, 27, 29, 5)	(3, 27, 29, 59, 61, 37, 35, 5)

m	25; (20, 4)	49; $\xi = (42, 6)$	10; (4, 2)	100; (20, 4)
n	0, 1, 2, 3	0, 1, 2, 3, 4, 5, 6	0, 1	0, 1, 2, 3
R_n	(3, 2, 8, 12)	3, (22, 15, 6, 20, 13, 8)	(3, 7)	(3, 27, 83, 87)

TABLE VII.—*Residues* (R_n) of H_n (mod p or p^*).

$$H_n = 5^N, \quad N = 5^n.$$

p	2; (1, 1)	3; (2, 1)	7; (6, 2)	11; (5, 1)	13; (4, 1)	17; (16, 4)	19; $\xi = (9, 6)$
n	0	0	0, 1	0, 1	0	0, 1, 2, 3	0, 1, 2, 3, 4, 5
R_n	(1)	(1)	(2, 3)	(5, 1)	(5)	(5, 3, 5, 3)	(5, 9, 3, 4, 2, 6)
p	23; $\xi = (22, 5)$	29; $\xi = (14, 6)$	31; (3, 2)	37; $\xi = (36, 6)$	41; (20, 1)	43; $\xi = (42, 6)$	
n	0, 1, 2, 3, 4	0, 1, 2, 3, 4, 5	0, 1	0, 1, 2, 3, 4, 5	0, 1	0, 1, 2, 3, 4, 5	
R_n	(5, 3, 10, 4, 11)	(5, 7, 13, 6, 4, 9)	(5, 6)	(5, 17, 18, 15, 13, 2)	5, (9)	(5, 14, 20, 17, 3, 15)	
p	47; $\xi = (46, 22)$	53; (52, 4)					
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21	0, 1, 2, 3					
R_n	(5, 23, 22, 12, 14, 3, 8, 9, 17, 13, 7, 19, 2, 15, 4, 10, 16, 6, 21, 11, 18, 20)	(5, 2, 21, 26)					
p	59; $\xi = (29, 14)$	61; (30, 2)	67; $\xi = (22, 5)$	71; (5, 1)			
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13	0, 1, 2	0, 1, 2, 3, 4	0, 1			
R_n	(5, 2, 27, 11, 19, 13, 6, 12, 29, 24, 16, 28, 9, 10)	5, (14, 13)	(5, 24, 9, 22, 8)	5, (1)			
p	73; $\xi = (72, 6)$	79; (39, 4)					
n	0, 1, 2, 3, 4, 5	0, 1, 2, 3					
R_n	(5, 14, 33, 20, 28, 34)	(5, 35, 11, 30)					
p	83; $\xi = (82, 20)$	89; $\xi = (44, 5)$					
n	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19	0, 1, 2, 3, 4					
R_n	(5, 29, 23, 25, 11, 31, 39, 38, 35, 27, 33, 20, 18, 10, 15, 8, 17, 24, 19, 40)	(5, 10, 36, 21, 20)					
p	97; $\xi = (96, 8)$	101; (25, 1)					
n	0, 1, 2, 3, 4, 5, 6, 7	0, 1, 2					
R_n	(5, 21, 13, 23, 5, 21, 13, 23)	5, 6, (1)					
p^*	4	8	16	32	64; (16, 4)	128; $\xi = (32, 8)$	
n	0	0	0	0, 1	0, 1	0, 1, 2, 3, 4, 5, 6, 7	
R_n	(1)	(3)	(5)	(5, 11)	(5, 11, 27, 21)	(5, 53, 37, 43, 59, 11, 27, 21)	
p^*	9; (6, 2)	27; $\xi = (18, 6)$	81; $\xi = (54, 18)$				
n	0, 1	0, 1, 2, 3, 4, 5	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17				
R_n	(4, 2)	(5, 7, 13, 11, 4, 2)	(5, 34, 13, 11, 23, 2, 32, 20, 14, 16, 31, 25, 22, 7, 40, 38, 4, 29)				
p^*	49; $\xi = (42, 6)$						
n	0, 1, 2, 3, 4, 5						
R_n	(5, 11, 12, 10, 9, 4)						

ON THE UNIFORM CONVERGENCE OF FOURIER'S SERIES

By E. W. HOBSON.

[Received and Read January 10th, 1907.]

THE definition of an integral introduced by Lebesgue has led to a notable increase in the scope of Fourier's series. A Fourier's series corresponding to a function $f(x)$ defined in the interval $(-\pi, \pi)$ may be said to exist, whenever the coefficients have a definite meaning, independently of any question as to the convergence of the series. A large class of functions are integrable in accordance with Lebesgue's definition which are not integrable in accordance with the definition of Riemann. For such functions the Fourier's coefficients exist, if the integrals by which they are expressed are interpreted as Lebesgue's integrals. Investigations relating to this extended class of Fourier's series have been carried out by Lebesgue.* Every summable limited function has a Lebesgue integral, and also some summable functions which are not limited. The latter integrals are always absolutely convergent, in the sense that the absolute values of the functions are also integrable. I have elsewhere shewn that the Lebesgue definition of such an integral is in agreement with the older definitions in cases when those definitions are all applicable. The only integrals which are not covered by Lebesgue's definition are those which are non-absolutely convergent improper integrals. A series in which the coefficients are expressed by such integrals, Lebesgue has proposed to call "generalized Fourier's series." Very little is known about such series, and they will not be referred to in the present communication, which is concerned with the class of Fourier's series in which the coefficients are expressed by Lebesgue integrals, the function employed being either limited or unlimited. Many of the older investigations connected with the theory of Fourier's series were directed to obtaining sufficient conditions for the convergence of the series at a particular point.

* His investigations are contained in two memoirs: "Sur les séries trigonométriques," in the *Annales Sc. de l'Ecole Normale*, Sér. 3, Vol. xx., 1903; and "Sur la convergence des séries de Fourier," *Math. Annalen*, Vol. lxxiv., 1905; also in the work *Leçons sur les séries trigonométriques*, 1906.

Later investigations were concerned with conditions that the series might converge uniformly in a whole interval contained in the interval $(-\pi, \pi)$. The present communication is concerned with the latter question, the function to be represented being taken to possess a Lebesgue integral, but not necessarily a Riemann integral, in the interval $(-\pi, \pi)$. A theorem of a general character has been established in § 2, and the detailed results are deduced from this theorem. It seems not unlikely that this theorem might prove useful in other investigations of a similar character in which the series are not trigonometrical ones, but involve other kinds of periodic functions.

1. The following theorem relating to linear sets of points will be required :—

Let G be a measurable set of points contained in an interval (a, b) , and let the set G be enclosed in the interiors of linear intervals forming a finite, or enumerably infinite, set H of non-overlapping intervals, such that $m(H) - m(G) = \eta$, where η is a positive number. If (c, d) be any sub-interval whatever, contained in (a, b) , and if G_1, H_1 be the parts of G and of H contained in (c, d) , then

$$m(H_1) - m(G_1) \leq \eta.$$

The term “measurable” is here used in the sense defined by Lebesgue, and the measure of a set G is denoted by $m(G)$.

To prove the theorem, let us assume that, if possible,

$$m(H_1) - m(G_1) = \eta + \alpha$$

where α is some positive number. Let the points of G_1 be enclosed in the interiors of non-overlapping intervals all contained in (c, d) , and forming a set H_2 , such that $m(H_2) < m(G_1) + \alpha$; this is possible, since G_1 is a measurable set. We have then $m(H_2) < m(H_1) - \eta$. Now consider the set of intervals $\bar{H}$ which consists, in (c, d) , of the set H_2 , and in the parts of (a, b) not interior to (c, d) of the same intervals that belong to the set H .

We have then

$$m(\bar{H}) = m(H) - m(H_1) + m(H_2) < m(H) - \eta < m(G).$$

It has thus been shewn that the set G can be enclosed in a set $\bar{H}$ of intervals, such that $m(\bar{H}) < m(G)$; but this is impossible, and hence no such positive number as α can exist. Therefore $m(H_1) - m(G_1) \leq \eta$.

2. Let $f(x)$ be a summable function, either limited or unlimited, defined for the interval $(-\pi, \pi)$. In case $f(x)$ be unlimited, it will be assumed to be such that the absolutely convergent integral $\int_{-\pi}^{\pi} f(x) dx$ exists in accordance with Lebesgue's definition. The Fourier's series corresponding to $f(x)$ then exists in the sense that the coefficients in the series all have a definite meaning.

Now let the definition of $f(x)$ be extended so as to apply to values of x lying outside the interval $(-\pi, \pi)$. This extension we make by taking $f(x)$ to be a periodic function of x , of period 2π , except that $f(\pi)$, $f(-\pi)$ may have different values, in which case it is immaterial what values we assign to $f(2\pi)$, $f(-2\pi)$, $f(3\pi)$, $f(-3\pi)$, ...

The following general theorem will now be established:—

The function $f(x)$ being of the character above described, each of the four integrals $\int_a^\beta f(x \pm 2z) \chi(z) \frac{\sin mz dz}{\cos}$, taken through any interval (a, β) such that $0 \leq a < \beta \leq \frac{1}{2}\pi$, converges to the limit zero, as the integer m is increased indefinitely, uniformly for all values of x contained in the interval $(-\pi, \pi)$; the function $\chi(z)$ being any function with limited total fluctuation (à variation bornée).

More generally, $\sin mz$ or $\cos mz$ may be replaced by $\phi(mz)$, where $\phi(z)$ is any limited summable function of which the integral, taken through any finite interval whatever, is less, in absolute magnitude, than some fixed finite number independent of the particular interval.

First, it will be assumed that $f(x)$ is a limited function. It is sufficient to consider the case of the integral $\int_a^\beta f(x+2z) \chi(z) \sin mz dz$; the proof in the case of the other three integrals being precisely similar. Also the substitution of $\phi(mz)$ for $\sin mz$ makes no essential difference in the proof.

Let U and L denote the upper and the lower limits of $f(x+2z)$, for all the values of x in the interval $(-\pi, \pi)$, and for all values of z in the interval (a, β) . Let the interval (L, U) be divided into p portions

$$(c_0, c_1), (c_1, c_2), \dots, (c_{q-1}, c_q), \dots, (c_{p-1}, c_p)$$

where $c_0 = L$, $c_p = U$, and $c_q - c_{q-1} < \epsilon$, for every value of q .

Let the function $f_1(x+2z)$ be defined as follows:—For those values of $x+2z$ for which $c_0 \leq f(x+2z) < c_1$, let $f_1(x+2z) = c_0$; for those values of $x+2z$ for which $c_1 \leq f(x+2z) < c_2$, let $f_1(x+2z) = c_1$; and

generally let $f_1(x+2z) = c_{q-1}$ for those values of $x+2z$ for which

$$c_{q-1} \leq f(x+2z) < c_q.$$

For any particular value of x , it may, for example, happen that there are no values of z such that $c_0 \leq f(x+2z) < c_1$: in that case there are no values of z , with the given value of x , for which $f_1(x+2z) = c_0$.

We have

$$\left| \int_a^\beta f(x+2z) \chi(z) \sin mz \, dz - \int_a^\beta f_1(x+2z) \chi(z) \sin mz \, dz \right| < \epsilon(\beta-a) \bar{\chi}$$

where $\bar{\chi}$ is the upper limit of $|\chi(z)|$ in the interval (α, β) ; and this holds for all values of x and of m .

The integral $\int_a^\beta f_1(x+2z) \chi(z) \sin mz \, dz$ is equal to

$$\sum_{q=0}^{q=p} c_q \int_{e_q} \chi(z) \sin mz \, dz$$

where e_q is that set of points z at which $c_q \leq f(x+2z) < c_{q+1}$; this set e_q depending upon the value of x .

In the interval $(-2\pi, 2\pi)$ of the variable x , let E_q be that set of points at each of which $c_q \leq f(x) < c_{q+1}$. Let the set E_q be enclosed in a finite, or enumerably infinite, set H_q of non-overlapping intervals such that $m(H_q) - m(E_q) = \eta$. For any fixed value of x , the set e_q consists of that part of E_q which lies in the interval $(x+2\alpha, x+2\beta)$ contained in $(-2\pi, 2\pi)$. In accordance with the theorem of § 1, the set of intervals F_q which consists of that part of H_q which lies in $(x+2\alpha, x+2\beta)$ is such that $m(F_q) - m(e_q) \leq \eta$. It is to be observed that the number η is independent of the value of x .

We have also

$$\left| \int_{e_q} \chi(z) \sin mz \, dz - \int_{F_q} \chi(z) \sin mz \, dz \right| < \bar{\chi} \eta.$$

Let now the intervals of H_q be arranged in diminishing order of length, and let them be denoted by $\gamma_1, \gamma_2, \gamma_3, \dots$. If $x+2\alpha$ or $x+2\beta$ be contained inside an interval of H_q , we divide that interval into two parts with different indices, each ending at $x+2\alpha$ or $x+2\beta$.

We may choose r so that $m(H_q) - (\gamma_1 + \gamma_2 + \dots + \gamma_r) < \eta$. Of the intervals $\gamma_1, \gamma_2, \gamma_3, \dots$, let those which fall in $(x+2\alpha, x+2\beta)$ be $\gamma_{s_1}, \gamma_{s_2}, \gamma_{s_3}, \dots$ where $s_1 < s_2 < s_3 \dots$; and let s_t be the greatest of these indices which does not exceed r . We have then $\gamma_{s_{t+1}} + \gamma_{s_{t+2}} + \dots < \eta$, and $m(F_q) - (\gamma_{s_1} + \gamma_{s_2} + \dots + \gamma_{s_t}) < \eta$; or, denoting by D_q the finite set of

intervals $\gamma_{s_1}, \gamma_{s_2}, \dots, \gamma_{s_t}$, we have $m(F_q) - m(D_q) < \eta$. The number of intervals in the set D_q cannot exceed the number r , which is independent of the value of x .

We now have

$$\left| \int_{F_q} \chi(z) \sin mz \, dz - \int_{D_q} \chi(z) \sin mz \, dz \right| < \eta \bar{\chi}.$$

The function $\chi(z)$ being a function with limited total fluctuation in the interval (α, β) , it may be expressed as the difference $\chi_1(z) - \chi_2(z)$ of two functions $\chi_1(z)$, $\chi_2(z)$, each of which is monotone throughout the interval (α, β) .

The integral $\int_{\mu}^{\lambda} \chi(z) \sin mz \, dz$, taken through an interval (μ, λ) , may by means of the second mean-value theorem be expressed as

$$\chi_1(\mu) \int_{\mu}^{\xi} \sin mz \, dz + \chi_1(\lambda) \int_{\xi}^{\lambda} \sin mz \, dz - \chi_2(\mu) \int_{\mu}^{\xi'} \sin mz \, dz - \chi_2(\lambda) \int_{\xi'}^{\lambda} \sin mz \, dz$$

where ξ , ξ' are two numbers in the interval (μ, λ) . We thus see that

$$\left| \int_{\mu}^{\lambda} \chi(z) \sin mz \, dz \right| < \frac{4}{m} (\bar{\chi}_1 + \bar{\chi}_2)$$

where $\bar{\chi}_1$, $\bar{\chi}_2$ are the upper limits of $\chi_1(z)$, $\chi_2(z)$ in (α, β) , the interval (μ, λ) being supposed to be contained in (α, β) .

We have now

$$\left| \int_{D_q} \chi(z) \sin mz \, dz \right| < \frac{4}{m} (\bar{\chi}_1 + \bar{\chi}_2) t < \frac{4}{m} (\bar{\chi}_1 + \bar{\chi}_2) r$$

where t is the number of intervals in D_q , and cannot exceed the number r .

Combining the inequalities which have been obtained, we find that

$$\left| \int_{\alpha}^{\beta} f(x+2z) \chi(z) \sin mz \, dz \right| < \epsilon(\beta-\alpha) \bar{\chi} + \left(2\bar{\chi}\eta + \frac{4r}{m} \bar{\chi}' \right) \sum_{q=0}^{q=p} c_q$$

where $\bar{\chi}' = \bar{\chi}_1 + \bar{\chi}_2$; and this holds for every value of x in $(-\pi, \pi)$. Let us now choose arbitrarily a positive number ξ ; we can then choose ϵ so so that $\epsilon(\beta-\alpha) \bar{\chi} < \frac{1}{3}\xi$. Having fixed ϵ accordingly, and consequently the numbers $c_0, c_1, \dots, c_q$, we next choose η so that $2\bar{\chi}\eta + \sum_{q=0}^{q=p} c_q < \frac{1}{3}\xi$; the number r is then fixed. We can now choose a value m_1 of m such that

$$\frac{4r}{m} \bar{\chi}' \sum_{q=0}^{q=p} c_q < \frac{1}{3}\xi,$$

for $m \geq m_1$.

It has now been shewn that, having given a positive number ξ , arbitrarily small, a number m_1 can be so determined that

$$\left| \int_a^\beta f(x+2z) \chi(z) \sin mz \, dz \right| < \xi,$$

for $m \geq m_1$, and for all values of x in the interval $(-\pi, \pi)$. It has therefore been shewn that, when $f(x)$ is a limited summable function,

$$\int_a^\beta f(x+2z) \chi(z) \sin mz \, dz$$

converges to the limit zero, as m is indefinitely increased, uniformly for all values of x in the interval $(-\pi, \pi)$, and consequently also for all values of x in any interval (a, b) contained in $(-\pi, \pi)$.

Next, let $f(x)$ be no longer limited, but still integrable in accordance with Lebesgue's definition.

If ξ be an arbitrarily small positive number, a positive number N can be so determined that

$$\int |f(x)| \, dx < \frac{1}{2} \xi / \bar{\chi},$$

the integral being taken over that set of points K_N in the interval $(-2\pi, 2\pi)$, for each of which $|f(x)| > N$. If k_N be that part of K_N which lies in the interval $(x+2a, x+2\beta)$, for any fixed value of x belonging to the interval $(-\pi, \pi)$, we have, *a fortiori*,

$$\int_{k_N} |f(x)| \, dx < \frac{1}{2} \xi / \bar{\chi}.$$

Let the function $f_2(x+2z)$ be defined by the specifications

$$f_2(x+2z) = f(x+2z), \quad \text{if } |f(x+2z)| \leq N,$$

$$\text{and} \quad f_2(x+2z) = 0, \quad \text{if } |f(x+2z)| > N.$$

Thus $f_2(x+2z)$ vanishes at all the points of the set k_N , and it is a limited summable function.

We have now

$$\begin{aligned} \int_a^\beta f(x+2z) \chi(z) \sin mz \, dz \\ = \int_a^\beta f(x+2z) \chi(z) \sin mz \, dz + \int_a^\beta f_2(x+2z) \chi(z) \sin mz \, dz. \end{aligned}$$

By the first part of the theorem, we see that a value m_1 of m can be determined so that

$$\left| \int_a^\beta f_2(x+2z) \chi(z) \sin mz \, dz \right| < \frac{1}{2} \xi,$$

for $m \geq m_1$, and for all values of x in $(-\pi, \pi)$.

$$\text{Also} \quad \left| \int_{k_N} f(x+2z) \chi(z) \sin mz \, dz \right| < \tfrac{1}{2}\xi;$$

hence we have shewn that

$$\left| \int_a^\beta f(x+2z) \chi(z) \sin mz \, dz \right| < \xi,$$

provided $m \geq m_1$, for all values of x in $(-\pi, \pi)$. The theorem has therefore been completely established.

3. Some particular cases of the general theorem established in § 2 will now be considered.

(1) If we consider only the single point $x = 0$, we see that, for $a = 0$, $\beta = \frac{1}{2}\pi$, and taking $\chi(z) = 1$, the integrals

$$\int_0^{\frac{1}{2}\pi} f(2z) \frac{\sin mz}{\cos mz} \, dz, \quad \int_0^{\frac{1}{2}\pi} f(-2z) \frac{\sin mz}{\cos mz} \, dz$$

both converge to the limit zero as m is indefinitely increased. Changing $f(2z)$ into $f(z)$, we see that, if $m = 2n$,

$$\int_{-\pi}^{\pi} f(z) \frac{\sin nz}{\cos nz} \, dz$$

converges to zero as n is indefinitely increased. The following theorem, which has been proved by Lebesgue, is therefore obtained:—

If $f(x)$ be any summable function, which, if unlimited, still possesses a Lebesgue integral, the coefficients in the corresponding Fourier's series converge to zero as the integer n is indefinitely increased.

(2) It is well known that the condition of convergence of a Fourier's series at a point x where $f(x)$ has an ordinary discontinuity, or is continuous, is that

$$\int_0^{\frac{1}{2}\pi} [f(x+2z) + f(x-2z) - f(x+0) - f(x-0)] \frac{\sin mz}{\sin z} \, dz$$

should converge to zero, as the odd integer m is indefinitely increased.

Let $\beta = \frac{1}{2}\pi$, and let $\chi(z) = \operatorname{cosec} z$, which is of limited fluctuation in the interval $(a, \frac{1}{2}\pi)$, if $0 < a$. We see then that, if (a, b) be an interval for x in which $f(x)$ is limited, and be also such that $f(a-0)$, $f(b+0)$ are finite, then

$$\int_a^{\frac{1}{2}\pi} [f(x+2z) + f(x-2z) - f(x+0) - f(x-0)] \frac{\sin mz}{\sin z} \, dz$$

converges uniformly to zero in the interval (a, b) of x .

For, by the theorem, the two integrals

$$\int_a^{\frac{1}{2}\pi} f(x+2z) \frac{\sin mz}{\sin z} dz, \quad \int_a^{\frac{1}{2}\pi} f(x-2z) \frac{\sin mz}{\sin z} dz$$

converge uniformly to zero in $(-\pi, \pi)$ and therefore in (a, b) ; also

$$|f(x+0) + f(x-0)|$$

is less for all values of x in (a, b) than some fixed finite number, and $\int_a^{\frac{1}{2}\pi} \frac{\sin mz}{\sin z} dz$ converges to the limit zero. We therefore see that, in order that a Fourier's series may converge uniformly in an interval (a, b) contained in $(-\pi, \pi)$, it is sufficient that the function be continuous in (a, b) , including the end-points, and that, for any arbitrarily small value of the positive number α , the integral

$$\int_0^\alpha \{f(x+2z) + f(x-2z) - 2f(x)\} \frac{\sin mz}{\sin z} dz$$

should converge to the limit zero, as m is indefinitely increased, uniformly for all values of x in (a, b) .

In this integral, the only functional values of $f(x)$ that are involved are those in the interval $(a-\alpha, b+\alpha)$, provided the values of x in the integrand are confined to those in the interval (a, b) . It thus appears that the question whether the Fourier's series converges uniformly in (a, b) depends only upon the nature of the function in the interval $(a-\alpha, b+\alpha)$ which contains (a, b) ; and the number α is arbitrarily small. We have therefore obtained the following theorem:—

If (a, b) be any interval contained in $(-\pi, \pi)$, such that $f(x)$ is continuous in (a, b) , including the end-points a and b , then the answer to the question whether the Fourier's series converges uniformly in (a, b) , or not, depends only upon the nature of the function $f(x)$ in an interval (a', b') which includes (a, b) in its interior and exceeds it in length by an arbitrarily small number. The function $f(x)$ may be of any character in the part of $(-\pi, \pi)$ outside (a', b') , subject only to its having a Lebesgue integral in $(-\pi, \pi)$.

This theorem contains, for the theory of uniform convergence, the parallel to the theorem relating to the convergence of the series at a single point, viz., that the convergence or non-convergence of the series at a single point depends only on the nature of the function in an arbitrarily small neighbourhood of that point; being independent of the nature of the function outside that neighbourhood, subject only to the existence

of the coefficients of the series. This latter theorem was proved by Riemann, for the case of functions satisfying his condition of integrability, and has been proved by Lebesgue for the case of all summable functions which are integrable in accordance with his definition of integrability. This theorem of Lebesgue is the particular case of the theorem here given, which arises when the interval (a, b) is reduced to a single point x .

(3) It is known that, if a function be of limited total fluctuation in $(-\pi, \pi)$, then the corresponding Fourier's series is uniformly convergent in any interval (a, b) which does not contain, either in its interior or at its ends, any point of discontinuity of the function. By applying the theorem obtained in (2), we now obtain the following theorem:—

The function $f(x)$ being summable, and, if unlimited, also integrable, in accordance with Lebesgue's definition; if (a', b') be any interval contained in $(-\pi, \pi)$, such that $f(x)$ is of limited total fluctuation in (a', b') , then the Fourier's series corresponding to $f(x)$ converges uniformly in any interval (a, b) , contained in (a', b') , and such that $f(x)$ is continuous in (a, b) , including the end-points.

(4) Let the function $\chi(z)$ be defined by

$$\chi(0) = 0 \quad \text{and} \quad \chi(z) = \frac{1}{z} - \frac{1}{\sin z},$$

for $z > 0$; also let $\alpha = 0$, $\beta = \mu < \frac{1}{2}\pi$; then we see that

$$\int_0^\mu [f(x+2z) + f(x-2z) - 2f(x)] \left(\frac{1}{z} - \frac{1}{\sin z} \right) \sin mz \, dz$$

converges uniformly to zero, as m is indefinitely increased, in any interval (a, b) in which $f(x)$ is limited. It thus appears that, if

$$\int_0^\mu [f(x+2z) + f(x-2z) - 2f(x)] \frac{\sin mz}{\sin z} \, dz$$

converges uniformly in (a, b) , so also does

$$\int_0^\mu [f(x+2z) + f(x-2z) - 2f(x)] \frac{\sin mz}{z} \, dz.$$

Therefore, the condition of uniform convergence of the series in an interval (a, b) , in which the function is continuous, including the end-points a and b , is that

$$\int_0^\mu [f(x+2z) + f(x-2z) - 2f(x)] \frac{\sin mz}{z} \, dz$$

should converge uniformly to zero in the interval.

The number μ may be taken to be arbitrarily small; in fact, in accordance with the theorem of § 2, if $0 < \mu_1 < \mu$, the integral

$$\int_{\mu_1}^{\mu} [f(x+2z) + f(x-2z) - 2f(x)] \frac{\sin mz}{z} dz$$

converges to zero, as m is indefinitely increased, uniformly in (a, b) .

4. We proceed to investigate sufficient conditions for the uniform convergence of the Fourier's series in an interval (a, b) in which the function $f(x)$ is continuous.

Writing $F(z) = f(x+2z) + f(x-2z) - 2f(x)$, we express the integral

$$\int_0^{\mu} \frac{F(z)}{z} \sin mz dz$$

in the form $\left[\int_0^{2\pi/m} + \int_{2\pi/m}^{3\pi/m} + \int_{3\pi/m}^{4\pi/m} + \dots + \int_{2r\pi/m}^{\mu} \right] \frac{F(z)}{z} \sin mz dz$

where r is an integer such that $0 \leq \mu - \frac{2r\pi}{m} < \frac{2\pi}{m}$.

We assume that (a, b) is contained in an interval (a', b') in which $f(x)$ is limited; if then we choose μ to be less than the smaller of the numbers $\frac{1}{2}(a-a')$, $\frac{1}{2}(b'-b)$, we see that $F(z)$ is limited for all values of z in $(0, \mu)$ and for all values of x in (a, b) . We have now

$$\left| \int_0^{2\pi/m} F(z) \frac{\sin mz}{z} dz \right| < m \int_0^{2\pi/m} |F(z)| dz \\ < 2\pi \times \text{upper limit of } |F(z)| \text{ in the interval } \left(0, \frac{2\pi}{m}\right).$$

Since a continuous function is uniformly continuous, the two functions $f(x+2z) - f(x)$, $f(x-2z) - f(x)$ converge to zero, as z converges to zero, uniformly for all values of x in (a, b) .

It follows that

$$\left| \int_0^{2\pi/m} F(z) \frac{\sin mz}{z} dz \right| < \eta_m$$

where η_m converges to zero, as m is indefinitely increased, and is independent of the value of x .

Next, we have

$$\left| \int_{2r\pi/m}^{\mu} F(z) \frac{\sin mz}{z} dz \right| < \frac{\frac{4}{m}}{\mu - \frac{2\pi}{m}} \times \text{upper limit of } |F(z)| \text{ in } \left(\frac{2r\pi}{m}, \mu\right) \\ < \frac{\frac{4}{m}}{\mu - \frac{2\pi}{m}} \times \text{upper limit of } |F(z)| \text{ in } (0, \mu) < \eta'_m$$

where η'_m converges to zero, as m is indefinitely increased, independently of the value of x .

The remaining part of the integral may be written in the form

$$\int_0^{\pi/m} \left\{ \frac{F\left(z + \frac{2\pi}{m}\right)}{z + \frac{2\pi}{m}} - \frac{F\left(z + \frac{3\pi}{m}\right)}{z + \frac{3\pi}{m}} + \frac{F\left(z + \frac{4\pi}{m}\right)}{z + \frac{4\pi}{m}} - \dots \right. \\ \left. + \frac{F\left(z + \frac{2r-2}{m}\pi\right)}{z + \frac{2r-2}{m}\pi} - \frac{F\left(z + \frac{2r-1}{m}\pi\right)}{z + \frac{2r-1}{m}\pi} \right\} \sin mz \, dz,$$

which is less, in absolute value, than

$$\int_0^{\pi} \left| \frac{F\left(\frac{z+2\pi}{m}\right)}{z+2\pi} - \frac{F\left(\frac{z+3\pi}{m}\right)}{z+3\pi} + \dots + \frac{F\left(\frac{z+2r-2}{m}\pi\right)}{z+2r-2\pi} - \frac{F\left(\frac{z+2r-1}{m}\pi\right)}{z+2r-1\pi} \right| dz,$$

which is not greater than

$$\sum_{s=1}^{s=r-1} \int_0^{\pi} \left| \frac{F\left(\frac{z+2s\pi}{m}\right)}{z+2s\pi} - \frac{F\left(\frac{z+2s+1}{m}\pi\right)}{z+2s+1\pi} \right| dz.$$

Now

$$\frac{F\left(\frac{z+2s\pi}{m}\right)}{z+2s\pi} - \frac{F\left(\frac{z+2s+1}{m}\pi\right)}{z+2s+1\pi} \\ = \frac{F\left(\frac{z+2s\pi}{m}\right) - F\left(\frac{z+2s+1}{m}\pi\right)}{z+2s+1\pi} + \pi \frac{F\left(\frac{z+2s\pi}{m}\right)}{(z+2s\pi)(z+2s+1\pi)},$$

hence

$$\left| \frac{F\left(\frac{z+2s\pi}{m}\right)}{z+2s\pi} - \frac{F\left(\frac{z+2s+1}{m}\pi\right)}{z+2s+1\pi} \right| \\ \leq \frac{\left| F\left(\frac{z+2s\pi}{m}\right) - F\left(\frac{z+2s+1}{m}\pi\right) \right|}{(2s+1)\pi} + \frac{1}{2s(2s+1)\pi} \left| F\left(\frac{z+2s\pi}{m}\right) \right|.$$

We now see that the part of the integral to be estimated is, in absolute value, less than

$$\left(\frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2r-1} \right) \Delta + \frac{1}{2} \times \text{upper limit of } |F(z)| \text{ in } (0, \mu)$$

where Δ is the greatest of the numbers $\left| F\left(\frac{z+2s\pi}{m}\right) - F\left(\frac{z+\overline{2s+1}\pi}{m}\right) \right|$ for $s = 1, 2, 3, \dots, r-1$.

The upper limit of $|F(z)|$ or $|f(x+2z)+f(x-2z)-2f(x)|$, for the interval $(0, \mu)$ of z , and for all values of x in (a, b) , depends upon μ , and is a number $2u(\mu)$, which may be made as small as we please by taking μ sufficiently small.

Also

$$\left(\frac{1}{3} + \frac{1}{5} + \dots + \frac{1}{2r-1}\right)\Delta < \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2r}\right)\Delta < (C_r + \log 2r)\Delta$$

where C_r is a number between 0 and 1 which converges to Mascheroni's constant. Let D_m denote the greatest value of the fluctuation of $f(x)$ in a sub-interval of length π/m contained in the interval $(a-\mu, b+\mu)$, for all possible positions of such sub-interval; then we have $\Delta \leq D_m$.

It has now been proved that, for $0 < \mu < \frac{1}{2}\pi$,

$$\left| \int_0^\mu [f(x+2z)+f(x-2z)-2f(x)] \frac{\sin mz}{z} dz \right| < \eta_m + \eta'_m + u(\mu) + D_m \left[C_r + \log \frac{m}{\pi} + \log \left(\mu - \frac{2\theta\pi}{m} \right) \right]$$

where θ is such that $0 < \theta < 1$, and such that $\mu - \frac{2r\pi}{m} = \theta \frac{2\pi}{m}$.

We have now to find a sufficient condition that it be possible, with μ fixed, to determine a value $\bar{m}$ of m , corresponding to an arbitrarily prescribed positive number ξ , such that

$$\left| \int_0^\mu F(z) \frac{\sin mz}{z} dz \right| < \xi,$$

for $m \geq \bar{m}$, and for all values of x in (a, b) .

If μ_1 be a number such that $0 < \mu_1 < \mu$, we can choose μ_1 so small that $u(\mu_1) < \frac{1}{6}\xi$. The number μ_1 having been so chosen, we can now choose a value m_1 of m , such that

$$\left| \int_{\mu_1}^\mu F(z) \frac{\sin mz}{z} dz \right| < \frac{1}{6}\xi,$$

for $m \geq m_1$, and for all values of x in (a, b) ; this follows from the uniform convergence of the integral to the limit zero.

We then have

$$\left| \int_0^\mu F(z) \frac{\sin mz}{z} dz \right| < \frac{2}{3}\xi + \eta_m + \eta'_m + D_m \left\{ C_r + \log \frac{m}{\pi} + \log \left(\mu_1 - \frac{2\theta_1\pi}{m} \right) \right\},$$

the numbers η'_m , D_m , θ_1 now having the values which correspond to μ_1 instead of μ . We can now choose m_2 so that $\eta_m < \frac{1}{6}\xi$, for $m \geq m_2$; also, we can choose m_3 so that $\eta'_m < \frac{1}{6}\xi$, for $m \geq m_3$. Again, since D_m converges to zero, as m is indefinitely increased, we can so determine m that $D_m \left\{ C_r + \log \left(\mu_1 - \frac{\theta_1 \pi}{m} \right) \right\} < \frac{1}{6}\xi$, for $m \geq m_4$. Let us now assume that it is possible to so choose m_5 that $D_m \log m/\pi < \frac{1}{6}\xi$, for $m \geq m_5$. Taking $\bar{m}$ to be the greatest of the numbers m_1, m_2, m_3, m_4, m_5 , we now have

$$\left| \int_0^\mu F(z) \frac{\sin mz}{z} dz \right| < \xi,$$

for $m \geq \bar{m}$. Therefore, with the assumption made, that $D_m \log m/\pi$ converges to zero, as m is indefinitely increased, it has been shewn that the convergence of the Fourier's series in (a, b) is uniform.

The following theorem has accordingly been established:—

It is a sufficient condition for the uniform convergence of the Fourier's series in an interval (a, b) , that an interval (a', b') can be found which encloses (a, b) in its interior, and is such that, if F_β denote the fluctuation of $f(x)$ in any sub-interval of length β contained in (a', b') , $F_\beta \log \beta$ converges to zero, uniformly for all such sub-intervals, as β is indefinitely diminished.

The condition may also be stated in the form, that *it is sufficient that $\{f(x+\beta) - f(x)\} \log \beta$ converge uniformly to zero, as β is indefinitely diminished, for all values of x , such that x and $x+\beta$ are in the interval (a', b') which encloses (a, b) .*

This is seen by referring back to the foregoing proof, where the difference of functional values at the ends of an interval β was replaced by the fluctuation in that interval.

The condition is satisfied if $|f(x \pm \beta) - f(x)| \leq C\beta^k$, where C and k are positive numbers, for all values of β less than some fixed value β_1 , for every value of x such that $x, x \pm \beta$ are in (a', b') , the numbers C and k being independent of x .

The particular cases of these theorems which arise when (a, b) is reduced to a single point are known sufficient conditions of convergence of the series at a single point, due to Lipschitz.

The function $f(x)$ has throughout been assumed to be restricted in the whole interval $(-\pi, \pi)$, only so far that it is either a limited summable function, or an unlimited summable function which possesses a Lebesgue integral in $(-\pi, \pi)$.

5. Sufficient conditions for the uniform convergence of the series in an interval (a, b) may also be obtained in the following simple manner:—

If $0 < \mu_1 < \mu$, we have

$$\left| \int_0^\mu \frac{F(z)}{z} \sin mz \, dz \right| \leq \left| \int_0^{\mu_1} \frac{F(z)}{z} \sin mz \, dz \right| + \left| \int_{\mu_1}^\mu \frac{F(z)}{z} \sin mz \, dz \right|.$$

Let it now be assumed that, for every value of x in (a, b) , $\int_0^{\mu_1} \frac{F(z)}{z} \, dz$ exists, and that $\int_0^{\mu_1} \left| \frac{F(z)}{z} \right| dz$ converges to the limit zero, as μ_1 is indefinitely diminished, uniformly for all values of x in (a, b) .

We have then

$$\left| \int_0^{\mu_1} \frac{F(z)}{z} \sin mz \, dz \right| < \int_0^{\mu_1} \left| \frac{F(z)}{z} \right| dz.$$

The number μ_1 can now be chosen so small that, if ξ be an arbitrarily fixed positive number, the inequality

$$\int_0^{\mu_1} \left| \frac{F(z)}{z} \right| dz < \frac{1}{2}\xi$$

is satisfied for this value of μ_1 , and for every value of x in (a, b) . The number μ_1 having been so fixed, we can fix a value m_1 of m such that

$$\left| \int_{\mu_1}^\mu \frac{F(z)}{z} \sin mz \, dz \right| < \frac{1}{2}\xi,$$

for $m \geq m_1$, and for every value of x in (a, b) . We have then

$$\left| \int_0^\mu \frac{F(z)}{z} \sin mz \, dz \right| < \xi,$$

for $m \geq m_1$, and for every value of x in (a, b) .

The following theorem has therefore been established:—

It is a sufficient condition for the uniform convergence of the Fourier's series in an interval (a, b) contained in an interval (a', b') in which $f(x)$ is limited that

$$\int_0^{\mu_1} \left| \frac{f(x+2z) + f(x-2z) - 2f(x)}{z} \right| dz$$

should exist for all values of x in (a, b) , and converge to zero as μ_1 is indefinitely diminished, uniformly for all values of x in (a, b) . The condition will be satisfied if the two integrals

$$\int_0^{\mu_1} \left| \frac{f(x+2z) - f(x)}{z} \right| dz, \quad \int_0^{\mu_1} \left| \frac{f(x-2z) - f(x)}{z} \right| dz$$

both exist and are uniformly convergent.

In particular, the series is uniformly convergent in any interval (a, b) in which one of the four derivatives, and therefore each of the other three derivatives, is limited, the end-points a and b being included. A special case is that in which $f(x)$ has a limited differential coefficient throughout (a, b) , including a and b .

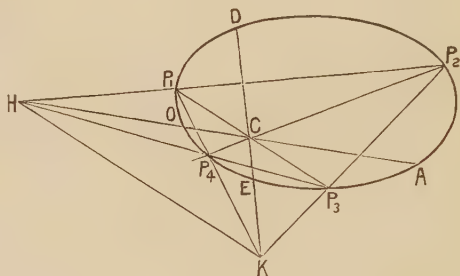
The condition is satisfied if $|f(x \pm \beta) - f(x)| < C\beta^k$, for all values of x in (a, b) , and for all values of β less than some fixed number; where C and k are fixed positive numbers independent of x . This condition is, at least in form, slightly simpler than the similar condition given in § 4. If (a, b) be reduced to a single point, it at once becomes the known condition of convergence at x , that $|f(x \pm \beta) - f(x)| < C\beta^k$.

ON THE PROJECTIVE GEOMETRY OF A BINARY QUARTIC AND ITS HESSIAN

By E. B. ELLIOTT.

[Received and Read February 14th, 1907.]

1. A conic drawn through the origin of coordinates x, y is chosen; and a quartic in x, y is represented by the quadrangle $P_1P_2P_3P_4$ of points in which the conic is met again by the four lines of which $u = 0$, the result of equating the quartic to zero, is the equation. Linear transformations of the quartic are obtained by taking any axes through any point of the conic, retaining the same quadrangle, and also by any projection of the figure. Real changes of origin and axes, and real projections, afford schemes of linear transformation with real coefficients.



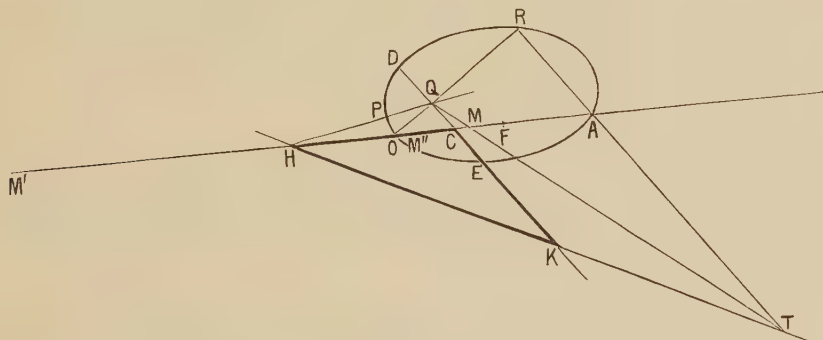
Let CHK be the harmonic triangle of the quadrangle. The sides of this are real if P_1, P_2, P_3, P_4 are all real or all imaginary in conjugate pairs, *i.e.*, if the quartic $u = 0$ with real coefficients has all its roots real or all imaginary. The sides of CHK cut the conic in three pairs of points, O and A , D and E , and a pair which in the case of a real CHK are imaginary, given by the equation $G = 0$, where G is the sextic covariant of u .

The quadrangle $u = 0$ is one of an infinity of quadrangles on the conic with the same harmonic triangle CHK . The vertices of any one are the points representative of some quartic of the pencil $\kappa u + \lambda H = 0$, where H is the Hessian of u . Given the triangle, one vertex P of any quadrangle of the pencil determines the other three as the second intersections of PC, PH, PK with the drawn conic.

To specify the particular quadrangle which gives the particular quartic of the pencil $H = 0$ itself is not so easy as the algebraic simplicity of

H in respect of u would lead us to expect, and I am not aware that any writer has taken the trouble to overcome the small difficulties of the specification of the Hessian by linear construction, though full attention has been paid to the expression of its properties in the language of polar pairs and triads. One difficulty arises from the fact that, if the roots of $u = 0$ are all real, those of $H = 0$ are all imaginary—it is easy to see that a binary quantic of any order with only real and unequal roots has a Hessian, a sum of squares, with no real roots—so that, starting from a quadrangle with real vertices as $u = 0$, we look for one with imaginary vertices as $H = 0$, a completed real exhibition of the H going with the u being thus an impossibility. There is also the complication that between quartics u and Hessians H there is not a one to one correspondence: a particular u has, of course, one Hessian H , but of u 's which possess a particular H there are two. These are apolar with one another.

2. Associate with each quadrangle, *i.e.*, each quartic of the pencil, a point M on CH as follows. Take P one vertex of the quadrangle, the other three being where PC, PH, PK cut the conic again. Let PH meet



KC in Q . Join OQ , and also join R , the point where OQ cuts the conic again, to A by a line meeting HK in T . QT cuts CH in the point M .

Now, take F the harmonic conjugate of H with regard to C and A . F has no reference to any particular quadrangle of the pencil.

Let M, M', M'' be the points of CH which are associated as above with the u -quadrangle, the u' -quadrangle of the pencil, taking u' apolar with u , and the H -quadrangle of u , respectively. The facts, to be established presently by aid of a canonical projection, are as follows.

- (i.) OA, FH, MM' are pairs of an involution.
- (ii.) CH, MM', OM'' are pairs of another involution.

In these there is complete symmetry as between M and M' ; the H -quadrangle is that of u' as well as that of u .

3. This association by means of harmonic properties and involutions of three quartics of a pencil $\kappa u + \lambda H$, with the same G , of which two are apolar and the third their common Hessian, is general. The following statement of realities of geometrical construction applies only when the harmonic triangle CHK is real, and so has no reference to quartics u with two roots real and two imaginary.

The above construction of a real M (or M' or M'') from a real P (or P' or P'') is linear. To construct P (or P' or P'') from a given real M (or M' or M'') we may join points of HK to M and to A , letting the connectors with M meet CH in points q , and the connectors with A meet the conic again in points whose connectors with O meet CH in points q' , and find, linearly by aid of the drawn conic, either double point Q of the involution of pairs qq' , finally joining QH to meet the conic in P . According to the position of M on CH , Q may be between D and E , or real and outside the segment DE , or imaginary. Only in one of the two former cases are the points P real; in the other they are imaginary on real lines through H ; in the third case they are imaginary on imaginary lines through H , but, as we shall see, on real lines through K .

Given either M or M' , we are told by (i.) that the other can be linearly constructed, and then by (ii.) that M'' can be.

Conversely, given M'' , M and M' constitute the common pair of two involutions, each given by two pairs, and can be linearly constructed by aid of the drawn conic. We shall see that a real M'' thus determines a real M and M' .

4. To justify the above statements, which are strictly projective in form, we have to prove them for a figure canonically simplified by any convenient projection. In the case, on which we are fixing attention, of a real CHK let us project the drawn conic into a circle in such a way that HK , the side of CHK which does not meet the conic in real points, goes to infinity. This is a real projection. We also choose for origin—which may be anywhere on the circle—an end O of the diameter OCA which is the projection of the side CH , and for axes OA and the tangent at O . The linear transformation which effects all this is a real one.

We are, in fact, thus led to the usual canonical form of a quartic. Slightly departing from the usual notation, we write for the canonical form of $u = 0$

$$4x^2y^2 = \mu (x^2 + y^2)^2. \quad (1)$$

The sextic covariant $G = 0$ is

$$xy(x^4 - y^4) = 0. \quad (2)$$

Every quartic of the pencil $\kappa u + \lambda H = 0$ has an equation of the form of (1). Take, in particular,

$$4x^2y^2 = \mu'(x^2 + y^2)^2 \quad (3)$$

for that quartic $u' = 0$ of the pencil which is apolar with $u = 0$. The condition to be satisfied, found by equating to zero the lineo-linear invariant of (1) and (3), is

$$(2\mu - 1)(2\mu' - 1) = -3. \quad (4)$$

Using this relation, we at once find that the Hessian $H = 0$ of u may be written

$$\begin{aligned} 4x^2y^2 &= -\mu\mu'(x^2 + y^2)^2 \\ &= \frac{1}{2}(2 - \mu - \mu')(x^2 + y^2)^2, \end{aligned} \quad (5)$$

the symmetry of which tells us that, as stated earlier, the two apolar quartics have the same Hessian.

It is interesting to notice incidentally that another statement of these facts is that the quartic equations in $\tan \theta$,

$$\cos 4\theta = \cos 4\alpha, \quad \cos 4\theta = \cos 4\beta, \quad \cos 4\theta = \cos 4\gamma,$$

where $\cos 4\alpha \cos 4\beta = -3$ and $\cos 4\alpha + \cos 4\beta + 2\cos 4\gamma = 0$,

are canonical forms of $u = 0$, $u' = 0$, $H = 0$ respectively, arrived at by real transformation, when $u = 0$ has only real or only imaginary roots. It is here clear that α and β cannot be both real, and readily shown that if one of the two, say α , is real, γ cannot be, except in the extreme cases of $\cos 4\alpha = \cos 4\gamma = \pm 1$, $\cos 4\beta = \mp 3$, in which cases u and H are identical, but for a constant factor, and perfect squares.

Returning to the notation of (1) to (5), the conic by which we cut the pencil of lines is the circle

$$x^2 + y^2 = 2x. \quad (6)$$

The vertices of the u -, the u' -, and the H -quadrangles are then by (1), (3), and (5) respectively on the line pairs

$$y^2 = \mu, \quad y^2 = \mu', \quad y^2 = -\mu\mu', \quad \text{subject to (4)}. \quad (7)$$

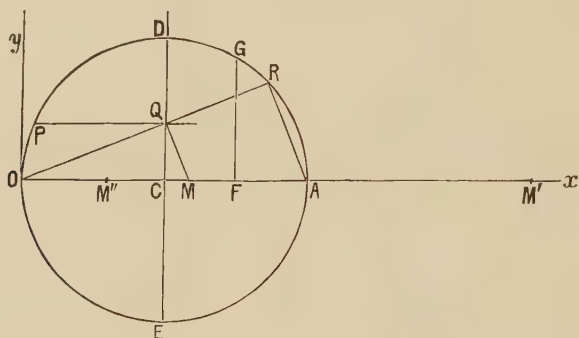
The quadrangles are now rectangles symmetrical about the lines $y = 0$ and $x = 1$. One vertex or side of a rectangle determines it completely.

The six points $G = 0$ are the ends of the rectangular diameters OCA , DCE and the circular points at infinity.

If we start from a real u -rectangle, all we can hope for in the way of real construction is, in virtue of a remark made above, to obtain real sides of the associated u' - and H -rectangles, whose vertices are imaginary except in the extreme cases of $\mu = 0, 1$.

Take P a vertex of the u -rectangle, and let a side $y = \sqrt{\mu}$ through it meet DE in Q . Join OQ , and let QM at right angles to it meet OA in M ; then

$$CM = CQ^2/OC = CQ^2 = \mu,$$



so that, if F is the middle point of CA , $FM = \mu - \frac{1}{2}$, in sign and magnitude. If, then, M' corresponds to vertices P' of the u' -rectangle, just as M does to P , we have, by (4), that

$$FM \cdot FM' = -\frac{3}{4} = FO \cdot FA,$$

i.e.,

$$MF \cdot FM' = FG^2,$$

so that MGM' is a right angle.

The constructions, or real steps towards construction, of the other three from a given one of P, P', M, M' are most naturally performed by drawing circles. But the effectiveness of methods which yield on projection the general linear ones expressed in §§ 2, 3 is clear. Given P , and so Q , a parallel through Q to RA cuts OA in M ; and M' is the conjugate of M in the involution of which F is centre and OA a pair [cf. § 2, (i.)]. M' stands to Q' , where DE (produced) meets a $y = \sqrt{\mu'}$ through a u' -point P' , just as M does to Q , i.e., OM' subtends a right angle at Q' . Given M (or M'), Q (or Q') is either double point of the involution on DE of which lines through M (or M') and the perpendiculars to them through O , i.e., the lines from O to the points where parallels to them through A cut the circle, determine pairs.

Again, take M'' on OA (produced if necessary) such that

$$CM'' = -\mu\mu',$$

i.e., that $CO \cdot CM'' = CM \cdot CM'$,

so that, in language suitable for projection, OM'' , MM' are pairs of an involution with C for centre [cf. § 2, (ii.)]. M'' goes with a Q'' (imaginary in the figure) on DE , and vertices P'' of the H -rectangle, just as M goes with Q and P .

5. The statements of §§ 2, 3 have now been justified, except that there remain for consideration questions of reality and imaginariness.

We are supposed to have started from a quartic $u=0$ with real coefficients, and have seen that when it has only real or only imaginary roots a real linear transformation gives it the form $4x^2y^2=\mu(x^2+y^2)^2$. In these cases then μ is real, and consequently by (4) so are μ' and $-\mu\mu'$. The points M , M' , M'' are real points on OA . Moreover, any real M (or M') on OA , infinitely produced, determines a real M' (or M) and a real M'' . Conversely, any real M'' determines a real M and M' : for, having $-\mu\mu'$ real, $=m$ say, (4) tells us that $\mu+\mu'=2(1-m)$, so that $(\mu-\mu')^2=4(1-m)^2+4m=(1-2m)^2+3$, and consequently $\mu-\mu'$ as well as $\mu+\mu'$ is real.

As M covers the whole line OA produced, proceeding from left to right and starting from $-\infty$, M' also covers it all from left to right, starting from F , while M'' covers it twice from right to left. We have, in fact, by (4), the following correspondence of parts of the ranges of μ , μ' , $-\mu\mu'$.

μ or $\mu' \dots$	$-\infty$ to -1	-1 to 0	0 to $\frac{1}{2}$	$\frac{1}{2}$ to 1	1 to 2	2 to ∞
μ' or $\mu \dots$	$\frac{1}{2}$ to 1	1 to 2	2 to ∞	$-\infty$ to -1	-1 to 0	0 to $\frac{1}{2}$
$-\mu\mu' \dots$	∞ to 1	1 to 0	0 to $-\infty$	∞ to 1	1 to 0	0 to $-\infty$

Half the table gives the whole, as it is symmetrical in its reference to μ , μ' . If B is the reflexion of C in A , we may also express it: according as

M or M' is to the left of O , between O and C , or between C and F ,

M' or M is between F and A , between A and B , or to the right of B ,

and M'' is to the right of A , between A and C , or to the left of C .

The speciality of the separating cases $-\mu\mu'=0, 1, \infty$ will be noticed.

Now, if by μ we mean either of μ , μ' , $-\mu\mu'$, the sides $y^2=\mu$ of the associated rectangle are real only if μ is not negative, i.e., if the associated M is not to the left of C , and the vertices of the rectangle are real only if

further μ does not exceed 1, *i.e.*, if M is not outside the segment CA . If, then, the u - (or the u' -) rectangle is one of distinct real vertices, the u' - (or the u -) rectangle has imaginary vertices, which lie on real or on imaginary parallels to the diameter OA according as M (or M') is between C and F or between F and A , and the H -rectangle has also imaginary vertices, on imaginary or real parallels to the diameter according to the same circumstances; and, if the H -rectangle is one of distinct real vertices, both the u - and the u' -rectangles have distinct imaginary vertices, those of one being on real and those of the other on imaginary parallels to the diameter. In the special separating cases of $-\mu\mu' = 0, 1$, *i.e.*, of M'' and either M or M' at C or A , the H -rectangle is identical with one of the u - and u' -rectangles, having real vertices united in pairs. In the third separating case of $-\mu\mu'$ infinite, the H -rectangle is also identical with one of the u - and u' -rectangles, having imaginary vertices united in pairs.

The summary of results as to reality is then as follows:—

- (i.) If the roots of u (or of u') are all real, the roots of u' (or of u) are all imaginary; and so are those of H , except that when u (or u') is the square of a quadratic with real roots H is the same square but for a constant factor.
- (ii.) If the roots of u (or of u') are all imaginary, either the roots of the other are all real and those of H all imaginary, or *vice versa*, except that in two separating cases the other and H are identical and squares of quadratics with real roots.
- (iii.) If the roots of H are all real and different, those of both u and u' are imaginary; and, if those of H are real and equal in pairs, those of one of u, u' are the same equal pairs and those of the other are imaginary.
- (iv.) If the roots of H are all imaginary, those of one of u, u' are real and those of the other imaginary; and in particular, if those of H are imaginary and equal in pairs, those of the one of u, u' which are imaginary are the same equal pairs.

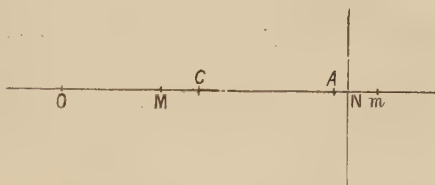
6. It has been stated (in § 3) that in a case when the vertices of one of the triangles lie on imaginary parallels to OA , *i.e.*, when the construction of Q (or Q' or Q'') from M (or M' or M'') is devoid of reality, they lie on real perpendiculars to OA . This is clear; for the points of the circle which lie on $y^2 = \mu$ also lie on $(x-1)^2 = 1-\mu$. The intersections N with OA of the required perpendiculars to OA are then given by

$$CN^2 = CA \cdot MA = CA \cdot Cm,$$

where

$$Am = MC.$$

The construction of the real sides of the quadrangle in the general figure of § 2 may then be stated. Find, linearly, m , the harmonic conjugate of M with regard to FH , and then find, linearly by aid of the drawn conic, the double points of the involution of which Am , CH are pairs: the connectors of these with K are the real sides required.



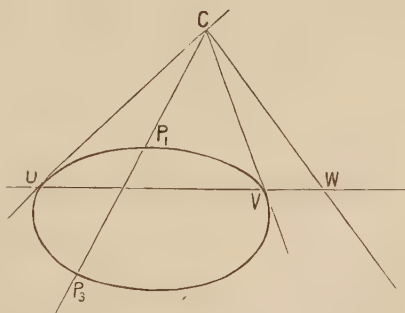
7. The canonical projection and reference of the above do not apply when u is a fourth power, or when it is the product of a square and a non-square. These cases are so simple and well known that they will not be dwelt upon.

Excluding these singular cases, it has been observed in § 3 that the association in projective terms of three allied quartics u , u' , H of a pencil $\kappa u + \lambda H$ with a given G is general. But the realities of construction detailed above have no applicability in cases when a chosen one of u , u' , H with real coefficients has two real and two imaginary roots. In such cases the projection adopted above is imaginary, or, in other words, the linear transformation by which it is effected involves imaginary coefficients.

We have already the ground for asserting that, if one of u , u' , H has two real and two imaginary roots, so also have the other two. For, if either had not, (i.) to (iv.) above would tell us that u had not.

For guidance to realities of construction in cases of two real and two imaginary roots a different canonical projection and reference are desirable, and to these we now proceed.

8. In a case when $u = 0$ has two real and two (conjugate) imaginary roots, two vertices P_1 , P_3 of its representative quadrangle on the chosen conic are real, and the other two are imaginary on a real line CW . One vertex C , and one side, the polar UVW of C , of the harmonic triangle are real, and the other two are the imaginary points on UV where it is met by the common pair of harmonic conjugates of CU , CV and CP_1 , CW .



The best canonical projection now is the real one by which UVW is sent to infinity, and the angles UCV , P_1CW are made right angles. The conic becomes a rectangular hyperbola with C for centre. Take the transverse axis, and O one of its ends, for axis of x and origin. The equation of the hyperbola, say

$$(x-1)^2 - y^2 = 1, \quad (8)$$

and that of P_1CP_3 and CW , say

$$\mu[(x-1)^2 - y^2] = 2(x-1)y, \quad (9)$$

give the vertices of the quadrangle. The lines to them from O are at once seen to be

$$4xy(x^2 + y^2) = \mu(x^2 - y^2)^2; \quad (10)$$

and this is accordingly the canonical form of $u = 0$ to which we are led by a linear transformation with real coefficients.

Every quartic of the pencil $\kappa u + \lambda H = 0$ derived from this is of the same form, with some μ or other. In particular the apolar $u' = 0$ of the pencil is

$$4xy(x^2 + y^2) = \mu'(x^2 - y^2)^2 \quad (11)$$

where

$$\mu\mu' = 3, \quad (12)$$

and the Hessian $H = 0$ is

$$4xy(x^2 + y^2) = -\frac{1}{2}(\mu + \mu')(x^2 - y^2)^2. \quad (13)$$

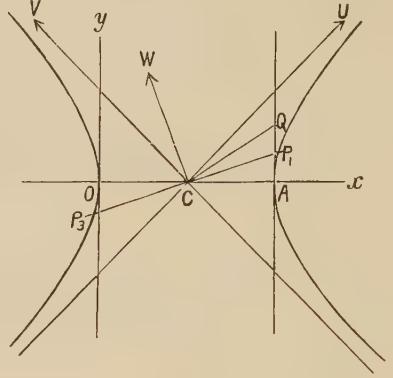
The sides of the common harmonic triangle of the quadrangles representative of the pencil are the line at infinity and the lines from C to the circular points, *i.e.*, the lines $(x-1)^2 + y^2 = 0$; and the sextic covariant $G = 0$, the equation of the lines from O to the intersections of these sides with the hyperbola, is

$$(x^2 - y^2)(x^4 + 6x^2y^2 + y^4) = 0. \quad (14)$$

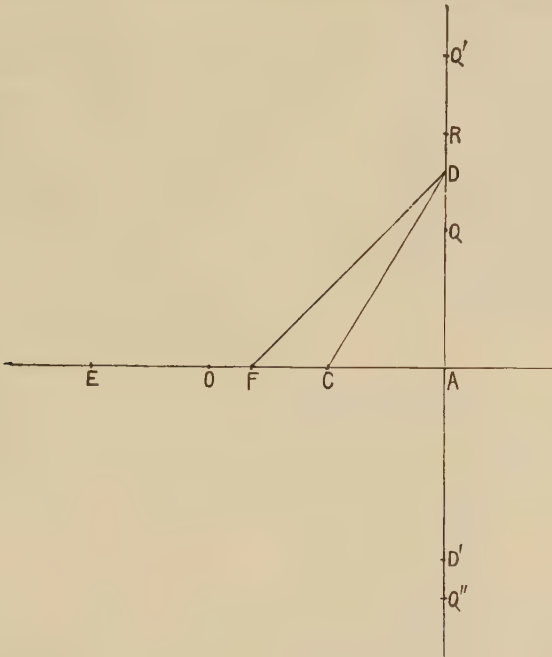
9. Now (9) gives that, if P_1 is a u -point on the hyperbola,

$$\tan 2ACP_1 = \mu. \quad (15)$$

Hence, if on the tangent at A we take an ordinate $AQ = \mu$, the two sides P_1CP_3 , CW through C of the u -quadrangle are the bisectors of the angle ACQ .



We are thus led to exhibit the relative positions of the real vertices P_1 and P_3 , P'_1 and P'_3 , P''_1 and P''_3 of the u -, the u' , and the H -quadrangles

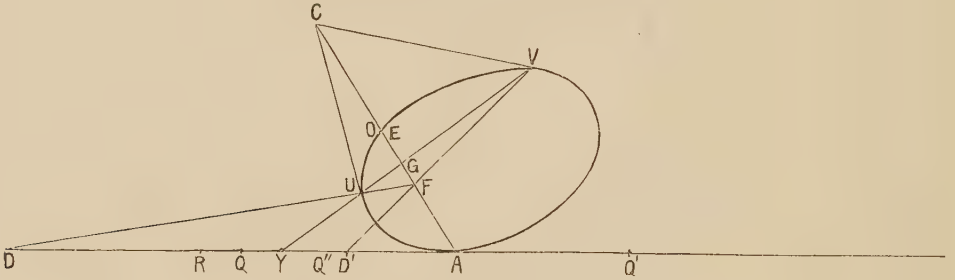


respectively, by means of corresponding points Q, Q', Q'' on the tangent at A . Taking $ACD = 60^\circ$, Q and Q' are such that $AQ \cdot AQ' = AD^2$; and Q'' , on the opposite side of A from Q and Q' , is such that $Q''A = AR$, where R is the middle point of QQ' . The lines $P_1CP_3, P'_1CP'_3, P''_1CP''_3$ are respectively the internal bisectors of the angles ACQ, ACQ', ACQ'' .

It remains to express in terms suitable for projection ways of finding the two others of Q, Q', Q'' when one of the three is given. First we must specify D in projective language. In imaginary terms we may say that CD forms with the perpendicular through C to CA and the lines from C to the circular points an equi-anharmonic pencil. But a real construction for D of the kind we require is obtained by taking $OE = CO$ on CO produced, finding F such that $AF^2 = AC \cdot AE$, i.e., finding a double point of the involution of which A is centre and C, E a pair, and drawing FD (or FD') parallel to an asymptote. The determination of Q' , and then of Q'' , when Q is given, is now immediate, the constructions being in projective language that of fourth harmonic points. When it is Q'' that we have given, its reflection R is found as a harmonic conjugate, and then Q and Q' as harmonically conjugate both with regard to D, D' and with regard to R and the point at infinity on AR .

Projecting the figure back into the general one in which CP_1P_3 and C_1W are two lines of which only the first meets the conic of reference in real points, we may now state as follows.

CP_1P_3 and CW , $CP_1'P_3'$ and CW' , $CP_1''P_3''$ and CW'' are pairs of an overlapping involution to which every quartic of the pencil $\kappa u + \lambda H$ contributes a pair, and of which CU and CV are one pair. This pair and any other determine it. Take COA the one which cuts the conic of the common pair of this involution and the, also overlapping, involution of pairs of conjugate lines with regard to the conic through C . Let Q on the tangent at A be where that tangent is met by the harmonic conjugate of CA with regard to CP_1 , CW . Points Q' , Q'' on the tangent are similarly associated with the CP_1' , CW' and CP_1'' , CW'' of u' and H .



Construct two points D , D' on the tangent at A , having no connection with the particular CP_1 or CP_1' or CP_1'' as follows. On COA , which meets UV in G , take E the harmonic conjugate of C with regard to O , G , and F either double point of the involution of which C , E and A , G are pairs. D and D' are where FU and FV meet the tangent.

Now, having Q on ADD' , take Q' the harmonic conjugate of Q with regard to D and D' . This is the point associated with the $CP_1'P_3'$ of the apolar u' just as Q is with the CP_1P_3 of u .

Again, take R the harmonic conjugate of Y , where UV cuts ADD' , with regard to Q and Q' , and then take Q'' the harmonic conjugate of R with regard to A and Y . Q'' has the same association with the $CP_1''P_3''$ of the Hessian of u , and of u' .

All the above constructions can be performed linearly by aid of the drawn conic. So can that of Q , Q' from a given Q'' , by finding R , the harmonic conjugate of Q'' with regard to A , Y , and then constructing the common pair of harmonic conjugates of D , D' and R , Y . So, finally, can that of CP_1P_3 and CW (or $CP_1'P_3'$ and CW' or $CP_1''P_3''$ and CW'') from CQ (or CQ' or CQ''), by finding the common pair of the involution of which CA , CQ are the double lines, and that of which CU , CV and CA , CY are pairs.

INVARIANTS OF THE GENERAL QUADRATIC FORM MODULO 2

By LEONARD EUGENE DICKSON.

[Received February 21st, 1907.—Read March 14th, 1907.]

1. The theory of invariants of homogeneous forms under linear transformation with coefficients in a finite field of order p^n offers decided contrasts* to the usual algebraic theory.

In the present paper I consider the invariants of the m -ary quadratic form† Q_m in the field of order 2, the most important case for the applications. For each value of m less than 6, I obtain a complete set (m in number) of independent invariants, as well as a complete set of linearly independent invariants. A number of the invariants for $m = 6$ are given in §§ 18, 19.

The minimum number of variables upon which a form Q_m can be expressed equals the *rank* r of the discriminantal determinant, r being defined as in the algebraic theory, except that for minors of odd order the factor 2 must first be deleted. When r is even there are two distinct canonical forms. It is shown in the present paper that the complete classification of quadratic forms can be accomplished by means of *invariant functions*.

2. The notation‡ employed for the m -ary quadratic form is

$$(1) \quad Q_m \equiv \sum c_{ij} X_i X_j + \sum b_i X_i^2 \quad (i, j = 1, \dots, m; i < j),$$

the coefficients being integers reduced modulo 2. Any m -ary linear homogeneous transformation with coefficients modulo 2 can be derived from the generators $(X_i X_j)$ and

$$(2) \quad X_1 = X_1' + X_2', \quad X_2 = X_2', \quad \dots, \quad X_m = X_m'.$$

* For binary forms these contrasts are emphasized in my paper in the *Transactions of the American Mathematical Society*, April, 1907.

† For binary quadratic forms in the field of order 2^n , see the paper just cited; for ternary forms, *American Journal of Mathematics*, January, 1908. In Vol. xxi. (1899), pp. 193–256, of the same journal I first gave the canonical forms of m -ary quadratic forms in a field of order p^n .

‡ Multinomial coefficients cannot be inserted, as in the algebraic theory.

Under the latter transformation, Q becomes Q' , in which

$$(3) \quad b'_2 = b_2 + b_1 + c_{12}, \quad c'_{2i} = c_{2i} + c_{1i} \quad (i = 3, \dots, m),$$

the remaining coefficients being unaltered. The effect of the substitution $(X_i X_j)$ is the same as that of the permutation $[ij]$ on the subscripts of the coefficients.

If we treat the form (1) algebraically, its discriminant is

$$(4) \quad \begin{vmatrix} 2b_1 & c_{12} & c_{13} & \dots & c_{1m} \\ c_{12} & 2b_2 & c_{23} & \dots & c_{2m} \\ \dots & \dots & \dots & \dots & \dots \\ c_{1m} & c_{2m} & c_{3m} & \dots & 2b_m \end{vmatrix}.$$

As shown in the second paper cited above, all terms in the expansion of (4) have even coefficients when m is odd; so that one-half of the expansion is a function with integral coefficients, called the *semi-discriminant* S_m of Q_m modulo 2. For example,*

$$(5) \quad S_3 = c_{12}c_{13}c_{23} + b_1c_{23}^2 + b_2c_{13}^2 + b_3c_{12}^2,$$

$$(6) \quad S_5 = \sum_{(12)} c_{12}c_{13}c_{24}c_{35}c_{45} + \sum_{(19)} c_{12}^2c_{34}c_{35}c_{45} + \sum_{(5)} b_1[2345]^2,$$

where the first sum extends over the twelve products in which each subscript occurs exactly twice; while, in the notation for Pfaffians,

$$[ijk\ell] = c_{ij}c_{kl} + c_{ik}c_{lj} + c_{il}c_{jk}.$$

For m even, it is evident that, when multiples of 2 are omitted, (4) equals the square of the Pfaffian $[12 \dots m]$. According as m is even or odd, the discriminant or the semi-discriminant is an invariant of Q_m under linear transformation modulo 2.

3. We may write down *a priori* two new invariants of Q_m :

$$(7) \quad A_m = \prod_{i < j}^{i, j=1, \dots, m} (c_{ij} + 1), \quad I_m = A_m \prod_{i=1}^m (b_i + 1).$$

That these functions are unaltered by the substitution (3) follows by inspection, since $c(c+1) \equiv 0 \pmod{2}$. The interpretation of these invariants is obvious. The form Q_m vanishes identically if, and only if, $I_m \equiv 1$; it is reducible to a unary form if, and only if, $I_m \equiv 0$, $A_m \equiv 1$.

* These results are valid in the $GF[2^n]$. For the present case $n = 1$, we may omit the exponents 2, since $c^2 \equiv c \pmod{2}$.

4. For the sake of comparison, certain results will be stated, in the present notation, which were proved in the earlier papers. As a complete set of linearly independent invariants of Q_m , for $m = 2$, we may take A_2 , I_2 , and

$$(8) \quad J_2 = (b_1 + c_{12} + 1)(b_2 + c_{12} + 1);$$

while, for $m = 3$, we may take A_3 , I_3 , S_3 , and

$$(9) \quad J_3 = \{b_1 + (c_{12} + 1)(c_{13} + 1)\} \{b_2 + (c_{12} + 1)(c_{23} + 1)\} \{b_3 + (c_{13} + 1)(c_{23} + 1)\}.$$

If we omit $I_m = A_m J_m$, we obtain independent invariants.

Invariants of the Quaternary Quadratic Form Q_4 , §§ 5-11.

5. We proceed to determine a set of linearly independent invariants of Q_4 . Let ϕ be a general polynomial in the ten coefficients c_{ij} , b_i . Since $a^2 \equiv a \pmod{2}$, we may assume that each exponent is 0 or 1. Under the substitution (3), ϕ becomes ϕ' , such that

$$\begin{aligned} \phi' - \phi = & \beta[b_2] + c_{13}[c_{23}] + c_{14}[c_{24}] + \beta c_{13}[b_2 c_{23}] + \beta c_{14}[b_2 c_{24}] + c_{13} c_{14}[c_{23} c_{24}] \\ & + \beta c_{13} c_{14}[b_2 c_{23} c_{24}], \end{aligned}$$

where $\beta = b_1 + c_{12}$, $[bc] = \partial^2 \phi / \partial b \partial c$, ... We may set*

$$(10) \quad \phi = B + C b_2 + D c_{23} + E c_{24} + G b_2 c_{23} + H b_2 c_{24} + J c_{23} c_{24} + K b_2 c_{23} c_{24},$$

where B , ..., K are independent of b_2 , c_{23} , c_{24} . Then

$$\begin{aligned} \phi' - \phi = & (K c_{14}) b_2 c_{23} + (K c_{13}) b_2 c_{24} + (K \beta) c_{23} c_{24} + (\beta G + J c_{14} + \beta K c_{14}) c_{23} \\ & + (\beta H + J c_{13} + \beta K c_{13}) c_{24} + (G c_{13} + H c_{14} + K c_{13} c_{14}) b_2 \\ & + \beta C + D c_{13} + E c_{14} + \beta G c_{13} + \beta H c_{14} + J c_{13} c_{14} + \beta K c_{13} c_{14}. \end{aligned}$$

The conditions that $\phi' - \phi \equiv 0$, identically in b_2 , c_{23} , c_{24} , reduce to

$$(11) \quad K c_{14} = 0, \quad K c_{13} = 0, \quad K \beta = 0, \quad \beta G = J c_{14}, \quad \beta H = J c_{13},$$

$$(12) \quad G c_{13} = H c_{14}, \quad \beta C + D c_{13} + E c_{14} + \beta H c_{14} = 0.$$

In view of the first three conditions (11),

$$K = (c_{13} + 1)(c_{14} + 1)[l(1 + b_1 + c_{12}) + m b_1 c_{12}],$$

where l and m are functions of c_{34} , b_3 , b_4 only. Let m' denote the coefficient of $b_3 b_4 c_{34}$ in m . Upon replacing ϕ by $\phi - m' I_4$, we may set $m' = 0$. Hence no term of the resulting ϕ has the factor

$$b_2 c_{23} c_{24} \cdot b_1 c_{12} \cdot b_3 b_4 c_{34}.$$

* It would be impracticable to deal with the complete expression of 2^{10} terms of ϕ .

Apply the permutation [13] of the subscripts. Hence no term has the factor $c_{12}c_{24}c_{23}c_{14}\Pi b_i$. Hence no term of m has the factor b_3b_4 ; so that no term of ϕ has the factor $c_{12}c_{23}c_{24}\Pi b_i$. Apply [24] to the latter. Hence no term has the factor $c_{14}c_{34}c_{24}\Pi b_i$; so that there is no term $b_3b_4c_{34}$ in l . Hence no term of ϕ has the factor $b_2c_{23}c_{24}.b_3b_4c_{34}$. Apply [13]. Thus no term of ϕ has the factor $b_2c_{12}c_{24}b_1b_4c_{14}$; so that b_4 does not occur in m . Hence no term has the factor $b_2c_{23}c_{24}b_3b_1c_{12}$. Apply [14]. Thus no term contains $b_2c_{23}c_{12}b_3b_4c_{24}$. Hence b_3b_4 does not occur in l ; so that no term of ϕ contains $b_2c_{23}c_{24}b_3b_4$. In view of the form of the invariants given below, we have now carried the above reduction to its limit:

(13) *no term of ϕ has a factor $b_ib_jb_kc_{ij}c_{ik}$.*

Since m does not contain b_3 or b_4 , while b_3b_4 does not occur in l , and, since the invariance of ϕ requires that K be unaltered by [34], we may set

$$m = m_1 + m_2c_{34}, \quad l = l_1 + l_2c_{34} + l_3(b_3 + b_4) + l_4(b_3 + b_4)c_{34}.$$

Apply the permutation [14] to ϕ ; the terms multiplying $b_2c_{23}c_{12}$ are

$$(c_{34} + 1)(c_{14} + 1)\{l'(1 + b_4 + c_{24}) + (m_1 + m_2c_{13})b_4c_{24}\},$$

where $l' = l_1 + l_2c_{13} + l_3(b_3 + b_1) + l_4(b_3 + b_1)c_{13}$.

The coefficient of $b_2c_{23}c_{24}c_{12}$ is

$$(c_{34} + 1)(c_{14} + 1)\{l' + (m_1 + m_2c_{13})b_4\}.$$

The corresponding coefficient in the initial form of ϕ is

$$(c_{13} + 1)(c_{14} + 1)\{l + (m_1 + m_2c_{34})b_1\}.$$

Identifying the two, we get $l_1 = l_2$, $l_3 = l_4 = m_1 = m_2$. Hence

$$K = (c_{13} + 1)(c_{14} + 1)(c_{34} + 1)[\{l_1 + m_1(b_3 + b_4)\}\{1 + b_1 + c_{12}\} + m_1b_1c_{12}].$$

If to K we apply the permutation [13], we obtain the terms of ϕ which multiply $b_2c_{12}c_{24}$. We drop those with the factor c_{23} . Hence

$$H = c_{12}(c_{13} + 1)(c_{34} + 1)(c_{14} + 1)\{l_1 + m_1(b_1 + b_4)\}\{1 + b_3\} + H_1,$$

where H_1 is a function of c_{13} , c_{14} , c_{34} , b_1 , b_3 , b_4 , only. In order to apply the last condition (11), we set

$$H_1 = \alpha + \gamma b_1 + \delta c_{13} + \epsilon b_1c_{13}, \quad \alpha, \gamma, \delta, \epsilon \text{ functions of } c_{14}, c_{34}, b_3, b_4.$$

Then $(b_1 + c_{12})H$ may be given the form

$$\begin{aligned} (\alpha + \gamma)b_1 + (\delta + \epsilon)b_1c_{13} + (\alpha + \pi)c_{12} + (\gamma + \pi)b_1c_{12} + (\delta + \pi)c_{12}c_{13} \\ + (\epsilon + \pi)b_1c_{12}c_{13}, \end{aligned}$$

where

$$\pi = (c_{34} + 1)(c_{14} + 1)(b_3 + 1)(l_1 + m_1b_4).$$

Since $(b_1 + c_{12})H$ must equal Jc_{13} , we have $\alpha = \gamma = \pi$, and

$$(14) \quad Jc_{13} = (\delta + \epsilon) b_1 c_{13} + (\delta + \pi) c_{12} c_{13} + (\epsilon + \pi) b_1 c_{12} c_{13},$$

the latter being noted for future use. In H_1 occurs $\gamma b_1 \equiv \pi b_1$, which contains the term $m_1 c_{34} c_{14} b_3 b_4 b_1$. Hence $m_1 = 0$ by (13). Thus

$$(15) \quad K = l_1 (c_{13} + 1)(c_{14} + 1)(c_{34} + 1)(1 + b_1 + c_{12}),$$

$$(16) \quad H = l_1 (c_{14} + 1)(c_{34} + 1)(b_3 + 1)(1 + b_1 + c_{12} + c_{12} c_{13}) + \delta c_{13} + \epsilon b_1 c_{13},$$

$$(17) \quad \pi = l_1 (c_{14} + 1)(c_{34} + 1)(b_3 + 1),$$

where l_1 is independent of the coefficients. Since $m = 0$,

$$(18) \quad \text{no term of } \phi \text{ has the factor } b_1 b_2 c_{12} c_{23} c_{24}.$$

6. We next determine the form of δ and ϵ . The terms of ϕ which multiply $b_2 c_{24}$ are $H + Kc_{23}$; they must be unaltered by the permutation [13]. Since this is already true* of the terms involving l_1 , it follows that $\delta + \epsilon b_1$ must be unaltered by [13]. We may set

$$\delta = \delta_1 + \delta_2 b_3, \quad \epsilon = \epsilon_1 + \epsilon_2 b_3, \quad \delta_1, \dots, \epsilon_2 \text{ functions of } c_{14}, c_{34}, b_4.$$

Then δ_1 and ϵ_2 are unaltered by [13], while δ_2 and ϵ_1 are interchanged.

For the moment, denote by e the factor of c_{14} in ϵ . Then the coefficient of $b_1 b_2 c_{14} c_{24}$ in ϕ , viz., that of $b_1 c_{14}$ in $H + Kc_{23}$, is

$$l_1 (c_{34} + 1)(b_3 + 1) + c_{13} e + l_1 c_{23} (c_{13} + 1)(c_{34} + 1).$$

This must be unaltered by the permutation [12], so that $e = l_1 (c_{34} + 1)$,

$$\epsilon_1 = d_4 + d_5 c_{34} + l_1 c_{14} (c_{34} + 1), \quad \epsilon_2 \text{ and } d\text{'s functions of } b_4 \text{ only,}$$

$$\delta_2 = d_4 + d_5 c_{14} + l_1 c_{34} (c_{14} + 1), \quad \delta_1 = d + d_1 (c_{14} + c_{34}) + d_2 c_{14} c_{34}.$$

In view of (10), G and H are interchanged by the permutation [34]. Hence the first condition (12) states merely that Hc_{14} shall be unaltered by [34], and hence that $\delta c_{13} c_{14}$ and $\epsilon c_{13} c_{14}$ shall be unaltered by [34]. Now

$$\delta c_{13} c_{14} = (d + d_1) c_{13} c_{14} + (d_1 + d_2) c_{13} c_{14} c_{34} + (d_4 + d_5) c_{13} c_{14} b_3,$$

$$\epsilon c_{13} c_{14} = (d_4 + l_1) c_{13} c_{14} + (d_5 + l_1) c_{13} c_{14} c_{34} + \epsilon_2 c_{13} c_{14} b_3.$$

If we set

$$d = f + g b_4, \quad \epsilon_2 = f_3 + g_3 b_4, \quad d_i = f_i + g_i b_4 \quad (i = 1, 2, 4, 5),$$

* Due to the above derivation of the terms multiplying c_{12} in H from K .

the f 's and g 's being constants, the conditions are

$$g_2 = g_1, \quad g_5 = 0, \quad g_4 = f_3, \quad f_5 + f_4 = g + g_1.$$

Hence the explicit expressions for δ and ϵ are

$$(19) \quad \delta = f + gb_4 + (f_1 + g_1 b_4)(c_{14} + c_{34}) + (f_2 + g_1 b_4) c_{14} c_{34} \\ + f_4 b_3 + f_3 b_3 b_4 + f_5 c_{14} b_3 + l_1 (c_{14} + 1) c_{34} b_3,$$

$$(20) \quad \epsilon = f_4 + f_3 b_4 + f_5 c_{34} + l_1 c_{14} (c_{34} + 1) + (f_3 + g_3 b_4) b_3,$$

where

$$(21) \quad f_5 = f_4 + g + g_1.$$

7. We next determine the form of C . Set $C = c + c'c_{12}$. Now the terms $C + Hc_{24}$ of ϕ , which have the factor b_2 but not c_{23} , must be unaltered by the permutation [14]. Hence c' is derived from the terms free of c_{12} in H by applying the permutation [14], while c is unaltered by [14]. Hence

$$(22) \quad c' = l_1 (c_{14} + 1)(c_{13} + 1)(b_3 + 1)(b_4 + 1) + \delta' c_{34} + \epsilon' b_4 c_{34},$$

where δ' , ϵ' are derived from δ , ϵ by the permutation [14]. Let, for the moment, $c_1 + c_2 b_1$ denote the terms of c free of c_{13} and c_{14} . Then, in the product $(b_1 + c_{12})C$, the terms involving neither c_{13} nor c_{14} are

$$(c_1 + c_2) b_1 + c_1 c_{12} + c_2 b_1 c_{12} + c_{12} (b_1 + 1)P,$$

where P denotes the second factor in (23). In view of the second condition (12), this sum must vanish identically in b_1 , c_{12} . Hence $c_1 = c_2 = P$; so that the terms of c free of c_{13} and c_{14} are

$$(23) \quad (b_1 + 1) \{ l_1 (b_3 + 1)(b_4 + 1) + c_{34} (f + f_4 b_3) + b_4 c_{34} (f_4 + f_3 b_3) \}.$$

Now c must be unaltered by the permutations [14] and [34]. Applying [14] to the function (23), we obtain as the terms of c with the factor c_{13} , but not c_{14} or c_{34} ,

$$(24) \quad c_{13} (b_4 + 1) (f + f_4 b_3 + f_4 b_1 + f_3 b_1 b_3).$$

Applying [34] to the latter, we obtain

$$(25) \quad c_{14} (b_3 + 1) (f + f_4 b_4 + f_4 b_1 + f_3 b_1 b_4).$$

The terms of c with the factor $c_{13}c_{14}$, but not c_{34} , must be unaltered by [34]; those with the factor $c_{13}c_{14}c_{34}$ must be unaltered by both [34] and [14]. In writing down the possible terms with new undetermined co-

efficients, we may omit $c_{13}c_{14}b_1b_3b_4$ in view of (13). There result

$$(26) \quad c_{13}c_{14} \{q_1 + q_2b_1 + q_3(b_3 + b_4) + q_4b_1(b_3 + b_4) + q_5b_3b_4\},$$

$$(27) \quad c_{13}c_{14}c_{24} \{q_6 + q_7(b_1 + b_3 + b_4) + q_8(b_1b_3 + b_1b_4 + b_3b_4)\}.$$

Applying [13] and [14] in turn to (26), we get

$$(28) \quad c_{13}c_{34} \{q_1 + q_2b_3 + q_3(b_1 + b_4) + q_4b_3(b_1 + b_4) + q_5b_1b_4\},$$

$$(29) \quad c_{14}c_{34} \{q_1 + q_2b_4 + q_3(b_3 + b_1) + q_4b_4(b_3 + b_1) + q_5b_1b_3\}.$$

Hence c is the sum of the terms (23)–(29).

8. We have now exhibited the terms of ϕ which multiply b_2 , and have imposed the conditions that they shall be unaltered by every permutation of the subscripts 1, 3, 4. It remains to require invariance under a permutation altering 2, say [12]. The terms multiplying b_1b_2 are obtained by inspection from the preceding formulæ. The factor of b_1b_2 is unaltered by [12], if, and only if,

$$(30) \quad g_1 = 0, \quad f_3 = 0, \quad f_4 = q_2 = q_3 = q_5 = q_7 = l_1, \quad q_4 = q_8 = 0.$$

We shall assume that formulæ (19), (20), (22)–(29) have been simplified by the insertion of the values of the parameters given by (30), (21).

By merely permuting the subscripts, we can deduce all the remaining terms of ϕ involving one or more b 's. We may proceed systematically as follows. To the above terms which multiply b_2 , but not b_1 , we apply [12] and obtain the new terms* multiplying b_1 . To those of the latter lacking b_3 , we apply [13] and obtain the terms which multiply b_3 , but neither b_2 nor b_1 . Of the resulting terms, we select those lacking b_4 and apply [34], obtaining the terms multiplying b_4 , but not b_1 , b_2 , or b_3 . We may now tabulate the terms of B , D , E , J of (10) which involve the b 's. Denote by B_1 , D_1 , E_1 , J_1 the terms, as yet unknown, of B , D , E , J , respectively, which are independent of the b 's. Then condition (14) is found to be equivalent to the following:—

$$(31) \quad f_1 = g_3 = q_6 = l_1, \quad q_1 = f, \quad f_2 = g + l_1;$$

$$(32) \quad J_1c_{13} = (f + l_1)c_{12}c_{13} + gc_{14}c_{34}c_{12}c_{13}.$$

Finally, there remains the second condition (12), which was partially

* We do not write out here the long expressions obtained so simply. If the reader will content himself with the mere verification that the resulting function (34) is invariant under (3), he may pass at once to § 9, where the method is similar, but the formulæ are simpler.

examined in § 7. In view of the values (21), (30), (31) of the parameters, this condition is found to reduce to

$$(33) \quad D_1 c_{13} + E_1 c_{14} = (f + l_1) c_{12} (c_{13} + c_{14}) (c_{34} + 1),$$

since the terms involving the b 's vanish identically.

We are consequently led to an invariant, all of whose terms involve one or more b 's, by setting

$$B_1 = D_1 = E_1 = J_1 = 0, \quad l_1 = f = 1, \quad g = 0,$$

the remaining parameters being then given by (21), (30), (31). As an illustration of the resulting simplifications, note that (19) and (20) become

$$\delta = (c_{14} + 1)(c_{34} + 1)(b_3 + 1), \quad \epsilon = (c_{14} + 1)(c_{34} + 1) + b_3 b_4.$$

Without giving further details, I will state that the sum of the resulting invariant and A_4 , given by (7), may be written in the elegant compact form

$$(34) \quad J_4 = \beta_1 \beta_2 \beta_3 \beta_4 + b_1 b_2 b_3 b_4 [1234],$$

the last factor denoting a Pfaffian, while

$$(35) \quad \beta_1 = b_1 + (c_{12} + 1)(c_{13} + 1)(c_{14} + 1), \quad \beta_2 = b_2 + (c_{12} + 1)(c_{23} + 1)(c_{24} + 1), \dots$$

Except for the final terms of (34), the invariant is analogous to the invariants (8) and (9) for binary and ternary forms.

9. To obtain the invariants linearly independent of I_4 and J_4 , we set $l_1 = 0$. Decided simplifications result. Thus, by (15)–(31),

$$(36) \quad K = 0, \quad H = f c_{13} + g(c_{14} c_{34} + b_4 + c_{14} b_3 + c_{34} b_1) c_{13},$$

$$(37) \quad G = f c_{14} + g(c_{13} c_{34} + b_3 + c_{13} b_4 + c_{34} b_1) c_{14},$$

$$(38) \quad C = f[c_{34}(b_1 + 1) + c_{13}(b_4 + 1) + c_{14}(b_3 + 1) + c_{12} c_{34} + c_{13} c_{14} + c_{13} c_{34} + c_{14} c_{34}] \\ + g c_{12} c_{34} (b_1 + b_3 c_{14} + b_4 c_{13} + c_{13} c_{14}).$$

By the method* explained at the middle of § 8, we deduce

$$(39) \quad J = b_1 \{f + g(c_{14} c_{34} + c_{14} b_3 + c_{13} c_{34} + c_{13} b_4 + c_{12} c_{34})\} \\ + g b_3 c_{12} c_{14} + g b_4 c_{12} c_{13} + J_1,$$

$$(40) \quad B = b_1 (f c_{12} c_{34} + f c_{34}) + b_3 \{f(c_{12} c_{34} + c_{12} c_{14} + c_{14} + c_{12} + b_4 c_{12}) + g b_4 c_{12} c_{34}\} \\ + b_4 f(c_{12} c_{34} + c_{12} c_{13} + c_{12} + c_{13}) + B_1,$$

* The arrangement of the terms agrees with the successive steps.

$$(41) \quad D = b_1 \{f(b_4+1+c_{34}+c_{14})+g(c_{14}+c_{12}c_{34})b_4\} + b_3 \{fc_{14}+gc_{12}c_{14}b_4\} \\ + b_4 \{f(1+c_{12}+c_{13}+c_{14})+g(c_{12}c_{13}c_{14}+c_{12}c_{13}c_{34})\} + D_1,$$

$$(42) \quad E = b_1 \{f(b_3+1+c_{34}+c_{13})+g(c_{13}+c_{12}c_{34})b_3\} \\ + b_3 \{f(1+c_{12}+c_{14}+c_{13})+gc_{12}(c_{14}c_{34}+c_{13}c_{14}+c_{13}b_4)\} + fb_4c_{13} + E_1.$$

It remains to determine the functions J_1 , B_1 , D_1 , E_1 of the c 's, so that (32) and (33) are satisfied and so that the terms

$$(43) \quad \phi_1 = B_1 + D_1c_{23} + E_1c_{24} + J_1c_{23}c_{24},$$

independent of the b 's, are unaltered by any permutation of the subscripts.

Now J_1 is a function of c_{12} , c_{13} , c_{14} , c_{34} , unaltered by [34] and satisfying condition (32) with $l_1 = 0$. Hence

$$(44) \quad J_1 = (c_{13}+1)(c_{14}+1)(j+j_1c_{34}+j_2c_{12}+j_3c_{12}c_{34})+fc_{12}+gc_{12}c_{13}c_{14}c_{34}.$$

When the permutation [14] is applied to (43), the terms of $J_1c_{23}c_{24}$ which have the factor c_{12} must remain unaltered, whence $j_3 = j_2$; while the terms independent of c_{12} must give rise to the terms of D_1c_{23} with the factor c_{12} . Similarly, under the permutation [23], the terms of $J_1c_{23}c_{24}$ which have the factor c_{34} must remain unaltered, whence $j_3 = j_1$; while the terms independent of c_{34} must give rise to the terms of D_1c_{23} with the factor c_{34} . Hence

$$(45) \quad D_1 = c_{12}(c_{34}+1)(c_{14}+1)(j+j_1c_{13})+c_{34}(c_{14}+1)(j+j_1c_{13})+fc_{34}c_{13}+D',$$

where D' is a function of c_{13} and c_{14} only. At the same time, the terms of D_1 independent of c_{12} must be unaltered by [14], those independent of c_{34} by [23]; by either condition, D' is of the form $j_3c_{13}(c_{14}+1)+\lambda+\mu c_{14}$.

We may now derive E_1 from D_1 by applying the permutation [34]:

$$(46) \quad E_1 = c_{12}(c_{34}+1)(c_{13}+1)(j+j_1c_{14})+c_{34}(c_{13}+1)(j+j_1c_{14}) \\ +fc_{14}c_{34}+j(c_{13}c_{14}+c_{14}+1)+\mu c_{13}.$$

Then condition (33) is satisfied if, and only if, $\lambda = j$, $j_1 + j = f$. Thus

$$(47) \quad D' = j(c_{13}c_{14}+c_{13}+1)+\mu c_{14}, \quad j_1 = j_2 = j_3 = j+f.$$

Now, by (43), $B_1 + E_1c_{24}$ must be unaltered by [14] and [23]. Proceeding as in the determination of D_1 , and choosing the constant term of ϕ to be j , we get

$$(48) \quad B_1 = c_{34} \{c_{13}(c_{12}+1)(j+j_1c_{14})+jc_{14}(c_{12}+1)+j+\mu c_{12}\} \\ + c_{12} \{c_{13}(j+j_1c_{14})+fc_{13}c_{14}+j(c_{14}+1)\} + jc_{13}(c_{14}+1) + kc_{14} + j.$$

To insure that (43) shall be unaltered by every permutation of the subscripts, it now suffices to require that B_1 be unaltered by [34], the condition for which is $k = j$.

In the final expression for the invariant (10) with the amplifications (36)–(48), the independent parameters are f, g, j, μ .

The coefficient of μ is [1234]. For $j = \mu = 1, f = g = 0$, the invariant is A_4 , given by (7).

For $f = 1, g = j = \mu = 0$, the invariant is the sum F_4 of the terms (49), (50):

$$(49) \quad \sum_{(6)} b_1 b_2 c_{34} + [1234] \sum_{(4)} b_1 + \sum_{(4)} b_1 (c_{23} c_{24} + c_{23} c_{34} + c_{24} c_{34} + c_{23} + c_{24} + c_{34}),$$

$$(50) \quad c_{23} c_{24} (c_{13} + 1)(c_{14} + 1)(c_{12} + c_{34} + c_{12} c_{34}) + c_{23} c_{24} c_{12} + c_{23} c_{12} c_{13} (c_{34} + 1)(c_{14} + 1) \\ + c_{24} c_{12} c_{14} (c_{34} + 1)(c_{13} + 1) + c_{13} c_{14} c_{34} (1 + c_{12} + c_{23} + c_{24}).$$

The expansion of (50) is symmetrical in the subscripts:

$$(50') \quad \sum_{(4)} c_{12} c_{13} c_{23} + \sum_{(12)} c_{12} c_{13} c_{14} c_{23} + \sum_{(6)} c_{12} c_{13} c_{14} c_{23} c_{24} + c_{12} c_{13} c_{14} c_{23} c_{24} c_{34}.$$

For $g = 1, f = j = \mu = 0$, the invariant is

$$(51) \quad [1234] \left\{ \sum_{(6)} b_1 b_2 c_{34} + \sum_{(4)} b_1 c_{23} c_{24} c_{34} \right\} + c_{12} c_{13} c_{14} c_{23} c_{24} c_{34}.$$

Using (50) in preference to (50'), we find that (51) equals [1234] F_4 .

THEOREM.—As a complete set of linearly independent invariants of the quaternary quadratic form Q_4 modulo 2, we may take

$$(52) \quad [1234], A_4, I_4, J_4, F_4, [1234] F_4.$$

10. On account of its low degree, we note the invariant

$$K_4 \equiv F_4 + A_4 + [1234] + 1,$$

whose terms are given by (49), and

$$(53) \quad \sum_{(6)} c_{12} + \sum_{(12)} c_{12} c_{13} + \sum_{(4)} c_{12} c_{13} c_{14} + \sum_{(12)} c_{12} c_{13} c_{24} + \sum_{(3)} c_{12} c_{13} c_{24} c_{34}.$$

11. THEOREM.—Every invariant of Q_4 is a rational integral function of the independent invariants [1234], A_4 , J_4 , K_4 .

Since every $A_4 c_{ij}$ vanishes, we have $A_4 J_4 = I_4$. In view of §§ 9, 10, it remains only to establish the independence of the four invariants. This is shown by the following table, since any chosen invariant takes two

distinct values, and each of the remaining three invariants the same value, for certain two canonical forms.

Canonical Form.*	[1234]	A ₄	J ₄	K ₄
$X_1 X_2 + X_3 X_4$	1	0	0	0
$X_1 X_2 + X_3 X_4 + X_1^2 + X_2^2$	1	0	0	1
$X_1 X_2 + X_3^2$	0	0	0	0
$X_1 X_2$	0	0	0	1
$X_1 X_2 + X_1^2 + X_2^2$	0	0	1	1
X_1^2	0	1	0	0
vanishing	0	1	1	0

The table also shows that no two of the canonical forms are equivalent.

Invariants of the Quinary Quadratic Form Q_5 , §§ 12-17.

12. We shall first determine the invariant J_5 corresponding to J_2 , J_3 , J_4 given by (8), (9), (34) respectively :

$$(54) \quad J_5 = (b_1 + p_1)(b_2 + p_2) \dots (b_5 + p_5) + b_1 b_2 b_3 b_4 b_5 \psi + \sum_{(5)} b_1 b_2 b_3 b_4 [1234] p_5,$$

where ψ is an unknown function of the c_{ij} , while

$$(55) \quad p_i = \Pi(c_{ij} + 1) \quad (j = 1, \dots, 5; j \neq i).$$

The first two parts of (54) are obviously called for in making the generalization. In including the final part, we have been guided somewhat by the principle that, when all the coefficients with a subscript 5 are suppressed, J_5 must reduce to J_4 , so that there must occur terms which reduce to $b_1 b_2 b_3 b_4 [1234]$.

In accord with the cases $m < 5$ being generalized, we shall assume that ψ itself is an invariant, and shall determine it so that the function (54) is invariant. Since the latter is unaltered by any permutation of the subscripts, it remains only to examine its invariance under the substitution (3).

Considering in turn the parts of the final sum in (54) in the order

* *American Journal of Mathematics*, Vol. xxi., 1899, p. 222.

p_5, p_4, p_3, p_2, p_1 , we see by inspection that their increments under (3) are

$$(56) \quad \begin{cases} b_1 b_3 b_4 [1234] (c_{12} + 1) p_5, \\ b_1 b_3 b_5 [1235] (c_{12} + 1) p_4, \\ b_1 b_4 b_5 [1245] (c_{12} + 1) p_3, \end{cases}$$

$$(57) \quad b_1 b_3 b_4 b_5 [1345] (c_{12} + 1) \delta, \quad b_1 b_3 b_4 b_5 [2345] p_1,$$

where δ denotes the increment to the final term of (58) :

$$(58) \quad \delta = (c_{23} + c_{13} + 1)(c_{24} + c_{14} + 1)(c_{25} + c_{15} + 1) + (c_{23} + 1)(c_{24} + 1)(c_{25} + 1).$$

The increment to the first product in (54) is found to be the sum of the three expressions (56) together with

$$(59) \quad b_1 b_3 b_4 b_5 (c_{12} + 1) \{ \delta + 1 + (c_{13} + 1)(c_{14} + 1)(c_{15} + 1) \}.$$

Since ψ is to be an invariant, the increment to the second part of (54) is

$$(60) \quad b_1 b_3 b_4 b_5 (c_{12} + 1) \lambda,$$

where we have set

$$(61) \quad \psi = \lambda + \lambda' c_{12}.$$

Hence J_5 is an invariant if, and only if, the terms (57), (59), and (60) cancel. Removing the factor $(c_{12} + 1)b_1 b_3 b_4 b_5$, we may write the condition in the form

$$(62) \quad \lambda = 1 + (c_{13} + 1)(c_{14} + 1)(c_{15} + 1) \{ 1 + [2345] \} + \delta \{ 1 + [1345] \}.$$

The remaining terms of ψ may be derived by permuting the subscripts :

$$(63) \quad \lambda' = (c_{34} + 1)(c_{35} + 1)(c_{45} + 1) + \{ 1 + [1345] \} \{ 1 + [2345] \}.$$

The complete (symmetrical) expression for ψ is

$$(64) \quad \psi = \sum_{(12)} c_{12} c_{13} c_{24} c_{35} c_{45} + \sum_{(10)} c_{12} c_{34} c_{35} c_{45} + \sum_{(30)} c_{12} c_{13} c_{45} + \sum_{(15)} c_{12} c_{34},$$

the first two sums occurring also in the semi-discriminant (6).

It is readily shown that (61) is invariant under transformation (3). Evidently the two products in (58) are merely interchanged by (3), so that δ is unaltered. Since the increment to [2345] is [1345], λ and λ' are unaltered. Hence ψ is an invariant.

13. Utilizing the known invariants S_5, A_5, I_5, J_5, ψ , given in §§ 2, 3, 12, and making use of certain devices, we shall be able to make a relatively simple determination of a complete set of linearly independent invariants of Q_5 . Any integral function of the coefficients may be given

the form

$$(65) \quad \phi = Ab_2c_{23}c_{24}c_{25} + Bb_2c_{23}c_{24} + Cb_2c_{23}c_{25} + Db_2c_{24}c_{25} + Eb_2c_{23} \\ + Gb_2c_{24} + Jb_2c_{25} + Kb_2 + Lc_{23}c_{24}c_{25} + Mc_{23}c_{24} \\ + Qc_{23}c_{25} + Rc_{24}c_{25} + Sc_{23} + Wc_{24} + Xc_{25} + Z,$$

where $A, \dots, Z$ are functions of the eleven coefficients other than $b_2, c_{23}, c_{24}, c_{25}$. The conditions for the invariance of ϕ under transformation (3) reduce to the following:—

$$(66) \quad Ac_{13} = Ac_{14} = Ac_{15} = A\beta = 0, \quad Ec_{13} + Gc_{14} + Jc_{15} + Bc_{13}c_{14} = 0,$$

$$(67) \quad \begin{cases} Bc_{14} = Cc_{15}, & Bc_{13} = Dc_{15}, & Cc_{13} = Dc_{14}, \\ B\beta = Lc_{15}, & C\beta = Lc_{14}, & D\beta = Lc_{13}, \end{cases}$$

$$(68) \quad E\beta + Mc_{14} + Qc_{15} = B\beta c_{14}, \quad G\beta + Mc_{13} + Rc_{15} = B\beta c_{13},$$

$$J\beta + Qc_{13} + Rc_{14} = C\beta c_{13},$$

$$(69) \quad K\beta + Sc_{13} + Wc_{14} + Xc_{15} + Mc_{13}c_{14} + Qc_{13}c_{15} + Rc_{14}c_{15} + Lc_{13}c_{14}c_{15} = 0,$$

where $\beta = b_1 + c_{12}$. In view of the first four conditions (66), we have

$$(70) \quad A = (c_{13} + 1)(c_{14} + 1)(c_{15} + 1)\{l(1 + b_1 + c_{12}) + mb_1c_{12}\},$$

where l and m are functions of $c_{34}, c_{35}, c_{45}, b_3, b_4, b_5$. By the argument in § 5 leading to (13), we here find that, after a suitable multiple of I_5 has been subtracted from ϕ , m does not contain the product of two of the coefficients b_3, b_4, b_5 , nor l the product of all three, while

$$(71) \quad \text{no term of } \phi \text{ has a factor } b_i b_j b_k b_l c_{ij} c_{ik} c_{il}.$$

In view of (65), l and m must remain unaltered under any permutation of the subscripts 3, 4, 5; while the factor $(c_{13} + 1)(c_{14} + 1)(c_{15} + 1)(l + mb_1)$ of $b_2c_{23}c_{24}c_{25}c_{12}$ must remain unaltered by [15]. Hence

$$l = \tau\{l_1 + x(b_3b_4 + b_3b_5 + b_4b_5) + y(b_3 + b_4 + b_5), \\ m = \tau\{x(b_3 + b_4 + b_5) + y\},$$

where $\tau = (c_{34} + 1)(c_{35} + 1)(c_{45} + 1)$. Applying [15] to the terms of A independent of c_{12} , we obtain the terms of B with the factor c_{12} . Hence we may set

$$B = c_{12}P\{l_1 + x(b_3b_4 + b_1b_3 + b_1b_4) + y(b_3 + b_4 + b_1)\}\{b_5 + 1\} \\ + a + \gamma b_1 + \delta c_{15} + \epsilon b_1c_{15},$$

where $P = \tau(c_{13} + 1)(c_{14} + 1)(c_{15} + 1)$, while $a, \gamma, \delta, \epsilon$ are functions of

$c_{13}, c_{14}, c_{34}, c_{35}, c_{45}, b_3, b_4, b_5$. To apply the fourth condition (67), set

$$\begin{aligned}\pi &= \tau(c_{13}+1)(c_{14}+1)(b_5+1), & \rho &= l_1 + xb_3b_4 + y(b_3+b_4), \\ \sigma &= x(b_3+b_4) + y.\end{aligned}$$

Then $(b_1+c_{12})B$ equals

$$\begin{aligned}(a+\gamma)b_1 + (\delta+\epsilon)b_1c_{15} + (a+\rho\pi)c_{12} + (\gamma+\rho\pi)b_1c_{12} \\ + (\delta+\rho\pi)c_{12}c_{15} + (\epsilon+\rho\pi)b_1c_{12}c_{15}.\end{aligned}$$

Since this must be a multiple of c_{15} , we have $a = \gamma = \rho\pi$. Since B has the term $\gamma b_1 = \rho\pi b_1$, ϕ has the term

$$b_2c_{23}c_{24} \cdot (xb_3+y)b_4 \cdot b_1 \cdot c_{13}c_{14}c_{34}c_{35}c_{45}b_5.$$

But by (71) the factor $b_4b_1b_2b_5c_{14}c_{24}c_{45}$ does not occur. Hence $x = y = 0$. Thus $m = 0$, $l = \tau l_1$. Hence, in ϕ , the coefficient of $b_2c_{23}c_{24}c_{25}$ is

$$A = l_1P(1+b_1+c_{12}).$$

Now J_5 (§ 12) has the term $Pb_1b_2c_{23}c_{24}c_{25}$. Hence $\phi - l_1J_5$ has $A \equiv 0$.

If from the general invariant we subtract suitable multiples of the invariants I_5, J_5 , no term of the resulting invariant ϕ' has a factor $b_i c_{ij} c_{ik} c_{il}$; furthermore, if a term has the factor $b_2 c_{23} c_{24}$, it has also the factor c_{15} .

In view of the notation (65) for the invariant ϕ , the transposition [45] interchanges B and C , G and J , M and Q , W and X ; while [35] interchanges B and D , E and J , M and R , S and X . We shall therefore not write out the explicit expressions for C, D, G, J, Q, R, W, X .

14. We first discuss the terms of ϕ' having the factor $b_1b_2b_3b_4b_5$ or $b_2b_3b_4b_5$, it being necessary to consider the latter with the former, since b_1 occurs explicitly in (66)–(69), in view of $\beta = b_1 + c_{12}$.

We desire the terms B_1 of B which have the factor $b_3b_4b_5$. By § 13, c_{15} occurs in every term, c_{12} in no term of B_1 . Thus in ϕ' we seek the factor of $b_2b_3b_4b_5c_{23}c_{24}c_{15}$, the factor being a function of $b_1, c_{13}, c_{14}, c_{34}, c_{35}, c_{45}$. By § 13, no one of the products

$$c_{13}c_{34}, \quad c_{13}c_{35}, \quad c_{34}c_{35}, \quad c_{14}c_{34}, \quad c_{14}c_{45}, \quad c_{34}c_{45}, \quad c_{35}c_{45}$$

can occur; nor $c_{13}c_{14}$, if b_1 occurs. If c_{13} , and hence also $b_3c_{13}c_{23}$, occurs, then must c_{45} occur. Inversely, if c_{45} , and hence $b_4c_{24}c_{45}$, occurs, then c_{13} must occur. Similarly, c_{14} implies c_{35} , and inversely. Further, B is

unaltered by [34]. Hence

$$B_1 = b_3 b_4 b_5 c_{15} \{s + s_1 c_{34} + s_2 (c_{13} c_{45} + c_{14} c_{35}) + b_1 [t + t_1 c_{34} + t_2 (c_{13} c_{45} + c_{14} c_{35})]\}.$$

By (67₁), Bc_{14} is unaltered by [45]. Hence $s_2 = s_1$, $t_2 = t_1$; so that

$$(72) \quad B_1 = b_3 b_4 b_5 c_{15} \{s + s_1 [1345]\} + b_1 b_3 b_4 b_5 c_{15} \{t + t_1 [1345]\}.$$

Similarly, the terms of B with the factor $b_1 b_3 b_5$ are

$$(73) \quad b_1 b_3 b_5 c_{15} \{r + r_1 (c_{14} + c_{45}) + r_2 c_{34} + r_3 (c_{13} c_{45} + c_{14} c_{35}) + r_4 c_{14} c_{45} \\ + (r_2 + r_3) (c_{14} + c_{45} + c_{14} c_{45}) c_{34}\}.$$

Denote the terms of E with the factor $b_3 b_4 b_5$ by

$$(74) \quad E_1 = b_3 b_4 b_5 \{b_1 c_{12} c_{45} (t + t_1 [1345]) + b_1 E' + c_{12} E'' + E'''\},$$

the first terms being derived from c_{24} times the final part of (72) by the permutation [14]. Since E'' is derived similarly from (73),

$$(75) \quad E'' = c_{45} \{r + r_1 (c_{14} + c_{15}) + r_2 c_{13} + r_3 (c_{15} c_{34} + c_{14} c_{35}) + r_4 c_{14} c_{15} \\ + (r_2 + r_3) (c_{14} + c_{15} + c_{14} c_{15}) c_{13}\}.$$

Now no terms of $E_1(b_1 + c_{12})$ are independent of c_{14} and c_{15} by (68₁). By (74),

$$(76) \quad E_1(b_1 + c_{12}) = b_3 b_4 b_5 \{b_1 c_{12} (E' + E'') + c_{12} (E'' + E''') + b_1 (E' + E''')\}.$$

Hence the terms free of c_{14} and c_{15} in E' , E'' , E''' are identical. By the last part of the theorem in § 13, E' contains none of the products

$$c_{13} c_{14}, \quad c_{13} c_{15}, \quad c_{13} c_{34}, \quad c_{13} c_{35}, \quad c_{14} c_{34}, \quad c_{15} c_{35}, \quad c_{34} c_{35}, \quad c_{34} c_{45}, \quad c_{35} c_{45},$$

while the factor c_{13} implies c_{45} , c_{34} implies c_{15} , c_{35} implies c_{14} . Also, E' must be symmetrical in 1, 4, 5. Finally, the terms free of c_{14} and c_{15} must be $c_{45}(r + r_2 c_{13})$. Hence

$$(77) \quad E' = r(c_{14} + c_{15} + c_{45}) + r_2 [1345] + r_5 (c_{14} c_{15} + c_{14} c_{45} + c_{15} c_{45}) + r_6 c_{14} c_{15} c_{45}.$$

In a similar manner, we get

$$(78) \quad E''' = c_{45} (r + r_2 c_{13}) + r_7 (c_{14} + c_{15}) + r_8 (c_{14} c_{35} + c_{15} c_{34}) + r_9 c_{14} c_{15} \\ + r_{10} (c_{14} c_{45} + c_{15} c_{45}) + r_{11} c_{13} c_{45} (c_{14} + c_{15}) + r_{12} c_{14} c_{15} c_{45} + r_{13} c_{13} c_{14} c_{15} c_{45}.$$

Then G and J are known (end of § 13). The conditions from the $b_3 b_4 b_5$ terms in (66₅) are

$$(79) \quad t = t_1 = r = r_2 = r_5 = r_6, \quad r_3 = r_1 + t, \quad r_4 = r_1, \quad r_9 = s, \\ r_{11} = r_8 + r_{10}, \quad r_{13} = r_{12} + s_1.$$

Applying [13] to $E'c_{23}$, we obtain certain terms of K :

$$(80) \quad tb_1 b_3 b_4 b_5 c_{12} \{c_{34} + c_{35} + c_{45} + c_{34} c_{35} + c_{34} c_{45} + c_{35} c_{45} + c_{34} c_{35} c_{45} + [1345]\}.$$

In the terms of K with the factor $b_1 b_3 b_4 b_5$, but free from c_{12} , the only possible terms are seen (by § 13, and symmetry in 1, 3, 4, 5) to be

$$(81) \quad b_1 b_3 b_4 b_5 \{ \rho + \rho_1 (c_{13} + c_{14} + c_{15} + c_{34} + c_{35} + c_{45}) + \rho_2 [1345] \}.$$

We have now determined all the terms of ϕ' with the factor $b_1 b_2 b_3 b_4 b_5$. They must be unaltered by every permutation of the subscripts. By considering the terms of the first and second degrees in the c_{ij} , we get

$$(82) \quad \rho_1 = 0, \quad \rho_2 = t.$$

Then the terms of ϕ' with the factor $b_1 \dots b_5$ are seen to equal

$$(83) \quad b_1 b_2 b_3 b_4 b_5 (\rho + t\psi),$$

where ψ is the invariant given by (64).

Denote the terms of K with the factor $b_3 b_4 b_5$, but free from b_1 , by

$$(84) \quad b_3 b_4 b_5 (c_{12} K' + K'').$$

In view of § 13 and the symmetry in 3, 4, 5, we get

$$(85) \quad K' = k'_1 + k'_2 (c_{13} + c_{14} + c_{15}) + k'_3 (c_{34} + c_{35} + c_{45}) + k'_4 (c_{13} c_{14} + c_{13} c_{15} + c_{14} c_{15}) \\ + k'_5 [1345] + k'_6 c_{13} c_{14} c_{15} + k'_7 (c_{34} c_{35} + c_{34} c_{45} + c_{35} c_{45}) + k'_8 c_{34} c_{35} c_{45},$$

$$(86)$$

$$K'' = k''_1 + k''_2 (c_{13} + \dots) + k''_3 (c_{34} + \dots) + k''_4 (c_{13} c_{14} + \dots) + k''_5 [1345] + k''_6 c_{13} c_{14} c_{15}.$$

By (69), there can be no terms in $K(b_1 + c_{12})$ independent of c_{13}, c_{14}, c_{15} . Hence, by (80), (81), (84), we get

$$(87) \quad k'_3 = k'_7 = k'_8 = k''_3 = 0, \quad k'_1 = k''_1 = \rho.$$

We have now determined all the terms of ϕ' with the factor $b_2 b_3 b_4 b_5$, but not b_1 . The conditions that they be symmetrical in 2, 3, 4, 5 are

$$(88) \quad k''_2 = \rho, \quad k''_4 = k'_2, \quad k''_6 = k'_4, \\ k'_5 = k''_5 = s = s_1 = r_7 = r_8 = 0, \quad r_1 = r_{10} = r_{12} = t.$$

Hence the factor of $b_2 b_3 b_4 b_5$ may be written in the form

$$(89) \quad \{ \rho + t[2345] \} \{ 1 + \Sigma c_{12} \} + \{ k'_2 + t[2345] \} \Sigma c_{12} c_{13} \\ + \{ k'_4 + t[2345] \} \Sigma c_{12} c_{13} c_{14} + \{ k'_6 + t[2345] \} c_{12} c_{13} c_{14} c_{15},$$

where the summations extend over the combinations of 2, 3, 4, 5.

Applying [13], we obtain the terms with the factor $b_1 b_2 b_4 b_5$. Thus, in B, G, J, E , the terms multiplying $b_1 b_4 b_5$, but not b_3 , are respectively

$$t\pi c_{15}, \quad t\pi c_{15}, \quad t\pi c_{14},$$

$$t\pi c_{12} c_{45} + \rho + k'_2 (c_{13} + c_{34} + c_{35}) + k'_4 (c_{13} c_{34} + c_{13} c_{35} + c_{34} c_{35}) + k'_6 c_{13} c_{34} c_{35},$$

where $\pi \equiv (c_{13}+1)(c_{34}+1)(c_{35}+1)$. If we now examine the terms of the last condition (66) which have the factor $b_1 b_4 b_5$, we see that the only terms not cancelling are those in Ec_{13} and that the condition requires

$$(90) \quad k'_2 = k'_4 = k'_6 = \rho.$$

We may now factor the function (89) and obtain

$$(91) \quad b_2 b_3 b_4 b_5 \{ \rho + t[2345] \} (c_{12}+1)(c_{13}+1)(c_{14}+1)(c_{15}+1)$$

as the terms of ϕ' with the factor $b_2 b_3 b_4 b_5$. Applying [12], we get the terms in $b_1 b_3 b_4 b_5$. Hence we obtain at once, as in § 12, the coefficient of $b_1 b_3 b_4 b_5$ in the increment to ϕ' under (3). This increment vanishes if, and only if,

$$(92) \quad \rho = t.$$

Finally, we shall prove that $t = 0$ by considering the general character, but not the explicit expressions, of certain terms involving only three b 's. Of the terms of $Eb_2 c_{23}$, we consider those with the factor $b_4 b_5$, but not b_3 . If such a term has the factor c_{12} and hence also $b_2 c_{23} c_{12}$, it must have the factor c_{45} by § 13. Again, if a term considered has the factor b_1 , and hence also $b_1 b_2 b_4 b_5$, it must occur in the expression derived from (91) by applying [13]. Hence the terms of E with the factor $b_4 b_5$, but not b_3 , are

$$\epsilon c_{12} c_{45} + \epsilon_1 + t(1 + c_{12} c_{45}) \pi b_1, \quad \pi \equiv (c_{13}+1)(c_{34}+1)(c_{35}+1),$$

where ϵ and ϵ_1 are independent of c_{12} and b_1 . By the first condition (68), the product of the preceding sum by $\beta \equiv b_1 + c_{12}$ must have no term independent of c_{14} and c_{15} . Let ϵ' and ϵ'_1 denote the terms of ϵ and ϵ_1 lacking c_{14} and c_{15} . Now $c_{12} b_1 (b_1 + c_{12}) = 0$, $x^2 = x$. Hence must

$$\epsilon' c_{45} c_{12} (b_1 + 1) + \epsilon'_1 (b_1 + c_{12}) + t \pi b_1 (1 + c_{12}) \equiv 0,$$

identically in c_{12} , b_1 . By the coefficient of $b_1 c_{12}$, $\epsilon' c_{45} = t \pi$. But π is independent of c_{45} . Hence $t = 0$. Also $\epsilon' = \epsilon'_1 = 0$.

THEOREM.—No term of the invariant ϕ' involves four of the b 's.

15. We next determine the form of the terms of ϕ' with the factor* $b_2 b_3 b_4$, carrying along the other terms needed for this purpose.

The terms of B have the factor c_{15} , but not c_{12} (§ 13). Hence in ϕ' we seek the factor of $b_2 b_3 b_4 c_{23} c_{24} c_{15}$, the factor being a function (symmetric in 3, 4) of c_{13} , c_{14} , c_{34} , c_{35} , c_{45} , in which (§ 13) c_{34} occurs singly, while both c_{13} and c_{45} occur or neither occurs, both c_{14} and c_{35} or neither. Hence the

* There are advantages in the use of a triple with b_2 , but not b_1 .

factor is a linear function of c_{34} , $c_{13}c_{45}+c_{14}c_{35}$, and $c_{13}c_{45}c_{14}c_{35}$. By (67₁), Bc_{14} must be unaltered by [45]. Hence the desired terms of B are

$$(93) \quad b_3 b_4 c_{15} \{d + d_1 [1345]\}.$$

We next exhibit the terms of E with the factor $b_3 b_4$ or $b_4 b_5$. Each of these terms involves either c_{14} or c_{15} (end of § 14). Applying § 13, we get

$$(94) \quad b_3 b_4 \{fc_{15} + f_1 c_{15}(c_{14} + c_{45}) + f_2 c_{14} c_{45} + f_3 c_{15} c_{34} + f_4 c_{14} c_{45}(c_{13} + c_{35}) \\ + f_5 c_{15}(c_{13} c_{45} + c_{14} c_{35}) + f_6 c_{14} c_{15} c_{45} + f_7 c_{14} c_{15} c_{45}(c_{13} + c_{35})\},$$

$$(95) \quad b_4 b_5 \{g(c_{14} + c_{15}) + g_1 c_{13}(c_{14} + c_{15}) + g_2 c_{14} c_{15} + g_3(c_{14} c_{35} + c_{15} c_{34}) \\ + g_4 c_{45}(c_{14} + c_{15}) + g_5 c_{13} c_{14} c_{15} + g_6 c_{13}(c_{14} c_{35} + c_{15} c_{34}) \\ + g_7 c_{13} c_{45}(c_{14} + c_{15}) + g_8 c_{14} c_{15} c_{45} + g_9 c_{13} c_{14} c_{15} c_{45}\}$$

as the possible terms independent of c_{12} , (94) being unaltered by [15], and (95) by [45]. We may write down similarly the terms involving c_{12} and hence with the factor c_{45} , since $b_2 c_{23} c_{12}$ implies c_{45} (§ 13). To the resulting terms of E with the respective factors $b_3 b_4 c_{12} c_{45}$, $b_4 b_5 c_{12} c_{45}$, we apply [14] and obtain the terms of Bc_{24} with the factors $b_1 b_3 c_{15}$, $b_1 b_5 c_{15}$ respectively. To these terms of B , we apply the conditions, from (67), that Bc_{14} shall be unaltered by [45], and Bc_{13} by [35]. After these simplifications, the terms of E in question become

$$(96) \quad b_3 b_4 c_{12} c_{45} c_{15} (e + ec_{34} + e_1 c_{13} + e_1 c_{14} c_{35}), \quad b_4 b_5 c_{12} c_{45} c_{15} (c_{14} c_{35} + c_{15} c_{34}) e_1.$$

By (69), the coefficient of $b_3 b_4$ in K must have no term independent of c_{13} , c_{14} , c_{15} . Applying [15] to (95), we get the terms of E with the factor $b_1 b_4$. To the corresponding terms of Ec_{23} , we apply [13] and get the terms of K with the factor $b_3 b_4 c_{12}$:

$$(97) \quad b_3 b_4 c_{12} \{c_{15}(c_{35} + c_{45})(g_1 + g_7 c_{34}) + (c_{13} c_{45} + c_{14} c_{35})(g_3 + g_6 c_{15}) \\ + c_{15} c_{35} c_{45} (g_5 + g_9 c_{34})\},$$

the terms independent of c_{13} , c_{14} , c_{15} being zero, so that

$$(98) \quad g = g_2 = g_4 = g_8 = 0.$$

The terms of K with the factor $b_3 b_4$, but not c_{12} , are (§ 13) of the form

$$(99) \quad b_3 b_4 \{kc_{15} + k_1(c_{13} c_{15} + c_{14} c_{15} + c_{15} c_{35} + c_{15} c_{45}) + k_2 c_{15} c_{34} \\ + (k_3 + k_4 c_{15})(c_{13} c_{45} + c_{14} c_{35}) + k_5 c_{15}(c_{13} c_{14} + c_{35} c_{45})\}.$$

Apply the permutation [1345] to the terms (96₁) multiplying $b_2 c_{23}$. We obtain

$$(100) \quad b_4 b_5 c_{13} c_{15} (e + ec_{45} + e_1 c_{34} + e_1 c_{14} c_{35})$$

times $b_2 c_{23} c_{24}$, viz., terms of B . Now Bc_{13} must be unaltered by [35]. In view of (93) and (100), we get

$$(101) \quad e = e_1 = d = d_1.$$

Applying [34] to (100), we get the $b_3 b_5$ terms of B . Since [45] and [35] replace B by C and D respectively, we get the $b_3 b_4$ terms of C and D :

$$(102) \quad eb_3 b_4 c_{14} c_{15} (1 + [1345]), \quad eb_3 b_4 c_{13} c_{15} (1 + [1345]).$$

Since [34] and [35] replace E by G and J , we obtain from (94)–(96) the $b_3 b_4$ terms of G and J . With these and (93), (94), (96₁), (97), (99), (102), we have, after inserting the proper multipliers from (65), all the terms of ϕ' with the factor $b_2 b_3 b_4$. In view of their origin, they are unaltered by [34]. They are seen to be unaltered by [23] if, and only if,

$$(103) \quad \begin{cases} k_1 = k_3 = k_5 = 0, & k_2 = f, & k_4 = g_1, & f_1 = f_2 = f_4 = f_6 = f_7 = 0, \\ f_3 = f_5 = g_7 = g_9 = e, & g_6 = g_5. \end{cases}$$

Then the terms are unaltered by [15]. The $b_3 b_4$ terms in the last condition (66) give

$$(104) \quad g_1 = f, \quad g_3 = 0, \quad g_5 = e.$$

The factor of $b_2 b_3 b_4$, involving only the parameters k, f, e , is thus

$$(105) \quad kc_{15} + fc_{15}\rho + ec_{15}(\sigma c_{12} + \tau),$$

where $\sigma = \lambda' + c_{35} + c_{45}$, where λ' is given by (63), and*

$$\rho = \rho' + c_{12}(c_{35} + c_{45}), \quad \rho' \equiv c_{23} + c_{24} + c_{25}(c_{13} + c_{14}) + [1345],$$

while τ has initially the form

$$(1 + [1345])(c_{23}c_{24} + c_{13}c_{24}c_{25} + c_{14}c_{23}c_{25}) + [1345](c_{23} + c_{24} + c_{25}) \\ + c_{25}c_{34}(c_{13} + 1)(c_{14} + 1) + c_{25}c_{13}c_{14}.$$

Hence
$$c_{15}(\tau + \rho') = c_{15} \{1 + (1 + [1345])\xi\},$$

$$\begin{aligned} \xi &\equiv (c_{23} + 1)(c_{24} + 1) + (c_{23} + 1)c_{25}c_{14} + (c_{24} + 1)c_{25}c_{13} + c_{25}c_{13}c_{14} \\ &= (c_{23} + 1 + c_{13})(c_{24} + 1 + c_{14})c_{25} + (c_{23} + 1)(c_{24} + 1)(c_{25} + 1), \end{aligned}$$

so that $c_{15}\xi = c_{15}\delta$, where δ is defined by (58). It now follows by inspection from (62) that $c_{15}(\tau + \rho') = c_{15}\lambda$. Hence the factor (105) of $b_2 b_3 b_4$ equals

$$(106) \quad c_{15} \{k + (e + f)\rho + e\psi\},$$

ψ being the invariant $\lambda + \lambda'c_{12}$ of § 12.

* A more symmetrical form for $c_{15}\rho$ is $c_{15} \{[1345] + [1235] + [1245]\}$.

Since there are no terms with four b 's, the coefficient of $b_1 b_3 b_4$ in the increment to ϕ' under (3) arises only from the $b_1 b_3 b_4$ and $b_2 b_3 b_4$ terms. Now (106) is unaltered by (3) since the increments to $[1235]$ and $[1245]$ are zero. Hence the coefficient of $b_1 b_3 b_4$ in the increment to the $b_2 b_3 b_4$ terms is (106) itself. Applying [12] to (106), we obtain

$$c_{25} \{k + e\psi + (e+f)([2345] + [1235] + [1245])\}$$

as the factor of $b_1 b_3 b_4$. The increment to this factor is obviously

$$c_{15}(k + e\psi) + (e+f) \{ (c_{25} + c_{15})[1345] + c_{15}[2345] + c_{15}[1235] + c_{15}[1245] \}.$$

Adding (106) to this, we obtain the total coefficient of $b_1 b_3 b_4$ in the increment:

$$(e+f) \{ c_{25}[1345] + c_{15}[2345] \}.$$

Since this must vanish, we have $e = f$.

THEOREM.—*The terms of the general invariant ϕ' of highest degree in the b 's are given by the product of the invariant* $k + e\psi$ and $\Sigma b_2 b_3 b_4 c_{15}$.*

16. The preceding result suggests a very decided simplification in the further work. It will suffice to determine the linearly independent invariants ϕ_i (with $k = e = 0$) lacking the products of three b 's, and a single invariant K_5 (with $k = 1, e = 0$). Indeed, we shall then have

$$\phi' = kK_5 + e\psi K_5 + \Sigma l_i \phi_i.$$

Since the simplified problem may be treated readily by the preceding methods, I will merely state the result that the invariants previously found together with K_5 form a complete set, the expression for K_5 being

$$(107) \quad K_5 \equiv \Sigma b_1 b_2 b_3 c_{45} + \Sigma b_1 b_2 (\lambda' + c_{34} c_{35} c_{45}) + \Sigma b_5 (\psi + \kappa + \rho) + \sigma,$$

where λ' is given by (63), κ by (53), while

$$(108) \quad \rho = (1 + [1234]) (c_{15} c_{23} c_{24} c_{34} + c_{25} c_{13} c_{14} c_{34} + c_{35} c_{12} c_{14} c_{24} + c_{45} c_{12} c_{13} c_{23}).$$

For the terms† σ independent of the b 's, expressed as the sum of the last eight terms (65), we have

$$L = c_{12}(1 + [1345]) + (c_{13} + 1)(c_{14} + 1)(c_{15} + 1),$$

$$M = c_{12}(c_{15} + 1)(c_{35} + 1)(c_{45} + 1) + (c_{13} + 1)(c_{14} + 1)(c_{35} + 1)(c_{45} + 1)$$

$$+ c_{15}(c_{13} + c_{14} + c_{35} + c_{45} + c_{13} c_{14} + c_{13} c_{45} + c_{14} c_{35} + c_{35} c_{45} + c_{13} c_{34} c_{35} + c_{14} c_{34} c_{45}),$$

* It will appear in the sequel that k and e are independent parameters.

† Under (3) the increment to σ equals c_{12} times the coefficient $\psi + \dots$ of b_2 in K_5 .

$$S = c_{12}M' + c_{13}c_{14}(c_{35}+1)(c_{45}+1) + c_{13}c_{15}(c_{34}+1)(c_{45}+1) \\ + c_{14}c_{15}(c_{34}+1)(c_{35}+1) + (1+c_{13}+c_{14}+c_{15})(c_{34}c_{35}+c_{34}c_{45}+c_{35}c_{45}) \\ + (1+c_{13})(1+c_{34}+c_{35}+c_{14}c_{15}) + c_{14}c_{34}+c_{15}c_{35},$$

$$Z = c_{12}S' + c_{13}c_{14}(c_{35}+1)(c_{45}+1) + c_{13}c_{15}(c_{34}+1)(c_{45}+1) \\ + c_{14}c_{15}(c_{34}+1)(c_{35}+1) + c_{13}c_{14}c_{15} \\ + (1+c_{13}+c_{14}+c_{15})(1+c_{34}+c_{35}+c_{45}+c_{34}c_{35}+c_{34}c_{45}+c_{35}c_{45}) \\ + [1345] + 1,$$

where M' and S' are derived from the terms independent of c_{12} in M and S by the respective permutations [14] and [13]. As at the end of § 13, we deduce Q , R , W , X .

When the terms of K_5 with subscript 5 are suppressed, there results the invariant K_4 of Q_4 (§ 10).

THEOREM.—*A complete set of linearly independent invariants of the quinary quadratic form Q_5 , modulo 2, is given by*

$$(109) \quad I_5, J_5, \psi, K_5, \psi.K_5, S_5, A_5.$$

17. Since $A_5c_{ij} = 0$, we evidently have $A_5J_5 = I_5$. In forming a table as in § 11 to prove that ψ, A_5, J_5, K_5, S_5 are independent, we note that, when every c_{i5} and b_5 are suppressed, these invariants reduce to [1234], A_4, J_4, K_4 , and zero, respectively. Hence we need only enlarge the table of § 11, by adding a column for S_5 with zero entries, and inserting the new line

$$X_1X_2+X_3X_4+X_5^2, \quad \psi = 1, \quad A_5 = 0, \quad J_5 = 0, \quad K_5 = 1, \quad S_5 = 1.$$

From the enlarged table we deduce at once the following:—

THEOREM.—*A complete set of independent invariants is given by ψ, A_5, J_5, K_5 , and S_5 . Each canonical form is completely characterized by the values of these invariants.*

Invariants of the Senary Quadratic Form Q_6 .

18. Proceeding as in § 12, we obtain the invariant

$$(110) \quad J_6 = (b_1+p_1) \dots (b_6+p_6) + b_1 \dots b_6\chi + \Sigma b_1b_2b_3b_4b_5\psi_6p_6 \\ + \Sigma b_1b_2b_3b_4[1234]p_5p_6,$$

in which ψ_6 is the function ψ (§ 12) of the c_{ij} ($i \neq 6, j \neq 6$), ψ_k being

derived from ψ by the permutation $[6k]$, while

$$(111) \quad p_i = \Pi(c_{ij} + 1) \quad (j = 1, \dots, 6; j \neq i).$$

Finally, χ is an invariant* (independent of the b 's) of Q_6 :

$$(112) \quad \chi = \mu + \mu' c_{12},$$

$$(113) \quad \mu = 1 + (c_{13} + 1)(c_{14} + 1)(c_{15} + 1)(c_{16} + 1)(1 + \psi_1) + \delta(1 + \psi_2),$$

$$(114) \quad \begin{aligned} \mu' = & (c_{13} + 1)(c_{34} + 1)(c_{35} + 1)(c_{36} + 1)C \\ & + c_{23} c_{56} (c_{14} + 1)(c_{34} + 1)(c_{45} + 1)(c_{46} + 1)(1 + [1356]) \\ & + (\psi_2 + 1) \{ \delta' + c_{23} (1 + [2456]) + c_{23} (c_{45} + 1)(c_{46} + 1) \}, \end{aligned}$$

in which δ' is derived from (58) by the permutation $[136]$, and C from (63) by $[36]$, while the expression for δ corresponds to (58), viz.,

$$(115) \quad \delta = \Pi(c_{2j} + c_{1j} + 1) + \Pi(c_{2j} + 1) \quad (j = 3, 4, 5, 6).$$

Comment is needed only on the origin of (114). The terms μ , independent of c_{12} , in the invariant χ were determined directly (as in § 12); the remaining terms $\mu' c_{12}$ are to be determined so that χ shall be symmetrical in the subscripts 1, ..., 6. Set $\mu' = g + g_1 c_{23}$, g and g_1 independent of c_{23} . Evidently g may be derived from the coefficient of c_{23} in μ by [13], so that g is the sum of the first term of (114) and $(\psi_2 + 1)\delta'$. Indeed, the coefficient of c_{23} in the final part of μ is evident from (115), while that in ψ_1 is obtained from the coefficient (63) of c_{12} in $\psi \equiv \psi_6$ by the permutation $[136]$. From the expression for the coefficient of c_{23} in μ we deduce the coefficient of $c_{23} c_{24}$, viz.,

$$\begin{aligned} (\psi_2 + 1) \{ & (c_{25} + 1) c_{16} + (c_{26} + 1) c_{15} + c_{15} c_{16} \} \\ & + c_{56} (c_{13} + 1)(c_{14} + 1)(c_{15} + 1)(c_{16} + 1)(1 + [3456]). \end{aligned}$$

Set $g_1 = k + k_1 c_{24}$. Then k is derived from the preceding function by [14]. Set $k_1 = l + l_1 c_{25}$. Then l is derived from the coefficient $(\psi_2 + 1) c_{16}$ of $c_{23} c_{24} c_{25}$ in μ by [15]. There are no terms $l_1 c_{12} c_{23} c_{24} c_{25}$, since [16] would replace such a term by a term $c_{23} c_{24} c_{25} c_{26}$ of μ ; whereas its coefficient is zero.

The resulting function (114) may be given an elegant form, affording a direct generalization of (63), viz.,

$$(116) \quad \mu' = (c_{34} + 1)(c_{35} + 1)(c_{36} + 1)(c_{45} + 1)(c_{46} + 1)(c_{56} + 1) + (1 + \psi_1)(1 + \psi_2).$$

* The terms of lowest degree are the 15 Pfaffians [1234].

To make this identification, it is convenient to employ for ψ_1 the function obtained from ψ by applying [136]. By inspection, [1345] ψ = [1345], so that

$$(117) \quad [3456](1 + \psi_2) = 0.$$

Hence we may suppress [3456] in the three terms in which it occurs in the final part of (116). Adding the resulting expression to the final part of (114), we see that a number of terms cancel at once. The sum is

$$(118) \quad (\psi_2 + 1)\pi(1 + [2456]) + (\psi_2 + 1)c_{23}c_{56}(c_{45} + 1)(c_{46} + 1),$$

where $\pi \equiv (c_{34} + 1)(c_{35} + 1)(c_{36} + 1)$. The second part equals the $c_{23}c_{56}$ terms of (114). To simplify the first part, we employ for ψ_2 the function obtained from ψ by the permutation [26][13]. Then $(\psi_2 + 1)\pi$ becomes

$$\pi \{c_{13}(c_{45} + 1)(c_{46} + 1)(c_{56} + 1) + (c_{13} + 1)(1 + [1456])\}.$$

The sum of the first terms of (118) and (114) is thus the first term of (116).

The invariance of the function χ , defined by (112), (113), (116), follows at once from the facts that δ is evidently unaltered by (3), that the increment to ψ_1 is

$$(119) \quad \psi_2 + 1 + \{1 + [123456]\} \{1 + [3456]\},$$

and that from (117), and the symmetry of ψ_2 in 1, 3, 4, 5, 6, follows

$$(120) \quad [123456](\psi_2 + 1) = 0.$$

19. In addition to the constructed invariants A_6 , $\Delta = [123456]$, J_6 , and χ , there should occur two invariants T_6 , K_6 , of degrees 2 and 4, respectively, in the b 's. These six should form a complete set of independent invariants of Q_6 ; while they, together with $I_6 = A_6J_6$, ΔT_6 , and ΔK_6 , should form a complete set of linearly independent invariants. I have not attempted to prove these conjectures.

In seeking an invariant of Q_6 , we note that the coefficient of b_6 must be a function such that, if the subscripts 6 be suppressed, there shall result an invariant of Q_5 . In T_6 , the coefficient C of b_6 must involve the b 's linearly; the function which results from C after the suppression of the c_{i6} must be a linear combination of the invariants S_5 , A_5 , and ψ of Q_5 , with S_5 necessarily present. To make the degree as low as possible, we take $S_5 + \psi \equiv \Sigma b_1[2345] + \rho_6$, where

$$(121) \quad \rho_6 = \sum_{(30)} c_{12}c_{13}c_{45} + \sum_{(15)} c_{12}c_{34} \quad (\text{permutations of } 1, 2, 3, 4, 5).$$

By these and other considerations,* I am led to the required invariant,

$$(122) \quad T_6 = \Sigma b_1 b_2 [3456] + \Sigma b_i (\rho_i + \Delta) + \tau,$$

where ρ_i is derived from ρ_6 by $[I_6]$, while τ is an unknown function of the c_{ij} , symmetric in the subscripts 1, ..., 6, and having under (3) the increment

$$(123) \quad c_{12} (\rho_2 + \Delta).$$

Since the increments to ρ_6 and ρ_1 are $c_{12} [1345]$ and $\rho_2 + \Delta + (c_{12} + 1) [3456]$, respectively, the increment to the b terms of T_6 is independent of the b 's and equals (123).

* Including a study of the b terms, $B = \Sigma b_1 b_2 [3456] + \Sigma b_i \sigma_i$, occurring with the factor 4, but not 8, in the expansion of the discriminant (4), σ_6 being formed of the first two sums in (64). Since $\rho_6 + \sigma_6 = \psi$, it follows from (120) that ΔB equals the b terms of ΔT_6 . The invariance of T_6 implies that of ΔT_6 . As a check, we note that the increment to ΔB is $c_{12} \sigma_2 \Delta$ and is thus independent of the b 's.

ON REPEATED INTEGRALS

By E. W. HOBSON.

[Received February 5th, 1907.—Read February 14th, 1907.]

IN my paper* “On Absolutely Convergent Improper Integrals,” I considered the relation of Lebesgue’s definition of integration to those definitions which have been obtained by generalizing Riemann’s definition in such a manner that they are applicable to improper double integrals. I also considered the relation of double integrals as defined, for the case of unlimited functions, by de la Vallée-Poussin and by Jordan, with the corresponding repeated integrals. I did not, however, consider the case of functions for which the double integral exists only when the definition of Lebesgue is adopted. This I propose to do in the present communication. It appears that the theory of Lebesgue’s integrals throws light upon those cases in which one of the repeated integrals, or both of them, exist in accordance with the definition of Riemann or its generalization, but in which the double integral according to the definitions of Jordan and de la Vallée-Poussin has no existence. Even in the case of limited functions, the consideration of the Lebesgue double integral helps to fill up gaps in the ordinary theory of integration with respect to a parameter under the sign of integration.

I shall assume, as in my former paper, that a function $\phi(x, y)$, defined for a limited domain G , is replaced by a function $f(x, y)$, defined everywhere in a rectangle bounded by $x = a$, $x = b$, $y = c$, $y = d$, which contains the domain G ; the function $f(x, y)$ being defined to be equal to $\phi(x, y)$ at every point of G , and to be zero at every point of the rectangle which does not belong to G . It is assumed that the frontier of G has the plane measure zero.

1. Let $f(x, y)$ be a limited summable function.

It has been pointed out by Lebesgue that, if E be a set of points measurable in the plane, it does not necessarily follow that the section of

* *Proc. London Math. Soc.*, Ser. 2, Vol. 4.

the set E by a straight line parallel to one of the axes is necessarily linearly measurable; unless, indeed, the unproved assumption be adopted that all linear sets of points have a linear measure. Accordingly, although $\int f(x, y)(dx dy)$ necessarily exists, the single integrals

$$\int_a^b f(x, y) dx, \quad \int_c^d f(x, y) dy,$$

and therefore also the repeated integrals

$$\int_a^b dx \int_c^d f(x, y) dy, \quad \int_c^d dy \int_a^b f(x, y) dx,$$

cannot be assumed necessarily to have definite meanings, Lebesgue has, for the case of a single integral of a function $\phi(x)$ which is not summable, defined an upper and a lower integral, denoted by $\int_{\sup} \phi(x) dx$, and by $\int_{\inf} \phi(x) dx$ respectively. These must not be confused with the upper and lower integrals as defined by Darboux. For such a summable function as is not integrable in accordance with Riemann's definition, the upper and lower integrals as defined by Lebesgue are identical in value, whereas the upper and lower integrals, as defined by Darboux, have different values.

Lebesgue* has then proved the general theorem,

$$\begin{aligned} \int f(x, y)(dx dy) &= \int_a^b dx \int_c^d f(x, y) dy = \int_a^b dx \int_c^d f(x, y) dy \\ &= \int_c^d dy \int_a^b f(x, y) dx = \int_c^d dy \int_a^b f(x, y) dx. \end{aligned}$$

This theorem reduces to the theorem that

$$\int f(x, y)(dx dy) = \int_a^b dx \int_c^d f(x, y) dy = \int_c^d dy \int_a^b f(x, y) dx,$$

whenever the repeated integrals have definite meanings, the double integral always existing, since $f(x, y)$ is a limited summable function.

In case $\int f(x, y)(dx dy)$ exist in accordance with the ordinary definition, which is an extension of Riemann's definition of a single integral, it is

* See his memoir "Intégral, Longueur, Aire," *Annali di Mat.*, Ser. 3, Vol. VII., 1902.

well known that both the repeated integrals exist, and are equal to the double integral.

Various examples have, however, been given of functions for which only one of the repeated integrals exists, according to Riemann's definition, or for which both of them exist, and yet for which no double integral exists in accordance with the ordinary definition. The above theorem of Lebesgue throws light on such cases; for the following results follow immediately from it:—

If the two repeated integrals exist, in accordance with the ordinary definition, they must have equal values, if $f(x, y)$ be summable as a function in the plane, and they are equal to the Lebesgue double integral of the function.

If only one of the repeated integrals exist, in accordance with the ordinary definition, then the other exists as a repeated Lebesgue integral, and the two are equal to the Lebesgue double integral; it being assumed that the function is summable in the plane, and also on any straight line parallel to either axis.

All functions defined by any of the ordinary means are summable; it is, in fact, not definitely known whether it is possible to define a function which is not summable. Accordingly, all cases which arise in practice are covered by the two theorems here given.

The following examples will illustrate the utility of these remarks in the direction of completing the ordinary theory:—

(1) For the rectangle* bounded by $x=0$, $x=1$, $y=0$, $y=1$, let $f(x, y) = 1$, for all rational values of x ; and let $f(x, y) = 2y$, for all irrational values of x . We have then $\int_0^1 f(x, y) dy = 1$, whatever value x may have; and hence $\int_0^1 dx \int_0^1 f(x, y) dy$ exists, in accordance with the ordinary definition, and $= 1$. The integral $\int_0^1 f(x, y) dx$ does not exist as a Riemann integral, except when $y = \frac{1}{2}$; for every point is a point of discontinuity. Consequently $\int_0^1 dy \int_0^1 f(x, y) dx$ does not exist in accordance with the ordinary definition. The double integral does not exist in accordance with the ordinary definition. The Lebesgue integral $\int f(x, y) (dx dy)$, however, exists, and is equal to 1. For the set of points at which $f(x, y) = 1$ has the

* This function was given by Thomae, *Schlömilch's Zeitschrift*, Vol. xxiii., p. 67.

measure zero; and therefore $f(x, y)$ has the same Lebesgue integral as the function $\phi(x, y)$, which is defined by $\phi(x, y) = 2y$ at all points. For the functional values at a set of points of zero measure are irrelevant in a Lebesgue integral. The repeated integral $\int_0^1 dy \int_0^1 f(x, y) dx$ also exists as a Lebesgue repeated integral, and is equal to 1; this may easily be verified directly.

(2) For the same rectangle as in (1), let a set* K of points be defined as follows:—The numbers x, y are expressed in the dyad scale, and only those values of x and y are taken which are expressed by terminating radix-fractions, the number of digits being the same for x as for y . Let the function $f(x, y)$ be defined as equal to c' at every point of K , and equal to c at every other point.

If x' denotes a terminating radix-fraction, there are only a finite number of points of K on the straight line $x = x'$; similarly there are only a finite number of points of K on the straight line $y = y'$, where y' denotes a terminating radix-fraction. It thus appears that both repeated integrals exist, in accordance with Riemann's definition, and that they are both equal to c . The double integral does not exist in accordance with the ordinary definition; for it can be shewn that the set K is everywhere-dense, and therefore the function $f(x, y)$ is totally discontinuous. Consider the straight line $y = x + a$, where a is a positive or negative radix-fraction with a finite number of digits; we see that, corresponding to any number x expressed by a finite number of digits greater than the number by which a is expressed, there is a point (x, y) on the straight line belonging to K . The component of K on this straight line is consequently everywhere-dense; thus, since the values of a are everywhere-dense in the interval $(-1, 1)$, it follows that the points of K are everywhere-dense in the rectangle. The Lebesgue double integral exists, and $= c$; for, as in (1), the points at which $f(x, y) = c'$ form a set of zero measure, and therefore the integral is the same as for a function which has everywhere the value c .

2. Let $f(x, y)$ be summable, but not limited. It has been established by Lebesgue that the equalities

$$\int f(x, y)(dx dy) = \int dx \int f(x, y) dy = \int dy \int f(x, y) dx$$

* See Pringsheim, *Munich Sitzungsberichte*, Vol. xxix., p. 48.

still hold whenever the integrals have meanings in accordance with his definition.

This theorem throws light upon the validity of the process of integration with respect to a parameter, under the integral sign, the limits of integration in both cases being finite. We may state the theorem as follows:—

Integration with respect to a parameter, under the integral sign, is always a valid process, provided the function is summable and integrable in the plane, if the result of the process have a definite meaning.

It must, however, be remembered that, even if $\int_c^{y_1} dy \int_a^b f(x, y) dx$ exist, in accordance with Riemann's definition, or one of its extensions to absolutely convergent integrals, $\int_a^b dx \int_c^{y_1} f(x, y) dy$ may exist, if it exist at all, only when Lebesgue's definition is employed.

The only case in which the repeated integrals of an unlimited function can both exist, but have unequal values, is when the function is either not summable in the plane, or else when it is summable but does not possess a Lebesgue integral.

In order to find sufficient conditions for the existence and equality of the repeated integrals of an unlimited function in cases when the corresponding double integral does not exist, either in accordance with the definition of Jordan, or with that of de la Vallée-Poussin, it is useful to introduce a definition of a double integral of a less stringent character than that of Jordan. This definition differs from that of Jordan, as given in my former paper (p. 137), in the one respect, that the domains D_n are restricted each to consist of a finite set of rectangles, the sides of each of which are parallel to those of the fundamental rectangle in which the function is defined.

A double integral which exists in accordance with this modified definition, I propose to speak of as a *restricted Jordan double integral*. Such a double integral may exist for a function which does not possess a Jordan double integral.

Assuming that the integral $\int_A f(x, y)(dx dy)$ exists, as a restricted Jordan double integral, let $f_n(x, y)$ be that limited function which, in the domain D_n , consisting of a finite set of rectangles, is equal to $f(x, y)$, and is zero in the complementary domain $C(D_n)$ which contains all the points of infinite discontinuity of $f(x, y)$.

We have then

$$\begin{aligned}\int_A f(x, y) (dx dy) &= \lim_{n=\infty} \int_A f_n(x, y) (dx dy) \\ &= \lim_{n=\infty} \int_a^b dx \int_c^d f_n(x, y) dy;\end{aligned}$$

and therefore we have

$$\int_A f(x, y) (dx dy) = \int_a^b dx \int_c^d f(x, y) dy,$$

provided

$$\lim_{n=\infty} \int_a^b dx \int_{\Lambda_n(x)} f(x, y) dy = 0,$$

where $\Lambda_n(x)$ denotes that finite set of intervals which forms the section of $C(D_n)$ by the ordinate corresponding to the abscissa x . From this result, the following theorem, very similar to one given by Jordan,* and specifying a particular mode of satisfying the last condition, may be deduced:—

For the existence and equality of the two repeated integrals

$$\int_a^b dx \int_c^d f(x, y) dy, \quad \int_c^d dy \int_a^b f(x, y) dx,$$

it is sufficient:

(1) *That the function $f(x, y)$ possess a restricted Jordan double integral in the fundamental rectangle.*

(2) *That the points of infinite discontinuity of $f(x, y)$ be distributed on a limited number of arcs of continuous curves representing monotone functions.*

(3) *That, corresponding to any positive number ϵ , positive numbers h_1, k_1 exist, such that*

$$\left| \int_x^{x+h} f(x, y) dx \right| < \epsilon, \quad \left| \int_y^{y+k} f(x, y) dy \right| < \epsilon,$$

for $|h| < h_1, |k| < k_1$, and for every value of x and y in the fundamental rectangle.

To shew that, under the conditions stated,

$$\lim_{n=\infty} \int_a^b dx \int_{\Lambda_n(x)} f(x, y) dy = 0,$$

* *Cours d'Analyse*, Vol. II., p. 67.

it is clear that the points of any one such curve can be enclosed in the interiors of a finite set of rectangles, the height of each of which is $< k_1$. Then $\Lambda_n(x)$ consists of a number of intervals not exceeding the number r of the curves on which the points of infinite discontinuity lie.

$$\text{We have then} \quad \left| \int_{\Lambda_n(x)} f(x, y) dy \right| < r\epsilon,$$

$$\text{and therefore} \quad \left| \int_a^b dx \int_{\Lambda_n(x)} f(x, y) dy \right|$$

is less than the arbitrarily small number $r\epsilon(b-a)$. Thus

$$\int_a^b dx \int_c^d f(x, y) dy$$

exists, and is equal to the restricted Jordan integral. Similarly it can be shewn that the other repeated integral has the same value.

The two examples given in my former paper (pp. 156, 157) will serve as illustrations of the greater completeness given to the theory by taking account of the existence of Lebesgue integrals.

(1) If $\mathbf{f}(x)$ be defined for the rectangle bounded by $x = 0, x = 1, y = 0, y = 1$, by the rule that $\mathbf{f}(x) = \frac{1}{2^n}$, for $x = \frac{2m+1}{2^n}$ ($n \geq 0$), and $\mathbf{f}(x) = 0$, for all other values of x , then

$$\int \left| \frac{1}{y} \sin \frac{1}{y} \right| \mathbf{f}(x) (dx dy) = 0.$$

The repeated integral $\int_0^1 dy \int_0^1 \left| \frac{1}{y} \sin \frac{1}{y} \right| \mathbf{f}(x) dx$ exists, and $= 0$. The other repeated integral $\int_0^1 dx \int_0^1 \mathbf{f}(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$, which was shewn not to exist in accordance with the earlier definitions, exists in accordance with Lebesgue's definition, and is also equal to zero. For, although $\int_0^1 \mathbf{f}(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$ diverges for an everywhere-dense set of values of x , that set of points has zero measure. Hence, since the Lebesgue integral is independent of the functional values at a set of points of zero measure, $\int_0^1 \mathbf{f}(x) \left| \frac{1}{y} \sin \frac{1}{y} \right| dy$ is integrable with respect to x in the interval $(0, 1)$, and has the value zero.

(2) If $f(x, y) = 0$ at all points in the rectangle bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, except at the points $x = \frac{2m+1}{2^s}$, $y \leq \frac{1}{2^s}$, where it has the improper value $+\infty$.

The repeated integral $\int_0^1 dx \int_0^1 f(x, y) dy$, which was shewn not to exist in accordance with Harnack's definition, exists in accordance with that of Lebesgue, and is equal to zero, the value of the double integral.

3. The question of the validity of differentiation under the integral sign has been treated by reducing the question to one of the validity of reversing the order of a repeated integral. Denoting by $Df(x, y)$ the derivative on the right, of $f(x, y)$ with respect to y , we have, under certain conditions,

$$f(x, y_0+h) - f(x, y_0) = \int_{y_0}^{y_0+h} Df(x, y) dy.$$

It may happen that $Df(x, y)$, although limited in the domain bounded by $x = a$, $x = b$, $y = y_0$, $y = y_0+h$, is not integrable with respect to y in accordance with Riemann's definition. Even when a differential coefficient everywhere exists, it is known that this may be the case. But the integral certainly exists; for $Df(x, y)$ is summable if $f(x, y)$ be so. It is here assumed that $Df(x, y)$ has a definite value for each value of x , for all values of y in (y_0, y_0+h) with the possible exception of a set of values of y of zero measure.

We have now, if u_y denote $\int_a^b f(x, y) dx$,

$$\frac{u_{y_0+h} - u_{y_0}}{h} = \frac{1}{h} \int_a^b dx \int_{y_0}^{y_0+h} Df(x, y) dy.$$

In accordance with Lebesgue's theorem, if $f(x, y)$ be summable as a function of the two variables (x, y) , the order of integration in the repeated integral may be reversed, or

$$\frac{u_{y_0+h} - u_{y_0}}{h} = \frac{1}{h} \int_{y_0}^{y_0+h} dy \int_a^b Df(x, y) dx,$$

provided the repeated integral have a definite meaning, and this is the case, since $\left| \int_a^b Df(x, y) dx \right|$ cannot exceed $b-a$ multiplied by the upper limit of $|Df(x, y)|$ in the two-dimensional domain.

We have now

$$Du_{y_0} = \int_a^b Df(x, y_0) dx,$$

provided $\int_a^b Df(x, y) dx$ be continuous with respect to y at y_0 on the right.

We have then the following theorem:—

If $Df(x, y)$ be limited in the domain bounded by $x = a$, $x = b$, $y = y_0$, $y = y_0 + a$, then, provided the Lebesgue integral $\int_a^b Df(x, y) dx$ be continuous at y_0 on the right, the derivative of $\int_a^b f(x, y) dx$ at y_0 on the right is $\int_a^b Df(x, y_0) dx$; it being assumed that $\int_{y_0}^{y_0+a} Df(x, y) dy$ exists at least as a Lebesgue integral.

In case $Df(x, y)$ be unlimited in the domain bounded by $x = a$, $x = b$, $y = y_0$, $y = y_0 + a$, it may still have a Lebesgue integral with respect to y in $(y_0, y_0 + a)$. In case the points of infinite discontinuity form, for each value of x , a reducible set in the interval $(y_0, y_0 + a)$, the equation

$$f(x, y_0 + h) - f(x, y_0) = \int_{y_0}^{y_0+h} Df(x, y) dy \quad (0 < h < a)$$

is still valid. If $Df(x, y)$ have a Lebesgue double integral in the two-dimensional domain, and $\int_{y_0}^{y_0+h} dy \int_a^b Df(x, y) dx$ have a definite meaning, as a repeated Lebesgue integral, then the above process is still valid. We therefore obtain the following theorem:—

If the points of infinite discontinuity of $f(x, y)$, considered as a function of y , form a reducible set, for each value of x , and $\int_{y_0}^{y_0+h} Df(x, y) dy$ exist as a Lebesgue integral, for each value of x ; if, further, $Df(x, y)$ have a Lebesgue double integral in the two-dimensional domain, and

$$\int_{y_0}^{y_0+h} dy \int_a^b Df(x, y) dx$$

exist as a repeated Lebesgue integral; and, if, lastly, $\int_a^b Df(x, y) dx$ be continuous on the right at $y = y_0$, then the derivative of $\int_a^b f(x, y) dx$ at y_0 on the right is $\int_a^b Df(x, y_0) dx$.

In any particular case, the Lebesgue integrals may exist in accordance with the earlier definitions.

As an example* of the application of this theorem, we may take the case of the differentiation of $\int_0^X (x-y)^{\frac{1}{2}} dx$ with respect to y . It can easily be shewn that $(x-y)^{-\frac{3}{2}}$ possesses a double integral in the domain bounded by $x=0$, $x=X$, $y=0$, $y=h$, and that the other conditions of the theorem are also satisfied.

* This example has been given by Mr. Hardy, *Messenger of Math.*, Vol. xxxiii., p. 63, as a case not covered by the ordinary criteria.

ON A FORMULA FOR THE SUM OF A FINITE NUMBER OF
TERMS OF THE HYPERGEOMETRIC SERIES WHEN THE
FOURTH ELEMENT IS EQUAL TO UNITY

By M. J. M. HILL.

[Received and Read February 14th, 1907.]

1. The series being

$$1 + \frac{a\beta}{1\gamma} + \frac{a(a+1)\beta(\beta+1)}{2! \gamma(\gamma+1)} + \dots,$$

let

$$a_r = a(a+1) \dots (a+r-1).$$

The $(r+1)$ -th term is $\frac{a_r \beta_r}{r! \gamma_r}$. Also

$$\frac{a_r \beta_r}{r! (\gamma-1)_r} - \frac{a_r \beta_r}{r! \gamma_r} = \frac{a\beta}{(\gamma-1)\gamma} \frac{(a+1)_{r-1} (\beta+1)_{r-1}}{(r-1)! (\gamma+1)_{r-1}}.$$

Now, let $G(a, \beta, \gamma, s)$ denote the sum of the first $(s+1)$ terms of the series. Then

$$\begin{aligned} G(a, \beta, \gamma-1, s) - G(a, \beta, \gamma, s) &= \sum_{r=1}^{r=s} \left(\frac{a_r \beta_r}{r! (\gamma-1)_r} - \frac{a_r \beta_r}{r! \gamma_r} \right) \\ &= \frac{a\beta}{(\gamma-1)\gamma} \sum_{r=1}^{r=s} \frac{(a+1)_{r-1} (\beta+1)_{r-1}}{(r-1)! (\gamma+1)_{r-1}}; \end{aligned}$$

and therefore

$$G(a, \beta, \gamma-1, s) - G(a, \beta, \gamma, s) = \frac{a\beta}{(\gamma-1)\gamma} G(a+1, \beta+1, \gamma+1, s-1). \quad (I.)$$

2. Consider next the expression

$$\begin{aligned} G(a, \beta, \gamma, s) - \frac{\gamma-a-\beta-1}{\gamma} G(a+1, \beta+1, \gamma+1, s-1) \\ = 1 + \frac{a\beta}{\gamma} + \frac{a_2 \beta_2}{2! \gamma_2} + \dots + \frac{a_s \beta_s}{s! \gamma_s} \\ - \frac{\gamma-a-\beta-1}{\gamma} \left(1 + \frac{(a+1)(\beta+1)}{(\gamma+1)} + \frac{(a+1)_2 (\beta+1)_2}{2! (\gamma+1)_2} + \dots \right. \\ \left. + \frac{(a+1)_{s-1} (\beta+1)_{s-1}}{(s-1)! (\gamma+1)_{s-1}} \right). \end{aligned}$$

The first and second terms of the first series, together with the first term of the second series, add up to

$$[(\alpha+1)(\beta+1)]/\gamma.$$

This together with one more term from each series add up to

$$[(\alpha+1)_2(\beta+1)_2]/2!\gamma_2.$$

This suggests that the sum of the first r terms of the first series and the first $(r-1)$ terms of the second series will add up to

$$\frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\gamma_{r-1}}.$$

Adding in one more term of each series, we have

$$\begin{aligned} \frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!\gamma_{r-1}} + \frac{\alpha_r\beta_r}{r!\gamma_r} - \frac{\gamma-\alpha-\beta-1}{\gamma} \frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{(r-1)!(\gamma+1)_{r-1}} \\ = \frac{(\alpha+1)_{r-1}(\beta+1)_{r-1}}{r!\gamma_r} [r(\gamma+r-1) + \alpha\beta - r(\gamma-\alpha-\beta-1)] \\ = \frac{(\alpha+1)_r(\beta+1)_r}{r!\gamma_r}. \end{aligned}$$

Hence the induction holds good, and therefore

$$G(\alpha, \beta, \gamma, s) - \frac{\gamma-\alpha-\beta-1}{\gamma} G(\alpha+1, \beta+1, \gamma+1, s-1) = \frac{(\alpha+1)_s(\beta+1)_s}{s!\gamma_s}. \quad (\text{II.})$$

3. It follows from (I.) and (II.) that

$$\begin{aligned} (\gamma-\alpha-1)(\gamma-\beta-1) G(\alpha, \beta, \gamma, s) - (\gamma-1)(\gamma-\alpha-\beta-1) G(\alpha, \beta, \gamma-1, s) \\ = \frac{\alpha_{s+1}\beta_{s+1}}{s!\gamma_s}. \quad (\text{III.}) \end{aligned}$$

Putting aside for the present the special cases $\gamma=1$, $\gamma=\alpha+\beta+1$, it follows that

$$\begin{aligned} G(\alpha, \beta, \gamma-1, s) \\ = \frac{(\gamma-\alpha-1)(\gamma-\beta-1)}{(\gamma-1)(\gamma-\alpha-\beta-1)} G(\alpha, \beta, \gamma, s) - \frac{\alpha_{s+1}\beta_{s+1}}{(\gamma-\alpha-\beta-1)s!(\gamma-1)_{s+1}}. \end{aligned}$$

Therefore

$$\begin{aligned}
 & G(a, \beta, \gamma, s) \\
 &= \frac{(\gamma-a)(\gamma-\beta)}{\gamma(\gamma-a-\beta)} G(a, \beta, \gamma+1, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta)s! \gamma_{s+1}} \\
 & G(a, \beta, \gamma+1, s) \\
 &= \frac{(\gamma-a+1)(\gamma-\beta+1)}{(\gamma+1)(\gamma-a-\beta+1)} G(a, \beta, \gamma+2, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta+1)s! (\gamma+1)_{s+1}} \\
 & G(a, \beta, \gamma+2, s) \\
 &= \frac{(\gamma-a+2)(\gamma-\beta+2)}{(\gamma+2)(\gamma-a-\beta+2)} G(a, \beta, \gamma+3, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta+2)s! (\gamma+2)_{s+1}} \\
 & \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 & G(a, \beta, \gamma+t, s) \\
 &= \frac{(\gamma-a+t)(\gamma-\beta+t)}{(\gamma+t)(\gamma-a-\beta+t)} G(a, \beta, \gamma+t+1, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta+t)s! (\gamma+t)_{s+1}}
 \end{aligned}
 \tag{IV.}$$

From (IV.) it follows that

$$\begin{aligned}
 & G(a, \beta, \gamma, s) \\
 &= \frac{(\gamma-a)_{t+1}(\gamma-\beta)_{t+1}}{\gamma_{t+1}(\gamma-a-\beta)_{t+1}} G(a, \beta, \gamma+t+1, s) - \frac{a_{s+1}\beta_{s+1}}{(\gamma-a-\beta)s! \gamma_{s+1}} f(a, \beta, \gamma, s, t),
 \end{aligned}
 \tag{V.}$$

where $f(a, \beta, \gamma, s, t)$

$$\begin{aligned}
 &= 1 + \frac{(\gamma-a)(\gamma-\beta)}{(\gamma-a-\beta+1)(\gamma+s+1)} + \frac{(\gamma-a)_2(\gamma-\beta)_2}{(\gamma-a-\beta+1)_2(\gamma+s+1)_2} + \dots \\
 &+ \frac{(\gamma-a)_t(\gamma-\beta)_t}{(\gamma-a-\beta+1)_t(\gamma+s+1)_t}.
 \end{aligned}
 \tag{VI.}$$

The series $f(a, \beta, \gamma, s, t)$ contains $t+1$ terms, and is, when t is infinite, of the form

$$1 + \frac{ab}{cd} + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)} + \dots$$

The condition for the convergency of this series is

$$a+b-c-d+1 < 0.$$

Hence, if t be made infinite in $f(a, \beta, \gamma, s, t)$, the condition that $f(a, \beta, \gamma, s, \infty)$ shall be finite is

$$(\gamma-a) + (\gamma-\beta) - (\gamma-a-\beta+1) - (\gamma+s+1) + 1 < 0,$$

$$\text{i.e.,} \quad -(s+1) < 0,$$

which is always satisfied, since s is a positive integer.

Hence $f(a, \beta, \gamma, s, \infty)$ is always finite. If s tend to infinity, it tends to the limit unity. Thus

$$f(a, \beta, \gamma, \infty, \infty) = 1.$$

Further, when t tends to infinity $G(a, \beta, \gamma + t + 1, s)$ tends to unity. Then (V.) becomes

$$G(a, \beta, \gamma, s) = \lim_{t \rightarrow \infty} \frac{(\gamma - a)_{t+1} (\gamma - \beta)_{t+1}}{\gamma_{t+1} (\gamma - a - \beta)_{t+1}} - \frac{a_{s+1} \beta_{s+1}}{(\gamma - a - \beta) s! \gamma_{s+1}} f(a, \beta, \gamma, s, \infty).$$

Adopting Gauss's notation,

$$\Pi(n, z) = \frac{n! n^z}{(z+1)(z+2) \dots (z+n)} = \frac{n^z \Gamma(n+1) \Gamma(z+1)}{\Gamma(n+z+1)}.$$

$$\frac{(\gamma - a)_{t+1} (\gamma - \beta)_{t+1}}{\gamma_{t+1} (\gamma - a - \beta)_{t+1}} = \frac{\Pi(t+1, \gamma-1) \Pi(t+1, \gamma-a-\beta-1)}{\Pi(t+1, \gamma-a-1) \Pi(t+1, \gamma-\beta-1)};$$

therefore $\lim_{t \rightarrow \infty} \frac{(\gamma - a)_{t+1} (\gamma - \beta)_{t+1}}{\gamma_{t+1} (\gamma - a - \beta)_{t+1}} = \frac{\Pi(\gamma-1) \Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1) \Pi(\gamma-\beta-1)}.$

Also $\frac{a_{s+1} \beta_{s+1}}{s! \gamma_{s+1}} = \frac{\Pi(s+1, \gamma-1)}{\Pi(s+1, a-1) \Pi(s+1, \beta-1)} (s+1)^{a+\beta-\gamma}.$

Hence (V.) gives on making t infinite

$$G(a, \beta, \gamma, s) = \frac{\Pi(\gamma-1) \Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1) \Pi(\gamma-\beta-1)} - \frac{\Pi(s+1, \gamma-1) (s+1)^{a+\beta-\gamma} f(a, \beta, \gamma, s, \infty)}{\Pi(s+1, a-1) \Pi(s+1, \beta-1) (\gamma-a-\beta)};$$

or, writing the symbols at length,

$$1 + \frac{a\beta}{\gamma} + \frac{a(a+1)\beta(\beta+1)}{2! \gamma(\gamma+1)} + \dots + \frac{a(a+1) \dots (a+s-1) \beta(\beta+1) \dots (\beta+s-1)}{s! \gamma(\gamma+1) \dots (\gamma+s-1)} \\ = \frac{\Pi(\gamma-1) \Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1) \Pi(\gamma-\beta-1)} - \frac{\Pi(s+1, \gamma-1) (s+1)^{a+\beta-\gamma} f(a, \beta, \gamma, s, \infty)}{(\gamma-a-\beta) \Pi(s+1, a-1) \Pi(s+1, \beta-1)}$$

where $f(a, \beta, \gamma, s, \infty)$ (VII.)

$$= \left(1 + \frac{(\gamma-a)(\gamma-\beta)}{(\gamma-a-\beta+1)(\gamma+s+1)} \right. \\ \left. + \frac{(\gamma-a)(\gamma-a+1)(\gamma-\beta)(\gamma-\beta+1)}{(\gamma-a-\beta+1)(\gamma-a-\beta+2)(\gamma+s+1)(\gamma+s+2)} + \dots \text{ to } \infty \right).$$

4. From this may be inferred, if the real part of $\gamma-a-\beta$ be positive, that the series is convergent, and that it converges to the limit

$$\frac{\Pi(\gamma-1) \Pi(\gamma-a-\beta-1)}{\Pi(\gamma-a-1) \Pi(\gamma-\beta-1)},$$

which is Gauss's formula.

In this case also, if s be finite, the second term gives an approximate value for the error made by leaving out the rest of the terms of the series.

If, however, the real part of $\gamma - \alpha - \beta$ be negative, then the series is asymptotic with

$$\frac{\Pi(\gamma-1)(s+1)^{a+\beta-\gamma}}{(a+\beta-\gamma)\Pi(a-1)\Pi(\beta-1)}$$

when s is large, and of course $(s+1)$ can be replaced by s in this expression.

The formula fails when $\gamma - \alpha - \beta$ is zero or a negative integer. It fails also if γ is a negative integer, but in the latter case all the terms in the series are infinite from a certain term onwards. In a subsequent communication I hope to discuss these cases of failure.

5. The following special cases may be noticed by way of verification.

(a) If $\gamma = \alpha + \beta + 1$, it follows from equation (II.) or (III.) that

$$G(\alpha, \beta, \alpha + \beta + 1, s) = \frac{(\alpha+1) \cdot (\beta+1)_s}{s! (\alpha+\beta+1)_s}.$$

Similar, but less convenient, formulæ can be deduced from (III.) for the case $\gamma = \alpha + \beta + a$ a positive integer.

(b) If a be a negative integer ($-r$), then the series contains a finite number of terms. Taking now $s \geq r$, it follows that $a_{s+1} = 0$,

$$\frac{\Pi(\gamma - \alpha - 1)}{\Pi(\gamma - 1)} = (\gamma - \alpha - 1)(\gamma - \alpha - 2) \dots (\gamma),$$

$$\frac{\Pi(\gamma - \alpha - \beta - 1)}{\Pi(\gamma - \beta - 1)} = (\gamma - \alpha - \beta - 1)(\gamma - \alpha - \beta - 2) \dots (\gamma - \beta);$$

and therefore the sum of the series is

$$\frac{(\gamma - \alpha - \beta - 1)(\gamma - \alpha - \beta - 2) \dots (\gamma - \beta)}{(\gamma - \alpha - 1)(\gamma - \alpha - 2) \dots (\gamma)}$$

where a is a negative integer.

(c) If $\gamma - \alpha$ be zero or a negative integer ($-r$), where r is a positive integer,

$$\Pi(\gamma - \alpha - 1) = \infty, \quad (\gamma - \alpha)_{r+1} = 0,$$

and so the series $f(\alpha, \beta, \gamma, s, \infty)$ contains only $(r+1)$ terms.

If $\gamma = \alpha$, the result is

$$1 + \frac{\beta}{1} + \frac{\beta(\beta+1)}{2!} + \dots + \frac{\beta(\beta+1) \dots (\beta+s-1)}{s!} = \frac{\beta(\beta+1) \dots (\beta+s)}{s!},$$

which is, of course, well known.

[I am indebted to the referee for the following remark.

Taking the case in which the real part of $\gamma - \alpha - \beta$ is positive, the equation (VII.) may, by substituting in (VII.) for

$$\frac{\Pi(\gamma-1) \Pi(\gamma-\alpha-\beta-1)}{\Pi(\gamma-\alpha-1) \Pi(\gamma-\beta-1)}$$

the series

$$1 + \frac{\alpha\beta}{1!\gamma} + \frac{\alpha_2\beta_2}{2!\gamma_2} + \dots,$$

transferring this series to the other side of the equation, and then dividing out by the factor

$$-\frac{\Pi(s+1, \gamma-1) (s+1)^{\alpha+\beta-\gamma}}{\Pi(s+1, \alpha-1) \Pi(s+1, \beta-1)},$$

be transformed into the following:—

$$\begin{aligned} & \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} \frac{\beta+s+1}{\gamma+s+1} + \frac{(\alpha+s+1)_2 (\beta+s+1)_2}{(s+2)_2 (\gamma+s+1)_2} + \dots \right) \\ &= \frac{1}{\gamma-\alpha-\beta} \left(1 + \frac{\gamma-\alpha}{\gamma-\alpha-\beta+1} \frac{\gamma-\beta}{\gamma+s+1} + \frac{(\gamma-\alpha)_2 (\gamma-\beta)_2}{(\gamma-\alpha-\beta+1)_2 (\gamma+s+1)_2} + \dots \right). \end{aligned} \quad \text{(VIII.)}$$

Equation (VIII.) has been proved on the hypothesis that $s+1$ was a positive integer. In the particular case, however, when $\beta = \gamma$, the equation becomes

$$\frac{1}{\alpha} + \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} + \frac{(\alpha+s+1)_2}{(s+2)_2} + \dots \right) = 0, \quad \text{(IX.)}$$

and the condition that the real part of $\gamma - \alpha - \beta > 0$ becomes that the real part of α is negative.

Now
$$\frac{1}{\alpha} + \frac{1}{s+1} = \frac{\alpha+s+1}{\alpha(s+1)},$$

$$\frac{1}{\alpha} + \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} \right) = \frac{(\alpha+s+1)_2}{\alpha(s+1)_2}.$$

This suggests that

$$\frac{1}{\alpha} + \frac{1}{s+1} \left(1 + \frac{\alpha+s+1}{s+2} + \dots + \frac{(\alpha+s+1)_{n-1}}{(s+2)_{n-1}} \right) = \frac{(\alpha+s+1)_n}{\alpha(s+1)_n},$$

which can be immediately verified by induction. But

$$\frac{(\alpha+s+1)_n}{\alpha(s+1)_n} = \frac{1}{\alpha} \frac{\Pi(n, s)}{\Pi(n, \alpha+s)} n^{\alpha}.$$

Hence the left-hand side of equation (IX.) is

$$\lim_{n=\infty} \frac{1}{a} \frac{\Pi(n, s)}{\Pi(n, a+s)} n^a,$$

and this vanishes, since the real part of a is negative.

Hence equation (IX.) is true independently of the integral character of $(s+1)$, and the referee suggests that equation (VIII.) may also be true independently of the integral character of $(s+1)$.

I hope to be able to include a discussion of this point in the communication promised above in regard to the cases of failure of equation (VII.).—*March 18th, 1907.*]

THE SINGULAR POINTS OF CERTAIN CLASSES OF FUNCTIONS OF SEVERAL VARIABLES

By G. H. HARDY.

[Received and Read January 10th, 1907.—Revised April, 1907.]

1. A great deal has been written about the singular points of particular classes of functions defined by Taylor's series in a single variable. Of the very numerous interesting results which have been obtained a large proportion have come from a development of an idea originally suggested by Hadamard. Hadamard first pointed out how, if we know the singular points of the function

$$V(x) = a_0 + a_1x + a_2x^2 + \dots,$$

we can discuss those of the function

$$f(x) = \int_0^1 \phi(t) V(tx) dt.$$

His original enunciation of the results thus obtained perhaps lacked something in precision, but all that was lacking has been amply supplied by later writers, and notably by Le Roy.* The result of the work in this and other directions of Hadamard and other writers has been that the theory of the singular points of Taylor's series in one variable may be said to be tolerably complete. Such general results as are likely to be proved by the methods at present at our disposal have been proved: and (what is almost equally important) large classes of particular functions have been discussed in detail; so that we are well supplied with interesting examples of all the theoretical possibilities, and can hope actually to determine, without serious difficulty, the nature of any special function which presents special points of interest.

The corresponding theory for functions of several variables is in a very different state. There are, of course, a number of general results which have been proved for functions of one variable and which can obviously be extended to those of several variables, and writers on the simpler theory have generally been content to point these out. Of tangible results which help us in the actual discussion of particular functions there are practically none.

* Some discussion of these results and some indications of further extensions will be found in two previous papers in these *Proceedings* (Ser. 2, Vol. 3, p. 381, and Vol. 5, p. 197).

Nothing is easier, on the other hand, than to write down any number of interesting special series which seem to invite discussion. Such series

$$\begin{aligned} \text{as } \sum \frac{x^\mu y^\nu}{(a + \mu\omega + \nu\omega')^a}, & \quad \sum \frac{\Gamma(a + \mu\omega + \nu\omega')}{\Gamma(a + b + \mu\omega + \nu\omega')} x^\mu y^\nu, \\ \sum \frac{\mu + \nu!}{\mu! \nu!} \frac{x^\mu y^\nu}{(a + \mu\omega + \nu\omega')^a}, & \quad \sum \frac{\mu + \nu!}{\mu! \nu!} \frac{\Gamma(a + \mu\omega + \nu\omega')}{\Gamma(a + b + \mu\omega + \nu\omega')} x^\mu y^\nu, \\ \sum \frac{\mu! \nu!}{\mu + \nu!} \frac{x^\mu y^\nu}{(a + \mu\omega + \nu\omega')^a}, & \quad \sum \frac{\mu! \nu!}{\mu + \nu!} \frac{\Gamma(a + \mu\omega + \nu\omega')}{\Gamma(a + b + \mu\omega + \nu\omega')} x^\mu y^\nu \end{aligned}$$

at once suggest themselves, with more general classes of functions of similar types.

2. The methods which I have employed in my former papers lend themselves naturally enough to the discussion of a large variety of classes of series in any number of variables of which those written above are, as particular examples, fairly typical. This I have indicated briefly in one of the papers referred to.*

I shall now consider the question in greater detail, beginning with some generalities.

3. Let us suppose that $V(x_1, x_2, \dots, x_n)$

is any function of $x_1, \dots, x_n$ that can be expanded in a Taylor's series

$$\sum c_{\mu_1, \dots, \mu_n} x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n}$$

convergent for $|x_1| < r_1, \dots, |x_n| < r_n$.

The associated radii $r_1, r_2, \dots, r_n$ are in general connected by a single functional relation, which may be obtained (theoretically, at any rate) by a method devised by Lemaire.†

Now suppose that $\omega_1, \omega_2, \dots, \omega_n$ are quantities whose real parts are positive, and that

$$a_{\mu_1, \mu_2, \dots, \mu_n}$$

is an analytic function of certain parameters α, β, γ , capable, when the real parts of γ and $\alpha + \beta - 1$ are positive, of being expressed in the form

$$a_{\mu_1, \dots, \mu_n} = \int_0^1 \left(\log \frac{1}{u} \right)^{\alpha-1} (1-u)^{\beta-1} u^{\gamma-1+\Omega} \phi(u) du$$

* *Proc. London Math. Soc.*, Ser. 2, Vol. 3, pp. 387-9. I shall refer to this paper as I, and to the second paper as II.

† For a detailed discussion of the nature of the region of convergence of the power series, I may refer to two memoirs by F. Hartogs (*Inaugural Dissertation*, published by Teubner, 1904, and *Math. Annalen*, Bd. Lxii.).

where

$$\begin{aligned}\Omega &= \mu_1 \omega_1 + \dots + \mu_n \omega_n, \\ u^{\gamma-1} &= e^{(\gamma-1) \log u}, \quad (1-u)^{\beta-1} = e^{(\beta-1) \log(1-u)} \\ \left(\log \frac{1}{u}\right)^{\alpha-1} &= e^{(\alpha-1) \log \log 1/u},\end{aligned}$$

and

$$u^{\mu_\nu \omega_\nu} = e^{\mu_\nu \omega_\nu \log u},$$

the logarithms being real. Let us, as in I (§ 1), consider the integral

$$\int_C (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1+\Omega} \phi(u) du$$

where now $(u-1)^{\beta-1} = e^{(\beta-1) \log(u-1)}$, $(\log u)^{\alpha-1} = e^{(\alpha-1) \log \log u}$,

the logarithms being real when u is real and greater than 1 and rendered uniform by a cut from 1 to $-\infty$ along the real axis, and where C is a loop from 0 enclosing the line $(0, 1)$. Then, for sufficiently small values of $x_1, x_2, \dots, x_n$,

$$\begin{aligned}(1) \quad & F(x_1, x_2, \dots, x_n) \\ &= \sum a_{\mu_1, \dots, \mu_n} c_{\mu_1, \dots, \mu_n} x_1^{\mu_1} \dots x_n^{\mu_n} \\ &= \frac{1}{2i \sin(\alpha+\beta)\pi} \int_C (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) V(x_1 u^{\omega_1}, x_2 u^{\omega_2}, \dots, x_n u^{\omega_n}) du,\end{aligned}$$

provided $R(\gamma) > 0$, and $\alpha+\beta$ is not integral,* and C does not include any singular points of $\phi(u)$ or of

$$V(x_1 u^{\omega_1}, x_2 u^{\omega_2}, \dots, x_n u^{\omega_n}).$$

The condition concerning $\phi(u)$ can certainly be satisfied if, as we shall assume, $\phi(u)$ is regular in a domain which includes the line $(0, 1)$ in its interior. The last condition is certainly satisfied for sufficiently small values of $x_1, \dots, x_n$. In these circumstances the equation (1) provides a representation of $F(x, \dots, x_n)$ certainly valid for sufficiently small values of the variables.

4. This representation of $F(x_1, \dots, x_n)$ is, of course, often valid for a range of values of the variables far wider than that for which the power series converges. In order to discuss for what range of values the equation (1) holds, I shall begin by considering the simplest case, that in which

$$c_{\mu_1, \mu_2, \dots, \mu_n} = 1,$$

$$V(x_1, x_2, \dots, x_n) = 1/(1-x_1)(1-x_2) \dots (1-x_n),$$

$$(2) \quad F(x_1, x_2, \dots, x_n) = \frac{1}{2i \sin(\alpha+\beta)\pi} \int_C \frac{(\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) du}{(1-x_1 u^{\omega_1})(1-x_2 u^{\omega_2}) \dots (1-x_n u^{\omega_n})},$$

* If $\alpha+\beta$ is an integer k , the formula may be replaced by

$$F(x_1, \dots, x_n) = \frac{(-1)^k}{2\pi i} \int (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \log(u-1) \phi(u) V(x_1 u^{\omega_1}, \dots, x_n u^{\omega_n}) du.$$

The loop C may be taken so as to enclose the line $(0, 1)$ as closely as we please. We must therefore consider first for what values of the variables (x) the part of the subject of integration which depends on (x) will have a singular point on the line $(0, 1)$.

Let us suppose therefore

$$x_1 u_1^{\omega_1} = 1 \quad (0 < u_1 \leq 1)$$

or

$$1/x_1 = e^{\omega_1 \log u_1}.$$

As u_1 varies from 0 to 1 the value of x_1 given by this equation varies from ∞ to 1. Its path is a certain equiangular spiral S_1 which, in the particular case in which ω_1 is real, reduces to the straight line $(1, +\infty)$. Similarly, we obtain spirals $S_2, \dots, S_n$ in the planes of $x_2, \dots, x_n$. By drawing barriers along these spirals we define a certain domain T for the variables (x) .

Let T'_1 be any finite domain in the plane of x_1 , all of whose points lie at a distance from S_1 greater than some arbitrarily small fixed positive quantity δ_1 . Similarly we define $T'_2, \dots, T'_n$. Let T' be the domain formed by the composition of $T'_1, T'_2, \dots, T'_n$.

Then $F(x_1, x_2, \dots, x_n)$ is regular in T' . For let $x_1^0, x_2^0, \dots, x_n^0$ be a system of values inside the domain of convergence of the original power series; and $x'_1, x'_2, \dots, x'_n$ any other system inside T' . We can suppose that (x) vary from (x^0) to (x') along a "path" which lies inside T' . If we suppose the loop C taken initially so closely surrounding $(0, 1)$ that no root of

$$x_\nu u^\nu = 1 \quad (\nu = 1, 2, \dots, n)$$

falls inside or on C for any set of values of (x) in T' , it is plain that the integral (2) gives the analytical continuation of $F(x_1, x_2, \dots, x_n)$ over a region which includes the path from (x^0) to (x') , and so over the whole region T' .

We may say shortly that $F(x_1, \dots, x_n)$ is regular *within* T' .

It is equally easy to define different, though similar, domains within which $F(x_1, \dots, x_n)$ is regular. We have only to take as fundamental a form of C enclosing, not the line $(0, 1)$, but some other path (such as an arc of a circle) from 0 to 1 in the u -plane. We then obtain a modified domain T' within which $F(x_1, \dots, x_n)$ is regular, bounded by a cut from 1 to ∞ in the plane of every x along a certain curve. To give a simple example, let us suppose every $\omega = 1$. Then it is easy to see that, if we take C to always include a fixed arc of a circle from 0 to 1, the cut in every x -plane must be along a straight line from 1 to ∞ , *these lines making in each plane the same angle with the real axis.*

For example, the function

$$(x-y)^{-1} \log \{(1-x)/(1-y)\},$$

which, as we shall see, falls under the class of functions here considered, is regular in any domain T formed by cutting the planes of x and y along straight lines from 1 to ∞ *making the same angle with the real axis*.

The last condition is essential; otherwise we could find a pair of values x, y lying inside T , and for which

$$x = y, \quad \log \{(1-x)/(1-y)\} = \pm 2\pi i,$$

and the function would no longer be regular inside T .

Let us return to the function $F(x_1, \dots, x_n)$. It will in general be many-valued, and the system of branches of F will be far more complex than that of the branches of a many-valued function of a single variable. The branch which is regular in T , *i.e.*, the branch represented near $(0, 0, \dots, 0)$ by the original power series, we call the *principal branch* and denote by $\bar{F}$.

We can now form some general conclusions with regard to the singularities of $\bar{F}$, or rather the *possible* singularities, since our method does not at present enable us actually to assert that any system of values does, in point of fact, correspond to a singularity.

(1) We can in no case form a domain within which $\bar{F}$ is regular and which includes any system of values for which any $x_\nu = 1$ or ∞ . Hence

$$x_1, x_2, \dots, x_\nu = 1, \dots, x_n,$$

$$x_1, x_2, \dots, x_\nu = \infty, \dots, x_n$$

correspond to possible singularities of the principal branch of F . In other words, *possible singularities are given by $x_\nu = 1, \infty$ for any values of the other variables*.

(2) No value of x_ν other than 1, ∞ can give a singularity of $\bar{F}$ for all possible values of the other variables. For, if we take any set of values of (x) of which none is 1 or ∞ , we can define a domain which includes the domain of convergence of the original power series, and this point, and throughout which our contour integral gives the analytical continuation of $\bar{F}$.

(3) Let us consider the domain T bounded by a definite set of barriers S_ν . It is possible that when x_ν tends to a certain value ξ_ν on the barrier S_ν (other than 1, ∞) $\bar{F}$ may tend to a singularity when the other variables have or tend to some particular values or system of values in some particular way. This is the case, for example, with the function

$$(x-y)^{-1} \log \{(1-x)/(1-y)\}$$

when x and y tend simultaneously and *from opposite sides* to equal values on the barriers in their respective planes.

Let us suppose, *e.g.*, that x_1 tends to a value ξ_1 upon S_1 . There is one, and only one, value u_1 ($0 < u_1 < 1$) such that

$$1 - \xi_1 u_1^{\omega_1} = 0,$$

and it is obvious that as x_1 approaches ξ_1 our representation of $\bar{F}$ ceases to be valid.

When x_1 is nearly, but not quite, equal to ξ_1 the subject of integration has a singularity $\bar{u}_1$ (see the figure, Fig. *a*) lying a little off $(0, 1)$. We can deform C slightly (Fig. *bc*) so as to leave $\bar{u}_1$, u_1 on one side of it. And the modified integral gives us the continuation of $\bar{F}$ over a region slightly passing the limits of T , in that its constituent part T_1 contains a small region *which includes* ξ_1 .

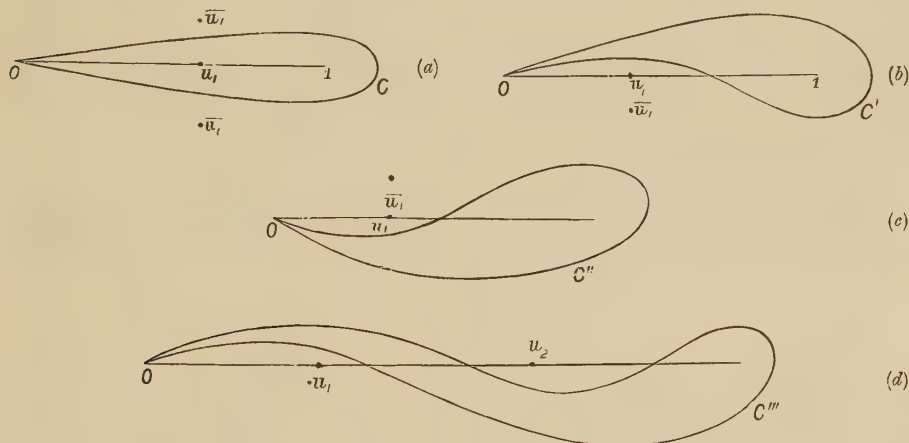


FIG. 1.

It follows that, if x_1 tends to ξ_1 , the other variables remaining off the barriers in their planes, $\bar{F}$ does not tend to a singularity. The same conclusion *generally* holds even if some or all of the other variables tend at the same time to points on their barriers, as appears from a reference to Fig. 1, (*d*), where is shown the modification necessary in C to meet the case in which two variables x_1, x_2 tend to values ξ_1, ξ_2 such that

$$1 - \xi_1 u_1^{\omega_1} = 0, \quad 1 - \xi_2 u_2^{\omega_2} = 0.$$

But there is an exception: If $u_1 = u_2$, the suggested modification is impossible; in other words, if x_1, x_2 vary in such a way that the contour C is *nipped* between two singularities of the subject of integration. In fact, the existence of singularities of $\bar{F}$ is indicated either (1) by the

approach of a singularity of the subject of integration to an end of the contour or (2) by the nipping of the contour as described above.*

5. We are thus led to discriminate two kinds of singularities of F from among those which appear as singularities of $\bar{F}$, viz., the singularities

$$x_\nu = 1, \infty$$

which we shall describe as *primary*, and the singularities given by

$$1 - \xi_\mu u_\mu^{\omega\mu} = 0, \quad 1 - \xi_\nu u_\nu^{\omega\nu} = 0$$

where μ, ν are any two of $1, 2, \dots, n$. These we shall describe as *secondary*. The distinction between these two classes is not precisely the same as that which I made in my earlier paper (I) between *principal* and *subsidiary* singularities in the case of functions of a single variable, since the secondary singularities here considered do, in fact, appear as singular when the variables (x) move in a prescribed manner towards special places on the boundary of T . Still, *qua* singularities of $\bar{F}$, they depend on the region adopted as fundamental in the definition of $\bar{F}$, and, from the point of view of the Taylor's series, there is a genuine distinction to be drawn between them and the primary singularities.

When we come to consider the function F as a whole these secondary singularities (as might be expected) also appear as a connected whole, viz., as the systems of values given by

$$x_\mu^{\omega\nu} = x_\nu^{\omega\mu} \quad (\mu, \nu = 1, 2, \dots, n).$$

Besides these two classes of singularities there remains the possibility of a third, viz., a class of singularities of other branches of F which do not appear at all as singularities of $\bar{F}$. Thus we shall find that often $x_\nu = 0$ defines a singularity of all branches of F other than the principal branch, just as $x = 0$ is a singularity of all branches of

$$\frac{x}{1^a} + \frac{x^2}{2^a} + \frac{x^3}{3^a} + \dots$$

other than the principal branch. Such singularities may more appropriately be called *subsidiary*.

6. The preceding arguments are merely an adaptation of those used by Hadamard and Le Roy for functions of a single variable. They use *line* instead of *loop* integrals; but, so far as these generalities are

* For the analogue of (1) for functions of one variable, see I and II', *passim*. For that of (2), see II, § 6. The phenomenon of nipping occurs also in Hadamard's proof of the "multiplication theorem" for singularities or ordinary Taylor's series.

concerned, one form of integral has little advantage over the other.* It is in the further development of the theory, especially in the obtaining of more precise information as to the nature of the singularities, that the great advantage of the loop integral declares itself.

7. The arguments which we have used for a special form of $V(x_1, \dots, x_n)$ may obviously be extended, *mutatis mutandis*, to the general case. But, in general, the statement is somewhat less simple. Let us therefore consider the case perhaps next in simplicity.

Let

$$c_{\mu_1, \mu_2, \dots, \mu_n} = \frac{\mu_1 + \mu_2 + \dots + \mu_n!}{\mu_1! \mu_2! \dots \mu_n!}; \dagger$$

so that

$$V(x_1, x_2, \dots, x_n) = 1/(1-x_1-x_2-\dots-x_n)$$

$$\text{and } F(x_1, x_2, \dots, x_n) = \frac{1}{2i \sin(\alpha + \beta)\pi} \int_C \frac{(\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) du}{1-x_1 u^{\omega_1} - \dots - x_n u^{\omega_n}}.$$

Then the argument used above shows how we may define a branch of $F(x_1, \dots, x_n)$ regular within any domain which excludes all sets of values of (x) for which

$$1-x_1 u^{\omega_1} - \dots - x_n u^{\omega_n} = 0$$

for a real $u < 1$. We cannot, however, draw fixed cuts in the planes of (x) in such a way as to secure this. Hence there is a difficulty in giving a precise definition of a branch of F analogous to $\bar{F}$ above.

Let $x_1^0, x_2^0, \dots, x_n^0$ be any system of values of the variables such that

$$(1) \quad |1-x_1^0 u^{\omega_1} - \dots - x_n^0 u^{\omega_n}| \geq \delta > 0$$

for $0 \leq u \leq 1$. And let $x_\nu = x_\nu^0 \lambda^{\omega_\nu}$

where $0 \leq \lambda \leq 1$ and λ^{ω_ν} , like u^{ω_ν} , has its principal value. As λ varies from 0 to 1 each x_ν varies from 0 to x_ν^0 along a certain path C_ν . The aggregate of corresponding points of every C_ν we shall call the path C of the variables (x) from (0) to (x^0) .

For all points of the path C

$$|1-x_1 u^{\omega_1} - \dots - x_n u^{\omega_n}| = |1-x_1^0 (\lambda u)^{\omega_1} - \dots - x_n^0 (\lambda u)^{\omega_n}| \geq \delta.$$

Let $P_1, \dots, P_n$ be a set of corresponding points on the paths C_ν . Round each P_ν describe a small circle of radius ϵ . The domain of the x 's formed by all systems of values of which each corresponds to a point

* Using the loop enables us to avoid the annoying and irrelevant condition $R(\alpha + \beta) > 1$. When $\alpha = 1$ the restriction on γ could be removed by the use of a double-circuit integral.

† The region of convergence of the original power series is in this case defined by

$$|x_1| + |x_2| + \dots + |x_n| < 1.$$

within a circle C_v we denote by $C_\epsilon(\lambda)$. As λ varies from 0 to 1, $C_\epsilon(\lambda)$ varies from a small region enclosing the origin to a small region enclosing $(x_1^0, \dots, x_n^0)$. At any point in $C_\epsilon(\lambda)$

$$x_v = x_v^0 \lambda^{\omega_v} + \xi_v$$

where $|\xi_v| < \epsilon$, and so

$$|\xi_1 u^{\omega_1} + \dots + \xi_n u^{\omega_n}| < |\xi_1| + \dots + |\xi_n| < n\epsilon.$$

Choose ϵ so that

$$n\epsilon < \frac{1}{2}\delta.$$

Then, for every point in $C_\epsilon(\lambda)$,

$$|1 - \Sigma x_v u^{\omega_v}| = |1 - \Sigma x_v^0 (u\lambda)^{\omega_v} - \Sigma \xi_v u^{\omega_v}| > \delta - \frac{1}{2}\delta = \frac{1}{2}\delta,$$

for $0 \leq u \leq 1$.

It follows that, as λ varies from 0 to 1, $C_\epsilon(\lambda)$ defines a series of domains through each of which in succession we may analytically continue F until we have passed from the neighbourhood of the origin to the neighbourhood of $x_1^0, x_2^0, \dots, x_n^0$. Hence *this system of values does not correspond to a primary singularity of F .*

We have thus excluded from the possible primary singularities of F all systems of values which satisfy the condition (1).

8. Now let us give $x_1, x_2, \dots, x_{n-1}$ fixed values $x_1^0, x_2^0, \dots, x_{n-1}^0$. The values of x_n for which

$$1 - x_1^0 u^{\omega_1} - \dots - x_{n-1}^0 u^{\omega_{n-1}} - x_n u^{\omega_n} = 0$$

for some value of u between 0 and 1 form a continuous curve stretching from $1 - x_1^0 - \dots - x_{n-1}^0$ to infinity. The values of x_n for which

$$|1 - x_1^0 u^{\omega_1} - \dots - x_{n-1}^0 u^{\omega_{n-1}} - x_n u^{\omega_n}| \leq \delta$$

form a continuous domain, which includes this curve. By choosing δ sufficiently small, we can exclude from this domain any point in the finite part of the x_n -plane which does not actually lie upon the curve.

Hence the possible primary singularities of F can only be sought among the systems given by

$$x_1^0, x_2^0, \dots, x_{n-1}^0, x_n = (1 - x_1^0 u^{\omega_1} - \dots - x_{n-1}^0 u^{\omega_{n-1}}) / u^{\omega_n},$$

for $0 \leq u \leq 1$.

But, as before, we are at liberty to vary our fundamental u -path from 0 to 1. And the only systems which arise, *however* we choose this path, are evidently

$$x_1^0, x_2^0, \dots, x_{n-1}^0, 1 - x_1^0 - x_2^0 - \dots - x_{n-1}^0; \quad x_1^0, x_2^0, \dots, x_{n-1}^0, \infty.$$

Hence the primary singularities of $F(x_1, x_2, \dots, x_n)$ are given by

$$x_1 + x_2 + \dots + x_n = 1$$

or

$$x_v = \infty.$$

As in the previous case, these are the values for which a pole of

$$1/1 - x_1 u^{\omega_1} - \dots - x_n u^{\omega_n}$$

approaches an end of C .

Secondary or subsidiary singularities will clearly be given by those values of (x) for which two roots of $1 - x_1 u^{\omega_1} - \dots - x_n u^{\omega_n} = 0$ become equal. In particular, when $\omega_1, \omega_2, \dots$ are real and rational (in which case we may, without loss of generality, suppose them integral), they will be given by the discriminant of this equation.

9. The arguments of the preceding sections may evidently be applied whatever be the function $V(x_1, x_2, \dots, x_n)$. The cases of most interest to us at present are those in which V is a rational function P/Q regular at the origin. The primary singularities of the associated functions F are in this case given by

$$Q(x_1, x_2, \dots, x_n) = 0.$$

If V is a many-valued function such as

$$1/(1-x_1)^{\delta_1}(1-x_2)^{\delta_2} \dots (1-x_n)^{\delta_n}, \quad 1/(1-x_1-x_2-\dots-x_n)^{\delta},$$

some additional complications are introduced, but the general principle of the method is none the less applicable.

10. We are thus able to define, by means of a contour integral, a family of functions associated with each given function V . Each of these functions is generally many-valued, but possesses a branch whose properties bear a considerable resemblance to those of V .

The other branches of F will, however (as can be seen clearly enough from the case of one variable), have properties differing widely from those of V . In order to obtain a more accurate idea of the nature of these branches, as well as to obtain more precise information as to the nature of the primary singularities of F , further analysis is necessary. It is natural to attempt to use the method which I adopted in my first paper. That method (in its simplest form) was based on a change from the contour C to another contour C' , between which and C lay a pole of the subject of integration. It is therefore best adapted to the case in which V is rational, and I shall, for the present, confine myself to that case.

10. Let, then, $V = P/Q$ be rational. And let

$$x_1^0, x_2^0, \dots, x_n^0$$

be a set of values of the variables for which $Q(x_1, x_2, \dots, x_n) = 0$, and let

$$x_\nu = x_\nu^0 + \xi_\nu$$

where ξ_ν is small.

By carrying out the process of continuation described in § 7, we arrive at an equation

$$F(x_1, x_2, \dots, x_n) = \frac{1}{2i \sin(\alpha + \beta)\pi} \int_C (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) \frac{P(x_1 u^{\omega_1}, \dots)}{Q(x_1 u^{\omega_1}, \dots)} du.$$

In general, there will be just one pole of P/Q (considered as a function of u) which tends to $u = 1$ as $\xi_1, \dots, \xi_\nu$ tend to zero.

For, let $u = 1+t$, and consider the equation

$$Q\{(x_\nu^0 + \xi_\nu)(1+t)^{\omega_\nu}\} = 0.$$

In general, this can be expanded in the form

$$0 = a_1 \xi_1 + a_2 \xi_2 + \dots + a_n \xi_n + \lambda t + \text{higher powers of } \xi_\nu, t \quad (\lambda \neq 0);$$

and it follows from the general implicit function theorem that this equation has just one solution of the type

$$t = b_1 \xi_1 + \dots + b_n \xi_n + \text{higher powers of } \xi_\nu.$$

But, of course, in special cases there may be several solutions of different forms. If, e.g.,

$$Q = (1-x_1)(1-x_2) \dots (1-x_n),$$

each of the equations

$$1 - x_n u^{\omega_n} = 0$$

or

$$1 - (1 + \xi_\nu)(1 + \omega_\nu t + \dots) = 0$$

has one root which vanishes with the ξ 's.

Considering the general case, let

$$u = u^0(x_1, \dots, x_n)$$

be the pole in question. Then

$$\int_C = \int_{C'} + 2\pi i R$$

where R is the residue for $u = u^0$. The second integral (cf. I, § 1) is regular near $x_1^0, \dots, x_n^0$. We can thus isolate the irregular part of F . In order to throw light on the results thus obtained (as well as for the sake of the interest of the actual results), I shall consider some particular cases in detail.

11. The simplest case is that in which

$$V(x_1, \dots, x_n) = 1/(1-x_1)(1-x_2) \dots (1-x_n).$$

In this case there are n poles between C and C' , viz.,

$$u = x_\nu^{-1/\omega_\nu} \quad (\nu = 1, 2, \dots, n).$$

The residue at this pole is

$$-(\omega_\nu)^{-\alpha} \frac{\left\{ \log \left(\frac{1}{x_\nu} \right) \right\}^{\alpha-1} (1-x_\nu^{1/\omega_\nu})^{\beta-1} x_\nu^{(1-\beta-\gamma)/\omega_\nu} \phi(x_\nu^{-1/\omega_\nu})}{\prod'_{(\mu)} (1-x_\mu x_\nu^{-\omega_\mu/\omega_\nu})}$$

where the dash denotes that the product applies to values of μ different from ν , and the values of the many-valued functions have been already specified. Let this quantity be denoted by Ω_ν . Then

$$F(x_1, x_2, \dots, x_n) = \frac{\pi}{\sin(\alpha+\beta)\pi} \Sigma \Omega_\nu + \frac{1}{2i \sin(\alpha+\beta)\pi} \int_{C'} ,$$

We thus isolate what we may call the part of $F(x_1, \dots, x_n)$ irregular near $(1, 1, \dots, 1)$. The form of Ω_ν exhibits explicitly the singularities of F other than the primary singularities. It will be noticed that besides the *secondary* singularities already noted, which correspond to the factors

$$1-x_\mu x_\nu^{-\omega_\mu/\omega_\nu},$$

there are *fixed subsidiary* singularities given, e.g., by $x_\nu = 0$.

12. Let us specialise further by supposing $n = 2$ and write x, y, ω, ω' for $x_1, x_2, \omega_1, \omega_2$ to avoid suffixes. There are two particularly interesting special cases :

(i.) Let $\phi(u) \equiv 1$, $\alpha = 1$, so that

$$F(x, y) = \Gamma(\beta) \Sigma \frac{\Gamma(\gamma + \mu\omega + \nu\omega')}{\Gamma(\beta + \gamma + \mu\omega + \nu\omega')} x^\mu y^\nu.$$

The primary singularities are given by $x = 1$, $y = 1$; the function

$$F(x, y) - \frac{\pi}{\sin \beta\pi} \left\{ \frac{(1-x^{1/\omega})^{\beta-1} x^{(1-\beta-\gamma)/\omega}}{\omega(1-yx^{-\omega'/\omega})} + \frac{(1-y^{1/\omega'})^{\beta-1} y^{(1-\beta-\gamma)/\omega'}}{\omega'(1-xy^{-\omega/\omega'})} \right\}$$

is regular near $(1, 1)$; secondary or subsidiary singularities are given by

$$x = 0, \quad y = 0, \quad 1-x^{1/\omega} = 0, \quad 1-y^{1/\omega'} = 0, \quad 1-yx^{-\omega'/\omega} = 0, \quad 1-xy^{-\omega/\omega'} = 0.$$

Thus, e.g., $x = e^{2\pi i\lambda}$, where $\lambda = k+k'\omega$, and k and k' are any integers, gives a subsidiary singularity.

These results are easily verified by supposing $\omega = \omega' = 1$, when the irregular part reduces to

$$\frac{\pi}{\sin \beta\pi} \frac{1}{x-y} \left\{ (1-x)^{\beta-1} x^{2-\beta-\gamma} - (1-y)^{\gamma-1} y^{2-\beta-\gamma} \right\};$$

for a function of the form

$$F(x, y) = \Sigma \frac{x^\mu y^\nu}{\phi(\mu + \nu)} = \Sigma_{k=0}^{\infty} \frac{1}{\phi(k)} \Sigma_{(\mu+\nu=k)} x^\mu y^\nu$$

may be expressed in the form

$$\{xf(x) - yf(y)\}/(x-y),$$

where

$$f(x) = \sum x^k / \phi(k);$$

and so we can use the known results for functions of a single variable.

In calculating the form of Ω_v , it should be observed, we have assumed that the poles between C and C' are all distinct. If some of them coincide, the form of the residues corresponding to them becomes illusory, and must be calculated afresh, *ab initio* or by a passage to the limit.

A particularly interesting case is that in which $\beta = 1$. In this case the preceding formulæ fail. The proper formulæ can be easily constructed by (a) a passage to the limit, or (b) by an independent investigation, starting from the integral

$$\frac{1}{2\pi i} \int_C \frac{\log(u-1) u^{-1} du}{(1-xu)(1-yu^{\omega})}.$$

The result is that the irregular part of $\sum \sum \frac{x^{\mu} y^{\nu}}{\gamma + \mu\omega + \nu\omega'}$

is
$$-\frac{1}{\omega} \frac{x^{-\gamma/\omega} \log(x^{-1/\omega} - 1)}{1 - yx^{-\omega'/\omega}} - \frac{1}{\omega'} \frac{y^{-\gamma/\omega'} \log(y^{-1/\omega'} - 1)}{1 - xy^{-\omega/\omega'}}.$$

It is instructive to consider the special case in which $\omega = \omega' = 1$, $\gamma = 1$. We then obtain the function

$$\sum \sum \frac{x^{\mu} y^{\nu}}{1 + \mu + \nu} = \frac{1}{x-y} \log \left(\frac{1-y}{1-x} \right).$$

The *irregular part* turns out to be $\frac{1}{x-y} \log \left\{ \frac{x(1-y)}{y(1-x)} \right\}$,

while the term due to the contour integral round C' , which is regular near 1, 1, must therefore be

$$-\frac{\log x - \log y}{x-y}.$$

We thus obtain the formula

$$\frac{i}{2\pi} \int_{C'} \frac{\log(u-1) du}{(1-xu)(1-yu)} = \frac{\log x - \log y}{x-y},$$

the path of integration being a loop from 0 round 1, $1/x$, and $1/y$. This may be verified independently. The decomposition

$$\frac{1}{x-y} \log \left(\frac{1-y}{1-x} \right) = \frac{1}{x-y} \log \left\{ \frac{x(1-y)}{y(1-x)} \right\} - \frac{1}{x-y} \log \left(\frac{x}{y} \right)$$

gives a very simple and tangible illustration of the relations between the functions defined by the two contour integrals and the terms arising from the residues.

(ii.) An even more interesting special case is that in which $\alpha = 1$, $\phi(u) \equiv 1$. We find then the corresponding results for the function

$$\sum \frac{x^{\mu} y^{\nu}}{(\gamma + \mu\omega + \nu\omega')^{\alpha}}.$$

The irregular part of the principal branch is

$$\Gamma(1-\alpha) \left\{ \omega^{-\alpha} \frac{(\log 1/x)^{\alpha-1} x^{(1-\gamma)/\omega}}{1 - yx^{-\omega'/\omega}} + \omega'^{-\alpha} \frac{(\log 1/y)^{\alpha-1} y^{(1-\gamma)/\omega'}}{1 - xy^{-\omega/\omega'}} \right\}.$$

13. The formulæ which we have obtained can of course be applied to find *asymptotic* formulæ of a simpler kind valid near $x = 1$, $y = 1$. Let us suppose (to take the simplest and most interesting case) that ω , ω' , and α are real and positive and $\alpha < 1$, and that x and y tend simultaneously to 1 by real quantities in such a way that the ratios

$$1-x : 1-y : \omega(1-y) - \omega'(1-x)$$

remain between certain positive constants H , K . Let $1-x = \xi$ and $1-y = \eta$. Then

$$\left(\log \frac{1}{x} \right)^{\alpha-1} = \left(\log \frac{1}{1-\xi} \right)^{\alpha-1} \sim \xi^{\alpha-1}, \quad \left(\log \frac{1}{y} \right)^{\alpha} \sim \eta^{\alpha-1},$$

$$1 - yx^{-\omega'/\omega} \sim (\omega\eta - \omega'\xi)/\omega, \quad 1 - xy^{-\omega/\omega'} \sim (\omega'\xi - \omega\eta)/\omega',$$

and
$$\sum \frac{x^\mu y^\nu}{(\gamma + \mu\omega + \nu\omega')^\alpha} \sim \frac{\Gamma(1-\alpha)}{\omega(1-y) - \omega'(1-x)} \left\{ \left(\frac{\omega}{1-x} \right)^{1-\alpha} - \left(\frac{\omega'}{1-y} \right)^{1-\alpha} \right\},$$

which is the analogue for two variables of Appell's well known formula

$$\sum \frac{x^\mu}{(\gamma + \mu)^\alpha} \sim \frac{\Gamma(1-\alpha)}{(1-x)^{1-\alpha}},$$

and may be proved independently by methods which do not involve complex variables.

14. Let us pass now to the function

$$(1) \quad F(x_1, x_2, \dots, x_n) = \frac{1}{2i \sin(\alpha + \beta) \pi} \int_C \frac{(\log u)^{\alpha-1} (u-1)^{\beta-1} u^{\gamma-1} \phi(u) du}{1-x_1 u^{\omega_1} - x_2 u^{\omega_2} - \dots - x_n u^{\omega_n}}.$$

We have already seen that, if

$$x_1 + x_2 + \dots + x_n \neq 1,$$

we can continue F from (0) to (x) , and that in the neighbourhood of (x) the branch $\bar{F}$ which we are considering will be given by an equation of the above form.

If now we suppose that $\sum x_\nu$ is nearly, though not quite, equal to 1, we can apply the argument of §§ 9, 10. This is a case in which there is only *one* pole between C and C' .

In order to obtain tangible results let us again take $n = 2$ and use the notation of § 12, and consider some cases in which the part of $\bar{F}$ irregular near $x_0, y_0 = 1 - x_0$ can be actually calculated in finite terms, which requires that we should be able to solve the equation

$$1 - xu^\omega - yu^{\omega'} = 0$$

explicitly. The simplest cases are given by (i.) $\omega = \omega'$, (ii.) ω (or ω') = 0, (iii.) $\omega = 1, \omega' = 2$.

If (i.) $\omega = \omega'$ or (ii.) ω or $\omega' = 0$, the results are of interest only for purposes of verification.

(i.) If $\omega = \omega'$, F is a function of $x + y$ only.

(ii.) If $\omega' = 0$ (in which case the expression of F as a contour integral requires a little reconsideration, as in obtaining it we supposed the real parts of ω and ω' positive), F is a function of the form

$$\sum \frac{\mu + \nu!}{\mu! \nu!} \frac{x^\mu y^\nu}{\phi(\mu)} = \frac{1}{1-y} \sum_{(\mu)} \frac{1}{\phi(\mu)} \left(\frac{x}{1-y} \right)^\mu.$$

(iii.) Let $\omega = 1, \omega' = 2$. Then $1 - xu - yu^2 = 0$ gives

$$u = u_0 = -\frac{1}{2y} \{x + \sqrt{(x^2 + 4y)}\},$$

the sign of the radical being such that u reduces to 1 when $y = 1 - x$. And the irregular part is

$$-\frac{1}{\sqrt{(x^2 + 4y)}} (\log u_0)^{\alpha-1} (u_0 - 1)^{\beta-1} u_0^{\gamma-1} \phi(u_0).$$

An example is given by

$$\sum \frac{\mu + \nu!}{\mu! \nu!} \frac{x^\mu y^\nu}{(\gamma + \mu + 2\nu)^\alpha},$$

of which the simplest case is

$$\sum \frac{\mu + \nu!}{\mu! \nu!} \frac{x^\mu y^\nu}{1 + \mu + 2\nu} = -\frac{1}{\sqrt{(x^2 + 4y)}} \log \left\{ \frac{-\sqrt{(x^2 + 4y)} + x + 2y}{-\sqrt{(x^2 + 4y)} - x - 2y} \cdot \frac{-\sqrt{(x^2 + 4y)} + x}{-\sqrt{(x^2 + 4y)} - x} \right\},$$

where $\sqrt{(x^2+4y)}$ is negative for small real values of x and y , and tends to $x-2$ as y approaches $1-x$.

15. In the general case we cannot solve the equation

$$1-xu^\omega-yu^{\omega'}=0$$

explicitly, but we can investigate the character of the root

$$u_0 = 1 + a(x-x_0) + b(y-y_0) + c(x-x_0)^2 + d(x-x_0)(y-y_0) + \dots$$

(where $x_0+y_0=1$) as a function of x_0 and y_0 , and so determine the general character of the irregular part of $\bar{F}$. Thus, if ω and ω' are *positive and rational*, u_0 is an *algebraic* function of x and y , and the research of the further singular points of $F(x, y)$, beyond its primary singularities, is reduced to that of finding those of a finite expression involving this algebraic function.

This last remark obviously applies to the general case in which $V(x, y)$ (here $1/1-x-y$) is any rational function $P(x, y)/Q(x, y)$: the root u_0 of

$$Q(xu^\omega, yu^{\omega'}) = 0$$

being an algebraical function of x and y .

To give a definite example, consider the function

$$\sum \frac{\mu+\nu!}{\mu! \nu!} \frac{x^\mu y^\nu}{(\gamma+m\mu+n\nu)^2}.$$

We are led to recognise as singular, besides the primary singularities $x+y=1$, $x=\infty$, $y=\infty$:

(a) Values of x and y which make a root of $1-xu^m-yu^n=0$ equal to 0, 1, or ∞ ; the only additional value thus given is $y=0$ (if $n>m$), $x=0$ (if $m>n$), or $x+y=0$ (if $m=n$).

(b) Values of x and y which make two roots equal; these are given by

$$D(1-xu^m-yu^n)=0$$

where D denotes the discriminant.

16. (i.) Whatever values ω and ω' may have, we can obtain a simple *asymptotic* formula for $F(x, y)$ valid near $x_0, 1-x_0$. For let

$$x = x_0 + \xi, \quad y = y_0 + \eta,$$

$$\delta = 1-x-y = -\xi-\eta,$$

and $u_0 = 1+t$. We find

$$t = \frac{\delta}{\omega x_0 + \omega' y_0} + P_2(\xi, \eta),$$

from which we can at once deduce the formula

$$F(x, y) \sim -\frac{\pi}{\sin(\alpha+\beta)\pi} \frac{\phi(1) \frac{\omega-\omega'}{\omega+\omega'}}{(\omega x_0 + \omega' y_0)^{\alpha+\beta-1}} \frac{1}{(1-x-y)^{2-\alpha-\beta}}.$$

If, in particular, $\phi \equiv 1$, $\beta = 1$, we find

$$\sum \frac{\mu+\nu!}{\mu! \nu!} \frac{x^\mu y^\nu}{(\gamma+\omega\mu+\omega'\nu)^a} \sim \frac{\Gamma(1-\alpha)}{(\omega x_0 + \omega' y_0)^a} \frac{1}{(1-x-y)^{1-\alpha}}.$$

Similarly we have

$$\sum \frac{\mu_1 + \mu_2 + \dots + \mu_n!}{\mu_1! \dots \mu_n!} \frac{x_1^{\mu_1} \dots x_n^{\mu_n}}{(\gamma + \omega_1 \mu_1 + \dots + \omega_n \mu_n)^a} \sim \frac{\Gamma(1-a)}{(\omega_1 x_1^{\omega_1} + \dots + \omega_n x_n^{\omega_n})^a} \frac{1}{(1-x_1-\dots-x_n)^{1-a}},$$

as $x_1, \dots, x_n$ approach values $x_1^0, \dots, x_n^0$ such that $\sum x_v^0 = 1$.

(ii.) It will be found instructive to verify all these conclusions on such functions as

$$\int_0^1 \frac{du}{1-xu^\alpha - yu^\beta - zu^\gamma}$$

where α, β, γ are any of 0, 1, 2.

17. The functions which we have considered in §§ 11-16 are all derived from the base series

$$1/(1-x)(1-y), \quad 1/(1-x-y).$$

Any other base series will give rise to a corresponding set of functions. The following are some of the simplest and most interesting forms:—

$$(i.) \quad V(x, y) = \sum F(-\mu, -\nu, 1, \lambda) x^\mu y^\nu = 1/\{(1-x)(1-y) - \lambda xy\};$$

$$(ii.) \quad V(x, y) = \sum \frac{\mu! \nu!}{\mu + \nu + 1!} x^\mu y^\nu = \frac{1}{x+y-xy} \log \left\{ \frac{1}{(1-x)(1-y)} \right\};$$

$$(iii.) \quad V(x, y) = \sum \frac{\mu! \nu!}{\mu + \nu!} x^\mu y^\nu \\ = \frac{1}{(1-x)(1-y)} - \frac{xy}{(x+y-xy)(1-x)(1-y)} \\ + \frac{xy}{(x+y-xy)^2} \log \left\{ \frac{1}{(1-x)(1-y)} \right\};$$

$$(iv.) \quad V(x, y) = \sum \frac{\Gamma(a+\mu) \Gamma(b+\nu)}{\Gamma(a+b+\mu+\nu)} x^\mu y^\nu \\ = \frac{1}{x+y-xy} \{x F_{a, b}(x) + y F_{b, a}(y)\},$$

where

$$F_{a, b}(x) = \Gamma(b) \sum \frac{\Gamma(a+\mu)}{\Gamma(a+b+\mu)} x^\mu.$$

In case (1) the finite primary singularities of all the derived functions are given by

$$(1-x)(1-y) - \lambda xy = 0;$$

in the other cases by $x = 1, \quad y = 1,$

the factor $x+y-xy = 0$ not yielding primary singularities, as is independently obvious.

Any number of other base series may be constructed without difficulty. Of course, when the base series itself does not behave like a rational function in the neighbourhood of its primary singularities, we cannot

investigate the precise nature of the corresponding singularities of the derived functions with the simplicity of §§ 10-16.

It is natural to attempt to construct series whose primary singularities are given by

$$1 - xy = 0.$$

If, however, we take $1/(1 - xy)$ as our base series, our method fails, all the derived series turning out to be mere functions of $1 - xy$.

It is, however, easy to prove that

$$\Sigma \theta(\mu - \nu) x^\mu y^\nu = \frac{1}{1 - xy} \left\{ \theta(0) + \sum_1^\infty \theta(k) x^k + \sum_1^\infty \theta(-k) y^k \right\},$$

and any series of this kind may be taken as a base series and will give results of the kind desired.

For example, let θ be an even function, such that

$$\Theta(x) = \Sigma \theta(k) x^k$$

is an integral function of x , and let

$$\Phi(x, y) = \theta(0) + \Theta(x) + \Theta(y);$$

$$\text{also let } F(x, y) = \frac{1}{2i \sin(\alpha + \beta) \pi} \int_C (\log u)^{\alpha-1} (u-1)^{\beta-1} u^{-1} \Phi(u) \frac{\Phi(xu^\omega, yu^{\omega'})}{1 - xy u^{\omega + \omega'}} du.$$

Then (i.) we can infer that the only primary singularities of F are given by

$$xy = 1,$$

and (ii.) we can write down the part of F irregular near (x_0, y_0) , where $x_0 y_0 = 1$. Thus, *e.g.*, if

$$\omega = \omega' = 1, \quad \beta = 1, \quad \phi \equiv 1,$$

we find that

$$\Sigma \frac{\theta(\mu - \nu)}{(\gamma + \mu + \nu)^\alpha} x^\mu y^\nu$$

is a function of this type, and that the irregular part is

$$2^{-\alpha} \Gamma(1 - \alpha) \left\{ \log \left(\frac{1}{xy} \right) \right\}^{\alpha-1} (xy)^{-\frac{1}{2}} \left\{ \theta(0) + \Theta \left(\sqrt{\frac{x}{y}} \right) + \Theta \left(\sqrt{\frac{y}{x}} \right) \right\}.$$

If Θ is not integral, we get other primary singularities: thus those of

$$\Sigma \frac{x^\mu y^\nu}{\{a^2 + (\mu - \nu)^2\}^\beta (\gamma + \mu\omega + \nu\omega')^\alpha}$$

are $x = 1$, $y = 1$, and $xy = 1$.

Similarly with

$$\Sigma \frac{\Gamma(a + \mu - \nu)}{\Gamma(a + b + \mu - \nu)} x^\mu y^\nu$$

(in this case θ is not an even function).

18. A ready method of constructing interesting base series is the following:—Consider the integral

$$\frac{1}{2\pi i} \int_C \frac{R(u) du}{\{1 - x S(u)\} \{1 - y T(u)\}},$$

where R, S, T are rational. Suppose that the contour C includes the poles $u = \lambda_1, \dots, \lambda_p$, and excludes the poles $u = \mu_1, \dots, \mu_q$. Then (generally under certain restrictions as to which poles lie inside C) the integral can be expanded as a power series $\Sigma a_{\mu, \nu} x^\mu y^\nu$, where

$$a_{\mu, \nu} = \frac{1}{2\pi i} \int R S^\mu T^\nu du,$$

which can often be calculated in a simple form. The singularities are easily assigned: the *primary* ones come from making

$$\lambda_i = \mu_j, \quad \lambda_i = \infty,$$

the rest from making two λ 's or two μ 's equal or a μ infinite.

We are thus able to construct base series whose singularities have a prescribed form. Thus the integral

$$\frac{1}{2\pi i} \int \frac{du}{(1-xu)(u-y)}$$

gives rise to $1/(1-xy)$.

Sometimes we can use integrals containing irrational functions. Thus the integrals

$$\frac{i}{2 \sin a\pi \Gamma(b)} \int \frac{(-u)^a (1-u)^{b-1}}{(1-xu)(u-y)} du, \quad \frac{1}{4 \sin a\pi \sin b\pi} \int \frac{(u-1)^{a-1} (-u)^{b-1}}{(1-xu)\{1-y(1-u)\}} du,$$

taken round appropriate contours, give rise to the functions

$$\sum \frac{\Gamma(a+\mu-\nu)}{\Gamma(a+b+\mu-\nu)} x^\mu y^\nu, \quad \sum \frac{\Gamma(a+\mu) \Gamma(b+\nu)}{\Gamma(a+b+\mu+\nu)} x^\mu y^\nu$$

considered above.

19. I conclude with three remarks of a general character:—

(1) Various generalisations of the procedure here adopted at once suggest themselves. It is sufficient to refer to §§ 4–7 of the paper II. My object has been to consider only a few of the very simplest cases with a view to making some sort of a beginning. The general problem of attempting to classify such simple types of power series in two or more variables as naturally suggest themselves, according to the nature of their singular points, seems to me an extremely attractive, but an extremely difficult, one.

(2) There is one particularly interesting type of series which does not seem to be amenable to analysis of so simple a character. I refer to the series obtained by replacing such a factor as $(\gamma + \omega\mu + \omega\nu)^{-a}$ —which has frequently occurred in this paper—by a factor such as

$$(a\mu^2 + 2h\mu\nu + b\nu^2 + 2g\mu + 2f\nu + c)^{-a}.$$

Even the simplest case, in which the factor is $(\mu^2 + \nu^2)^{-a}$, defies us, owing to the non-existence of any simple expression of this quantity as a definite integral of the type required. Even if $a = 1$, it only appears to lend itself to analysis of an awkward and unsymmetrical type.

In order to deal with such a series as

$$\sum \frac{x^\mu y^\nu}{(\mu^2 + \nu^2)^a}$$

it seems necessary to have recourse to entirely different methods. There is no difficulty in establishing *asymptotic* formulæ analogous to those of §§ 13 and 16 (i.). Thus, if

$$\lim x = 1, \quad \lim y = 1, \quad \lim (1-y)/(1-x) = \tan \xi \quad (x, y \text{ real and } < 1),$$

we find

$$\lim \{ (1-x)^2 + (1-y)^2 \}^s \Sigma \Sigma \frac{x^\mu y^\nu}{(\theta + a\mu^2 + 2\beta\mu\nu + \gamma\nu^2)^s}$$

$$= \Gamma \{ 2(1-s) \} \int_0^{\frac{1}{2}\pi} \frac{d\lambda}{(a \cos^2 \lambda + 2\beta \cos \lambda \sin \lambda + \gamma \sin^2 \lambda)^s \{ \cos^2 (\lambda - \zeta) \}^{1-s}},$$

and in particular

$$\lim [-\log \{ (1-x)^2 + (1-y)^2 \}] \Sigma \Sigma \frac{x^\mu y^\nu}{a^2 + \mu^2 + \nu^2} = \frac{1}{4}\pi.$$

These results I have proved by methods depending on real variables only; and I have not yet been completely successful in finding a method which furnishes, with regard to these functions, results analogous to those obtained in this paper. I have, however, made sufficient progress to shew, for example, that singularities of

$$\Sigma \frac{x^\mu y^\nu}{(\theta + a\mu^2 + 2\beta\mu\nu + \gamma\nu^2)^s}$$

are given by $x = 1$, $y = 1$, and $\gamma (\log x)^2 - 2\beta \log x \log y + a (\log y)^2 = 0$.

(3) It might be thought that a great deal might be learnt by means of Hadamard's "multiplication of singularities" theorem, which holds for functions of any number of variables. But (as Hadamard has himself pointed out) this is not the case. Thus

$$x = 1 \text{ (any } y), \quad y = 1 \text{ (any } x)$$

are singular points for

$$\Sigma \frac{x^\mu y^\nu}{\gamma + \mu + \nu}, \quad \Sigma \frac{x^\mu y^\nu}{\delta + \mu + \nu}.$$

Multiplying, we obtain $\text{any } x, \text{ any } y$

as possible singularities of

$$\Sigma \frac{x^\mu y^\nu}{(\gamma + \mu + \nu)(\delta + \mu + \nu)},$$

and our knowledge is in no way advanced.

ON THE ASYMPTOTIC APPROXIMATION TO INTEGRAL
FUNCTIONS OF ZERO ORDER

By J. E. LITTLEWOOD.

(Communicated by Rev. E. W. BARNES.)

[Received December 19th, 1906.—Read January 10th, 1907.—Revised April–May,* 1907.]

1. In spite of the considerable attention which the theory of integral functions of finite order has received for the last fifteen years, the theory of the functions of zero order has been somewhat neglected. Analysis which applies to the theory of finite order usually breaks down in the case of zero order; moreover, the greater part of the few general theorems which exist in the latter theory are particular cases of general theorems applicable to all integral functions.

In his first memoir† on integral functions, M. Hadamard obtained an upper limit for the modulus of the general integral function defined by a Taylor series,‡ and he has given in a subsequent memoir§ a new means of finding such an upper limit. The latter method gives an approximation T for $M(r)$ [the maximum modulus of $|F(z)|$ on the circle $|z| = r$], such that $[M(r)]^{1+\epsilon} > T > [M(r)]^{1-\epsilon}$, when r is sufficiently large, where ϵ is as small as we please. This method applies to the case of functions of zero order.

M. Le Roy|| has given a method by which we can obtain asymptotic expressions for certain functions defined by a Taylor series with real and positive coefficients, the variable also being supposed real and positive. Among these functions are certain functions of zero order, *e.g.*, the function

$$\sum_0^{\infty} e^{-n^p} z^n \quad (1 < p < 2).$$

The asymptotic expressions of M. Le Roy for these functions afford a

* The paper has been practically rewritten.

† Memoir crowned by the French Academy, *Jour. de Math. (Liouville)*, t. ix., 1893.

‡ Hadamard also obtained other general results which hold in the case of zero order.

§ shall, however, only refer to such results as are connected with the subject of this paper.

|| *Bull. de la Soc. Math.*, t. xxiv., p. 186.

|| *Bull. des Sciences Math.*, 2me sér., t. xxiv.

nearer approximation for $M(r)$ than those obtained by the method of M. Hadamard.

M. Lindelöf* has considered a similar class of functions and obtains similar results.

M. Maillet† has extended as follows the definition of *order* in the case of functions of zero order.

$$\text{Let} \quad e_{k+1}(m) = a_1 a_2 \dots a_{k+1}^m,$$

$$\text{where} \quad a_1 = a_2 = \dots = e.$$

$$\text{Let} \quad E(x, k, \rho) = \sum_{m=0}^{\infty} \frac{x^m}{[e_k(m)]^{m/\rho}},$$

and let $M(r)$ be the maximum modulus of a function $f(z)$ on the circle $|z| = r$.

Then, if $M(r) < E(r, k, \rho + \epsilon)$, for all values of r , and if

$$M(r) = E(r, k, \rho - \epsilon_1)$$

for values of r as great as we please, where ϵ and ϵ_1 tend to zero as r tends to infinity, then we say that $f(z)$ is of index k , and of order $(0, k, \rho)$.

He has established theorems concerning the relations between the modulus of the function and the coefficients of the Taylor series, when these coefficients approximate to the form of those of $E(x, k, \rho)$. He also defines "irregular growth," and establishes theorems concerning functions with this property.

The sole particular functions of zero order (the argument being supposed complex) which have been considered in detail, are the function

$$\prod_{n=0}^{\infty} (1 + q^n x) \text{ and its generalisations}$$

$$\prod_{n=1}^{\infty} (1 + q^n x)^{n^p} \quad \text{and} \quad \prod_{n_1, n_2, \dots = 0}^{\infty} [1 + q_1^{n_1} q_2^{n_2} \dots q_s^{n_s} x].$$

M. Mellin‡ has given formulæ for the logarithms of these functions. In particular, his formula for $F(x) = \log \prod (1 + q^n x)$ provides an asymptotic expansion for $\log F(x)$ in the case when q is real. We shall give this result in § 12.

* *Acta Soc. Fenn.*, t. XXXI.

† *Journal de l'Ecole Polytechnique*, 1904 and 1905; *Jour. de Math. (Liouville)*, t. XI., 1903-4.

‡ *Acta Soc. Fenn.*, t. XXIX., p. 14, formulæ (23), (25).

This function $F(x)$ has also been considered by Dr. Barnes* and Mr. Hardy.†

Mr. Hardy obtains the formula (in our notation)

$$F(x) = \frac{H}{\sqrt{a}} e^{(\log a)^2 / \log(q^{-1})} \sin\left(\frac{\pi \log a}{\log(q^{-1})}\right),$$

where $-a = x = -re^{i\phi}$, $-\delta < \phi < \delta$, where δ is a small positive number, $\log a = \log r + i\phi$, and $K_1 < H < K_2$, where K_1, K_2 are constants. This formula gives an approximation for $F(x)$ near the line of zeros. The part of the plane for which this formula holds includes the part of the plane from which x is supposed excluded in the asymptotic expansion of M. Mellin (cf. § 12).

Dr. Barnes‡ has shown that in the case of certain functions of finite non-zero order we can obtain a complete asymptotic expansion of the logarithm of the function, *i.e.*, an expansion proceeding in descending powers of z . The analysis, however, breaks down for zero order. Moreover Dr. Barnes concludes§ that the complete asymptotic expansion of $\log F(z)$ in the case of functions $F(z)$, the modulus of whose n -th zero is a function of n of high order in n , is impossible without the introduction of functions at present unknown in analysis.

It is the object of the present paper to show that the functions of zero order have simple and characteristic properties, which do not generally hold for functions of finite order, that they should be studied by different methods than those applicable to other integral functions, and that the arithmetical method is specially applicable to their theory.

I wish to acknowledge my indebtedness to Dr. Barnes for his kind assistance and advice. My thanks are also due to one of the referees, who has given me a number of suggestions with reference to arrangement and choice of expression.

2. The most general integral function of zero order is

$$Fz = C \prod_1^{\infty} (1 + z/a_s),$$

where $a_1, a_2, \dots, a_s, \dots$ is the most general sequence of numbers arranged in

* *Phil. Trans. Roy. Soc. (A)*, Vol. 199, pp. 411-500; *Camb. Phil. Trans.*, Vol. XIX., pp. 322-355.

† *Quarterly Journal*, 1905.

‡ *Loc. cit.* and *Proc. London Math. Soc.*, Ser. 2, Vol. 3, Part 4.

§ *Phil. Trans. and Camb. Phil. Trans.*, *loc. cit.*

order of increasing moduli, subject to the condition that, if λ be any positive number, then $\sum_{s=1}^{\infty} |a_s|^{-\lambda}$ is convergent, however near to zero λ may be.

Let α_n be written for $|a_n|$, and let $\log \alpha_n / \log n = \phi(n)$.

Then $a_n = n^{\phi(n)*}$

Now we must have, for a function of zero order,

$$\lim_{n \rightarrow \infty} \phi(n) = \infty.$$

For suppose this were not the case; then we should have a sequence $n_1, n_2, \dots, n_r, \dots$ of values of n , for which $\phi(n) < h$, when h is some finite number.

The series $\sum_{n=1}^{\infty} \frac{1}{\alpha_n^{1/2h}}$ must converge. But the n_r -th term $> \frac{1}{n_r^{h/2h}} > \frac{1}{n_r^{\frac{1}{2}}}$, and therefore, since the terms of the series do not increase, the sum of the first n_r terms $> n_r \frac{1}{n_r^{\frac{1}{2}}} > n_r^{\frac{1}{2}}$. Thus, as n_r can be taken as large as we please, the series $\sum 1/\alpha_n^{1/2h}$ diverges, which is contrary to our hypothesis.

Hence $\lim_{n \rightarrow \infty} \phi(n) = \infty$.

In the case of functions of zero order, defined by a product form, which arise naturally, the moduli $a_1, a_2, \dots, a_n, \dots$ of the zeros will increase in a regular manner (the word "regular" being used in a rough sense), and, since $\alpha_n = n^{\phi(n)}$, and $\phi(n)$ ultimately becomes infinite, in the natural cases $\phi(n)$ will increase uniformly after a certain value of n , e.g.,

$$\alpha_n = n^{\log n}, \quad \alpha_n = e^n = n^{n/\log n}.$$

This, of course, only applies to functions which arise from the consideration of the product form. If a function of zero order be defined by a Taylor series, then, however regularly the coefficients of this series may increase, there seems no legitimate presumption that the moduli of the zeros of the function increase at all regularly.

In the present paper we shall chiefly be concerned with functions which are such that $\phi(n)$ increases uniformly with n , after some finite value of n , the word "increasing" always being understood to include the case "not decreasing."

Functions of this class we shall call "functions of standard type."

* The standard function of finite order ρ ($\rho < 1$) is $\prod_{s=1}^{\infty} (1 + zs^{-1/\rho})$. When $\rho = 0$ this form does not exist, and we introduce the function $\phi(n)$ to replace, to some extent, the constant $1/\rho$.

3. We shall use the following notation:—

$$F(z) = \prod_{s=1}^{\infty} \left(1 + \frac{z}{a_s}\right), \quad |a_s| = a_s = s^{\phi(s)} \quad (s > 1), \quad |z| = r.$$

Let n be the integer such that $a_n \leq |z| < a_{n+1}$. Then n depends uniquely on r . We shall always use the symbol n with this meaning.

We have

$$\begin{aligned} \log F(z) = & \log \left[\prod_{s=1}^n \left(\frac{z}{a_s}\right) \right] + \log \left[\prod_{s=n+2}^{\infty} \left(1 + \frac{z}{a_s}\right) \right] \\ & + \log \left[\prod_{s=1}^{n-1} \left(1 + \frac{a_s}{z}\right) \right] + \log \left(1 + \frac{z}{a_{n+1}}\right) + \log \left(1 + \frac{a_n}{z}\right), \quad (1) \end{aligned}$$

where n has its special meaning.

We call the first three terms of the right hand of (1), respectively P, R, S . We also denote the real parts of P, R, S , by $\bar{P}, \bar{R}, \bar{S}$.

$$\text{Then } \bar{P} = n \log r - \sum_{s=1}^n \log a_s, \quad \bar{R} = \sum_{s=2}^{\infty} \log \left| 1 + \frac{z}{a_{n+s}} \right|,$$

$$\bar{S} = \sum_{s=1}^{n-1} \log \left| 1 + \frac{a_s}{z} \right|.$$

Throughout this paper we shall be primarily concerned with $F(z)$, and not with $\log F(z)$, although most of our results are stated in terms of the latter function. We shall, then, not concern ourselves with the proper multiple of $2\pi i$ which occurs in equations like (1).

When we deduce an asymptotic expression for $F(z)$ from a corresponding expression for $\log F(z)$ the imaginary part of the latter expression yields a factor e^{ψ} in $F(z)$, where we may suppose $|\psi| \leq \pi$.

We shall then regard all possible values of $\log F(z)$ (for the same z) as equivalent, and we shall regard the imaginary part of $\log F(z)$, however it may be expressed, as a finite term in the expression for $\log F(z)$.

We may then re-write (1), after separating the right-hand side into its real and imaginary parts,

$$\log F(z) = \bar{P} + \bar{R} + \bar{S} + \log \left| 1 + \frac{z}{a_{n+1}} \right| + \log \left| 1 + \frac{a_n}{z} \right| + i\gamma_z, \quad (2)$$

where we suppose that $|\gamma_z| \leq \pi$.

4. We shall establish the following general theorem:—

Let $F(z) = C \cdot \prod_{s=1}^{\infty} (1 + z/a_s)$ be any integral function of zero order,

where $a_1, a_2, \dots$ are arranged in order of non-decreasing moduli. Let $M(r)$, $m(r)$ denote the maximum and minimum moduli of $F(z)$ on the circle $|z| = r$.

Then there exists a sequence of circles $|z| = r$, $r = r_1, r_2, \dots$, $r_1 < r_2 < \dots$, $\lim_{s \rightarrow \infty} r_s = \infty$, with the following properties.

If any positive number ϵ be assigned, however small, there is an integer μ depending on ϵ , such that, when $s > \mu$, on the circle $|z| = r_s$ we have

$$|F(z)| = \exp[\bar{P}\{1 + \eta(z)\}],$$

where

$$\bar{P} = n \log r - \sum_{s=1}^n \log a_s,$$

n having its special meaning, and where $|\eta(z)| < \frac{1}{2}\epsilon$.

Moreover, the sequence $r_1, r_2, \dots$ depends only on $|a_1|, |a_2|, \dots$, and is independent of the arguments of the zeros.

Hence, if $F_1(z)$ be any integral function of zero order, the sequence of the moduli of whose zeros is the same as the corresponding sequence for $F(z)$, and for which the coefficient of z^0 in the Taylor series is C , then

$$|F_1(z)| = \exp[\bar{P}\{1 + \eta_1(z)\}],$$

where $|\eta_1(z)| < \frac{1}{2}\epsilon$.

Again, when $s > \mu$, we have

$$\begin{aligned} m(r_s) &> [M(r_s)]^{1-\epsilon}, & m_1(r_s) &> [M_1(r_s)]^{1-\epsilon}, & m(r_s) &> [M_1(r_s)]^{1-\epsilon}, \\ m_1(r_s) &> [M(r_s)]^{1-\epsilon}. \end{aligned}$$

We shall require the two following lemmas:—

LEMMA I.—If we are given in the x -plane a circle with its centre at the origin and with radius l , and are given any m points $a_1, a_2, \dots, a_m$ in the plane, each of which may be within, without, or on the boundary of the circle, then there exists a concentric circle of radius $\leq l$, such that for every point x on it,

$$\left| \prod_{s=1}^m (x - a_s) \right| > (l/2e)^m.$$

Moreover, such a circle can be found whose position (*i.e.*, whose radius) depends only on $|a_1| \dots |a_m|$, and not on the arguments of the a 's.

$$\text{If } |a_s| = \alpha_s, \quad |x| = \rho, \quad \left| \prod_1^m (x - a_s) \right| \geq \left| \prod_1^m (\rho - \alpha_s) \right|,$$

whatever be the arguments of the a 's. It is therefore sufficient to prove

that, if $\alpha_1, \alpha_2, \dots, \alpha_m$ be any m points on the positive real axis, we can find a point ρ on this half axis such that $\rho < l$, and

$$\left| \prod_1^m (\rho - \alpha_s) \right| \geq (l/2e)^m.$$

Let a be any point on the positive real axis.

Consider
$$I_a \equiv \int_0^l \log |x - a| \, dx.$$

If $a \geq l$,
$$I_a \geq \int_0^l \log (l - x) \, dx \text{ (algebraically)}$$
$$\geq \int_0^l \log x \, dx \geq l \log l - l \geq l \log (l/e) > l \log (l/2e). \tag{1}$$

Next let $a < l$; then

$$\begin{aligned} I_a &= \int_0^a \log (a - x) \, dx + \int_a^l \log (x - a) \, dx \\ &= \left[\int_0^a + \int_0^{l-a} \right] \log x \, dx \\ &= a \log a - a + (l - x) \log (l - a) - l + a. \end{aligned}$$

Let $a = \frac{1}{2}l(1 + \beta)$, where $-1 < \beta < 1$. Then

$$\begin{aligned} I_a &= l \left[\frac{1}{2}(1 + \beta) \{ \log (l/2) + \log (1 + \beta) \} + \frac{1}{2}(1 - \beta) \{ \log (l/2) + \log (1 - \beta) \} - 1 \right] \\ &= l \log (l/2) - l + \left(\frac{1}{2}l \right) [(1 + \beta) \log (1 + \beta) + (1 - \beta) \log (1 - \beta)]. \end{aligned} \tag{2}$$

If $|\beta| < 1$,

$$\begin{aligned} &(1 + \beta) \log (1 + \beta) + (1 - \beta) \log (1 - \beta) \\ &= \log (1 - \beta^2) + \beta \log \frac{1 + \beta}{1 - \beta} = - \sum_1^\infty \frac{\beta^{2s}}{s} + \sum_1^\infty \frac{2\beta^{2s}}{2s - 1} = \sum_1^\infty \frac{\beta^{2s}}{s(2s - 1)} > 0. \end{aligned} \tag{3}$$

If $\beta = \pm 1$, it is easily seen that $(1 + \beta) \log (1 + \beta) + (1 - \beta) \log (1 - \beta)$ tends to the limit $2 \log 2 > 0$. (4)

From (2), (3), (4),

$$I_a \geq l \log (l/2) - l \geq l \log l/2e \quad (a < l). \tag{5}$$

From (1) and (5) we have $I_a \geq l \log l/2e$ for all values of a .

Giving a the values $\alpha_1, \alpha_2, \dots, \alpha_m$ and adding, we obtain

$$\int_0^l \log |(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_m)| \, dx \geq ml \log (l/2e).$$

Let M be the maximum value of $|(\rho - \alpha_1) \dots (\rho - \alpha_m)|$ for $0 \leq \rho \leq l$. Then $l \log M = \int_0^l \log M dx > \int_0^l \log |(x - \alpha_1) \dots (x - \alpha_m)| dx > ml \log (l/2e)$.

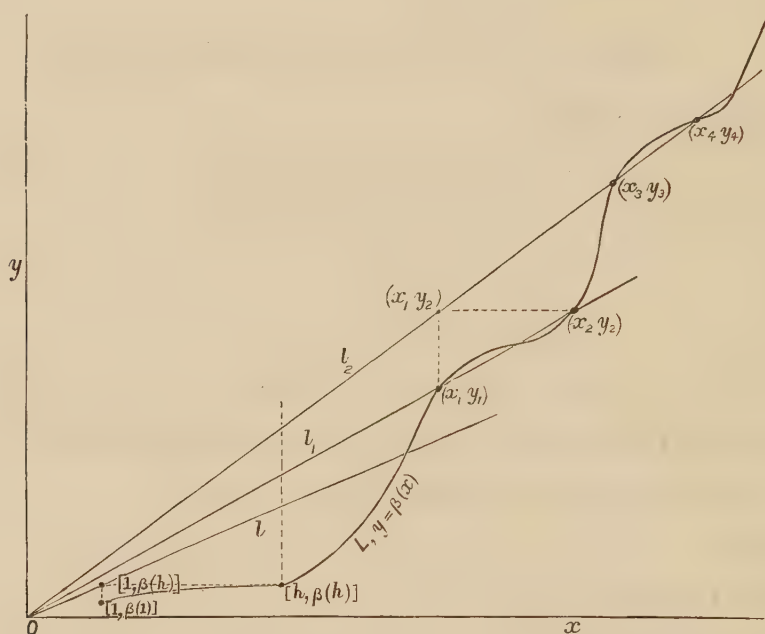
Therefore $\log M > m \log (l/2e)$,

and $M > (l/2e)^m$.

Thus there exists a ρ such that

$$|(\rho - \alpha_1) \dots (\rho - \alpha_m)| > (l/2e)^m,$$

and the Lemma is proved.



LEMMA II.—Let $\alpha_1, \alpha_2, \dots$ be a sequence of real positive numbers, such that $\alpha_{s+1} \geq \alpha_s$, and such that $\alpha_s = s^{\phi(s)}$, where $\lim_{s \rightarrow \infty} \phi(s) = \infty$.

Let $a(x)$ be the function defined by

$$a(x) = (x-s)\alpha_{s+1} + (s+1-x)\alpha_s,$$

when

$$s+1 \geq x \geq s.$$

Then $a(s) = \alpha_s$, $a(x)$ is a continuous non-decreasing function of x , and $a(x) = x^{\phi(x)}$, where $\lim_{x \rightarrow \infty} \phi(x) = \infty$.

Suppose now that we are given a number ν as large as we please. Then if any (large) number h be assigned, there exists a point (X, Y) of

the curve $y = a(x)$, such that $X > h$, and such that the part of the curve C , $(y/Y)^{1/\nu} = x/X$ [which evidently passes through (X, Y)], for which $1 < x < X$, is entirely above the corresponding part of the curve $y = a(x)$, while the part of C for which* $x > X+1$ is entirely below the corresponding part of $y = a(x)$.

Effect the transformation

$$x' = x, \quad y'^{\nu} = y.$$

The curve C becomes C' , $y'/Y' = x'/X'$, *i.e.*, a straight line through the origin, and $y = a(x)$ becomes $y' = \beta(x')$, where $\beta(x) = [a(x)]^{1/\nu}$, so that $\beta(x')$ is a non-decreasing function.

If in the original plane a point A is above another point B of the same abscissa, the transformed point A' will be above B' in the transformed plane.

Hence to prove the Lemma it is sufficient to prove that a point (X', Y') exists on $y' = \beta(x')$, such that $X' > h$, and such that the part of the line C' for which $1 < x' < X'$ is entirely above, and the part of C' for which $x' > X'+1$ is entirely below, the corresponding part of $y' = \beta(x')$.

Suppose the contrary case. (1)

Call the curve† $y = \beta(x)$, L .

This curve L starts at the point $[1, \beta(1)]$.

Since $\beta(x)$ is non-decreasing it is evident from the figure that the line l joining the origin O to $[1, \beta(h)]$ does not cut the curve L in any point of abscissa $< h$.

Let l_1 be any line through O , lying between l and Oy .

Since $\beta(x) = x^{\phi(x)/\nu}$ and $\lim_{x \rightarrow \infty} \phi(x)/\nu = \infty$, it is easily seen that every ray through the origin, lying in the first quadrant, and sufficiently near the axis Oy , but not coincident with that axis, must intersect the curve L .

Let then (x_1, y_1) be that intersection of l_1 with L which is nearest to the origin.

Then, from the figure, $x_1 > h$, and the point $[h, \beta(h)]$ is below the point of l_1 whose abscissa is h . Hence the part of L for which $x < x_1$ is below the corresponding part of l_1 . Again,

$$\beta(x) = x^{\phi(x)/\nu} > \frac{y_1}{x_1} x,$$

* The result can be extended to the case when this inequality is replaced by $x > X$, but the proof is somewhat troublesome, and the above form is sufficient for our purpose.

† For convenience we shall drop the accents.

when x is sufficiently large, so that the distant points of L are above the corresponding points of l_1 .

Then it follows from the supposition (1) that there is an intersection (x_2y_2) of l_1 with L such that $x_2 - x_1 \geq 1$.

For, if this were not the case, (x_1y_1) would be a point of L satisfying all the requirements for $(X'Y')$.

Let l_2 be the line joining O and (x_1y_2) , and let (x_3y_3) be that intersection of l_2 and L which is nearest to the origin. From the figure $x_3 > x_2$, and x_2 takes the place of h in the reasoning above.

Then, as before, there exists an intersection of l_2 and L such that $x_4 - x_3 > 1$. We proceed similarly with the line l_3 joining O and (x_3y_4) , and so on.

$$\text{Now } \frac{y_2}{x_2} = \frac{y_1}{x_1},$$

$$\frac{y_4}{x_4} = \frac{y_3}{x_3} = \frac{y_2}{x_2} = \frac{y_1}{x_1} \frac{x_2}{x_1},$$

$$\frac{y_6}{x_6} = \frac{y_5}{x_5} = \frac{y_4}{x_4} = \frac{y_1}{x_1} \frac{x_2x_4}{x_1x_3},$$

$$\dots \quad \dots \quad \dots \quad \dots$$

$$\frac{y_{2p}}{x_{2p}} = \frac{y_1}{x_1} \frac{x_2x_4 \dots x_{2p}}{x_1x_3 \dots x_{2p-1}} = \frac{y_1}{x_1^2} \frac{x_2}{x_3} \frac{x_4}{x_5} \dots \frac{x_{2p-2}}{x_{2p-1}} x_{2p} < \frac{y_1}{x_1^2} x_{2p},$$

since $x_{s+1} > x_s$. Therefore

$$y_{2p} < \left(\frac{y_1}{x_1^2} \right) x_{2p}^2 \text{ for all values of } p. \quad (2)$$

$$\text{Now } \frac{\beta(x)}{x^2} = x^{\phi(x)/\nu-2},$$

and tends to infinity with x . Therefore a number k exists such that, when $x > k$,

$$\frac{\beta(x)}{x^2} > \frac{y_1}{x_1^2}. \quad (3)$$

But, since $x_{2s} \geq x_{2s-1} + 1$, $x_{2s-1} > x_{2s-2}$, when $p > k$, we have $x_{2p} > k$.

Hence, since $(x_{2p}y_{2p})$ is a point of L , (2) and (3) are incompatible.

Hence the supposition (1) is false, and the Lemma is proved.

We can now proceed to the proof of the theorem.

Take for the sequence $\alpha_1, \alpha_2, \dots$ of Lemma II. the sequence of the moduli of the zeros of $F(z)$.

If we are given a number h , however large, and when we have assigned

a ν we can, by means of Lemma II., find an $X > h$ such that the point $[X, \alpha(X)]$ has the properties of the lemma.

Let m be the greatest integer contained in X . By Lemma I. we can find a number $r \leq \frac{1}{2}\alpha(X)$ such that, for all points z on the circle $|z| = r$,

$$\left| \prod_1^m (a_s + z) \right| > \left[\frac{\alpha(X)}{4e} \right]^m,$$

and such that this inequality obtains without alteration of r , when the arguments of the α 's are varied; so that

$$\left| \prod_1^m \left(1 + \frac{z}{a_s} \right) \right| > \frac{[\alpha(X)/4e]^m}{\alpha_1 \alpha_2 \dots \alpha_m}$$

$$\text{and} \quad \log \left| \prod_1^m \left(1 + \frac{z}{a_s} \right) \right| > P_1 - m \log(4e) \quad (1)$$

$$\text{where} \quad P_1 = m \log \alpha(X) - \log(\alpha_1 \alpha_2 \dots \alpha_m). \quad (2)$$

$$\text{Again,} \quad \log \left| \prod_1^m \left(1 + \frac{z}{a_s} \right) \right| < \log \prod_1^m \left[\frac{\alpha(X) + \alpha(X)}{\alpha_s} \right] < P_1 + m \log 2. \quad (3)$$

From (1) and (3) it follows that

$$\log \left| \prod_1^m \left(1 + \frac{z}{a_s} \right) \right| = P_1 + \theta m \quad \text{where } |\theta| \text{ is finite.} \quad (4)$$

$$\text{When } |x| \geq \frac{1}{2}, \quad |[\log |1+x|]| < 2|x|.$$

$$\text{Therefore, since } |z| < \frac{1}{2}\alpha(X), \text{ so that } \left| \frac{z}{a_{m+s}} \right| \leq \frac{1}{2} \quad (s \geq 1),$$

$$\begin{aligned} \left| \left[\log \left| \prod_1^\infty \left(1 + \frac{z}{a_{m+s}} \right) \right| \right] \right| &\leq \sum_1^\infty \left| \left[\log \left| 1 + \frac{z}{a_{m+s}} \right| \right] \right| < 2 \sum_1^\infty \left| \frac{z}{a_{m+s}} \right| \\ &< \alpha(X) \sum_1^\infty \frac{1}{\alpha_{m+s}}. \end{aligned} \quad (5)$$

Now by the property of Lemma II., when $m+s > X+1$, i.e., when $s > 1$,

$$\begin{aligned} \alpha_{m+s} &> \text{the ordinate of } [y/\alpha(X)]^{1/\nu} = x/X \text{ corresponding to } x = m+s \\ &> \frac{(m+s)^\nu}{X^\nu} \alpha(X). \end{aligned}$$

Hence

$$\begin{aligned} \alpha(X) \sum_1^\infty \frac{1}{\alpha_{m+s}} &< 1 + \alpha(X) \sum_2^\infty \frac{1}{\alpha_{m+s}} < 1 + X^\nu \sum_2^\infty \frac{1}{(m+s)^\nu} \\ &< 1 + X^\nu \int_1^\infty \frac{dt}{(m+t)^\nu} < 1 + X^\nu \frac{1}{(\nu-1)(m+1)^{\nu-1}} < 1 + \frac{m+1}{\nu-1}. \end{aligned}$$

Hence, from (5), when $m > \nu > 2$,

$$\left| \left[\log \left| \prod_1^\infty \left(1 + \frac{z}{a_{m+s'}} \right) \right| \right] \right| < 1 + \frac{m+1}{\nu-1} = \theta' m, \quad \text{where } |\theta'| < \frac{3}{\nu-1} < 3. \quad (6)$$

Next consider P_1 . We have

$$P_1 = m \log \frac{a(X)}{(a_1 a_2 \dots a_m)^{1/m}} > m \log \left[\frac{m a(X)}{a_1 + a_2 + \dots + a_m} \right], \quad (7)$$

since the arithmetic mean is greater than the geometric mean. Now by the property of Lemma II., when $1 < s < X$,

$$\begin{aligned} \alpha_s &< \text{the ordinate of } [y/a(X)]^{1/\nu} = x/X \text{ corresponding to } x = s, \\ &< a(X) \frac{s^\nu}{X^\nu}. \end{aligned}$$

$$\text{Hence } \sum_1^{m-1} \alpha_s < \frac{a(X)}{X^\nu} \sum_1^{m-1} s^\nu < \frac{a(X)}{X^\nu} \int_0^m t^\nu dt < \frac{a(X)}{X^\nu} \frac{m^{\nu+1}}{\nu+1} < \frac{a(X)}{\nu} m.$$

$$\text{Therefore } \left[\sum_1^m \alpha_s \right] / [m a(X)] < \frac{1}{m} + \frac{1}{\nu}.$$

Choose ν so large that $1/\nu < \frac{1}{2}\eta$, where $\eta = e^{-1/\epsilon_1}$ and ϵ_1 will presently be chosen, and choose $h > \nu$ so that $m > \nu$, $1/m < \frac{1}{2}\eta$. Then

$$\left[\sum_1^m \alpha_s \right] / [m a(X)] < \eta,$$

$$\text{and therefore } \log \left\{ \frac{m a(X)}{\sum_1^m \alpha_s} \right\} > \log \frac{1}{\eta} > \frac{1}{\epsilon_1}.$$

$$\text{Then, from (7), } P_1 > m/\epsilon_1. \quad (8)$$

$$\text{From (4) and (6), } \log |F(z)| = P_1 + (\theta + \theta')m + \log C.$$

Now

$$\left| \frac{(\theta + \theta')m + \log C}{P_1} \right| < \frac{m[|\theta| + |\theta'| + |\log C|]}{m/\epsilon_1} < \epsilon_1[|\theta| + |\theta'| + |\log C|].$$

Choose ϵ_1 so that this last expression is less than $\frac{1}{8}\epsilon$. Then

$$\log |F(z)| = P_1 [1 + \eta_1(z)], \quad \text{where } |\eta_1(z)| < \frac{1}{8}\epsilon \quad (9)$$

for every point of the circle $|z| = r$.

By choosing h sufficiently large, we can make m as large as we please, and hence P_1 , which is certainly positive, as large as we please.

Now r cannot be equal to α_s for any value of s , for then $\log |F(z)|$ would be $-\infty$ for $z = -\alpha_s$. Hence there is an n such that $\alpha_n < r < \alpha_{n+1}$, both limits excluded.

$$\text{Then} \quad \log |F(z)| = \bar{P} + \log Q, \quad (10)$$

$$\text{where} \quad \bar{P} = n \log r - \sum_1^n \log a_s$$

$$\text{and} \quad \log Q = \log \left| \left[\prod_1^n \left(1 + \frac{a_s}{z} \right) \prod_1^\infty \left(1 + \frac{z}{a_{n+s}} \right) \right] \right|.$$

We shall show that there is a point z_1 on $|z| = r$ for which $Q \leq 1$; so that $[\log Q]_{z_1} \leq 0$ algebraically.

Suppose the contrary case. Then, since Q is continuous for points z on the circle, and, since a continuous function attains its limits, there is a number $\lambda > 1$ such that $Q > \lambda$ for every point z of the circle.

[λ may depend on n , and tend to zero with $1/n$, but we only require λ to be independent of the argument of z .]

Hence, if p be any integer, and z' any point on the circle,

$$\prod_{t=0}^{p-1} [Q(z' e^{2\pi i t/p})] > \lambda^{p+1}. \quad (11)$$

But, since $\prod_1^\infty \left(1 + \frac{z}{a_{n+s}} \right)$ is absolutely convergent and $\prod_1^n \left(1 + \frac{a_s}{z} \right)$ is a finite product,

$$\begin{aligned} \prod_{t=0}^{p-1} [Q(z' e^{2\pi i t/p})] &= \prod_{s=1}^n \left[\prod_{t=0}^{p-1} \left\{ 1 + \frac{a_s}{z' e^{2\pi i t/p}} \right\} \right] \prod_{s=1}^\infty \left[\prod_{t=0}^{p-1} \left\{ 1 + \frac{z' e^{2\pi i t/p}}{a_{n+s}} \right\} \right] \\ &= \prod_{s=1}^n \left[\left| 1 + \left(\frac{a_s}{z'} \right)^p \right| \right] \prod_{s=1}^\infty \left[\left| 1 + \left(\frac{z'}{a_{n+s}} \right)^p \right| \right] \\ &< \prod_{s=1}^n \left[1 + \left(\frac{a_s}{r} \right)^p \right] \prod_{s=1}^\infty \left[1 + \left(\frac{r}{a_{n+s}} \right)^p \right]^* \\ &< \prod_{s=1}^n \left(1 + \frac{a_s}{r} \right) \prod_{s=1}^\infty \left(1 + \frac{r}{a_{n+s}} \right) \\ &< U, \text{ where } U \text{ is independent of } p, \text{ though not of } n. \quad (12) \end{aligned}$$

But, if p be taken sufficiently large, $\lambda^{p+1} > U$.

Hence (11) and (12) are incompatible, our supposition was false, and it follows that there exists a point z_1 on $|z| = r$ such that $Q(z_1) \leq 1$ and $[\log Q]_{z_1} \leq 0$.

In a similar manner we can prove that there exists a point z_2 such that $[\log Q]_{z_2} \geq 0$.

The number $\lambda > 1$ is replaced by $\mu < 1$. We consider the same product of Q 's as before, and the inequality indicated by the star is replaced by

$$\begin{aligned} \prod_{t=0}^{p-1} [Q(z' e^{2\pi i t/p})] &> \prod_{s=1}^n \left[1 - \left(\frac{a_s}{r} \right)^p \right] \prod_{s=1}^\infty \left[1 - \left(\frac{r}{a_{n+s}} \right)^p \right] \\ &> \prod_{s=1}^n \left(1 - \frac{a_s}{r} \right) \prod_{s=1}^\infty \left(1 - \frac{r}{a_{n+s}} \right) > V. \end{aligned}$$

We thus obtain an inequality incompatible with $\prod_{t=0}^{p-1} [Q(z'e^{2\pi it/p})] < \mu^{p+1}$ when p is sufficiently large.

Now $\bar{P}$ is independent of the argument of z , and is the same for z_1 and z_2 . The same thing is true for P_1 .

$$\text{Now} \quad [\log F(z) - \bar{P}]_{z=z_1} \leq 0 \quad (\text{algebraically}).$$

$$\text{From (9),} \quad [\log F(z) - (1 - \frac{1}{8}\epsilon)P_1]_{z=z_1} > 0.$$

$$\text{Hence, subtracting,} \quad (1 - \frac{1}{8}\epsilon)P_1 - \bar{P} < 0.$$

Similarly, by putting $z = z_2$, we obtain

$$(1 + \frac{1}{8}\epsilon)P_1 - \bar{P} > 0,$$

whence $P_1 = \bar{P}/(1 + \frac{1}{8}\epsilon\kappa_z)$, where $|\kappa_z| < 1$ for all points z of the circle, and

$$\log |F(z)| = P_1[1 + \eta_1(z)] = \bar{P} \left[\frac{1 + \eta_1(z)}{1 + \frac{1}{8}\epsilon\kappa_z} \right].$$

$$\text{Now} \quad \left| 1 - \frac{1 + \eta_1(z)}{1 + \frac{1}{8}\epsilon\kappa_z} \right| < (\frac{1}{8}\epsilon + \frac{1}{8}\epsilon)/(1 - \frac{1}{8}\epsilon) < \frac{1}{2}\epsilon, \quad \text{when } \epsilon < 4.$$

$$\text{Then} \quad \log |F(z)| = \bar{P}[1 + \eta(z)], \quad \text{where } |\eta(z)| < \frac{1}{2}\epsilon. \quad (13)$$

The complete form of the theorem is established by the following considerations:—

By choosing h sufficiently large, we can ensure that m is as large as we please, and that therefore X , therefore $\alpha(X)$, therefore P_1 , therefore $M(r)$, and therefore r or n , is as large as we please.

We have then determined an r such that

$$|F(z)| = \exp[\bar{P}\{1 + \eta(z)\}], \quad |\eta(z)| < \frac{1}{2}\epsilon.$$

Moreover, $P = n \log r - \sum_1^n \log \alpha_s$ is evidently independent of the arguments of z and of the zeros. In the course of the work r was determined (by means of Lemma I.) solely by means of the moduli $\alpha_1, \alpha_2, \dots$, and is independent of the arguments of the α 's; moreover, the analysis throughout assumes these arguments to be arbitrary.

Thus on the circle $|z| = r$

$$|F_1(z)| = \exp[\bar{P}\{1 + \eta_1(z)\}], \quad |\eta_1(z)| < \frac{1}{2}\epsilon.$$

We determine the sequence $r_1, r_2, \dots$ as follows:—

Let r_1 be a value of r which has the above properties when $\epsilon = \frac{1}{2}$.

Let r_2 be the least possible value of r such that $r_2 > r_1 + 1$, and such that r_2 has the above properties for $\epsilon = 1/2^2$.

Similarly, let r_3 be the least possible $r > r_2 + 1$ and corresponding to $\epsilon = 1/2^3$, and so on. We evidently have $\lim_{s \rightarrow \infty} r_s = \infty$. Then, when an

ϵ is assigned, if μ be the least integer such that $2^{-\mu} < \epsilon$, when $s > \mu$, on the circle $|z| = r_s$, we have

$$|F(z)| = \exp[\bar{P}\{1 + \eta(z)\}],$$

where
$$|\eta(z)| < \frac{1}{2^{s+1}} < \frac{1}{2}\epsilon.$$

A similar result evidently obtains for $F_1(z)$.

Finally, we have to show that $m(r_s) > [M(r_s)]^{1-\epsilon}$, ... We have

$$\log m(r_s) > \bar{P}(1 - \frac{1}{2}\epsilon) > \bar{P}(1 + \frac{1}{2}\epsilon)(1 - \epsilon) > [\log M(r_s)](1 - \epsilon).$$

Therefore
$$m(r_s) > [M(r_s)]^{1-\epsilon}.$$

Similarly, the other inequalities of the theorem are proved.

5. The theorem of the preceding article does not in general hold for functions of finite or infinite order, and it does not hold for any of the ordinary functions of finite order which occur in analysis, *e.g.*, e^z , $\sin z$, $1/\Gamma(z)$, $\sin z$.

It does not hold, moreover, for the functions* $P_\rho(z)$ of Dr. Barnes, which may be regarded as the standard type of functions of finite order.

The most precise completely general theorem† which is known concerning the relations of $m(r)$ and $M(r)$ for functions of finite order ρ , is that, on certain circles $|z| = r$, where r is as large as we please, $m(r) > e^{-r^{\rho+\epsilon}}$, where ϵ has its usual meaning. It seems certain, moreover, that when $\rho > 1$ we cannot find a lower limit for $m(r)$ which is of higher order than is implied in this inequality, unless we make further assumptions‡ as to the nature of the zeros of the function.

Again, on the circles $|z| = r$ of § 4, $\log |F(z)|$ has a dominant term § $\bar{P}$ which is independent at once of the argument of z and of the arguments of the zeros, and $|F(z)|$ has a dominant factor $\exp(\bar{P})$ which is similarly independent.

* Barnes, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, Part 4.

† [Note added August 17th.—I have proved the following theorem:—If $\rho < \frac{1}{2}$, a sequence of circles exists for which $m(r) > [M(r)]^{\cos 2\pi\rho - \epsilon}$, which is a generalisation of the theorem of § 4. I hope to publish the proof in another paper.]

‡ Cf. Lindelöf, *Acta Soc. Fenn.*, Vol. xxxi. (1), 1902, pp. 1-79, where an extensive theory of this kind is elaborated.

§ In saying that P is the dominant term of $f(z)$, it is meant that $f(z) = P + Q$, where

$\text{Lt}_{|z|=\infty} \left[\frac{Q}{P} \right] = 0$. Similarly, P is a dominant factor of $f(z)$ when $f(z) = P \cdot Q$, where

$$\text{Lt}_{z=\infty} \frac{|Q|^{\pm 1}}{|P|} = 0.$$

This again is not true for the standard functions of finite order.
Consider the function

$$F(z) = z \prod_1^{\infty} \left[\left(1 - \frac{z}{n^2}\right) \left(1 + \frac{z}{n^2}\right) \right] = \frac{\sin(\pi z^{\frac{1}{2}}) \sinh(\pi z^{\frac{1}{2}})}{\pi^2},$$

whose order is $\frac{1}{2}$.

If $z^{\frac{1}{2}} = \xi + i\eta$, the dominant term of $\log |F(z)|$ is $\pi(|\xi| + |\eta|)$. This evidently depends on the argument of z , and varies from $\sqrt{2}\pi r^{\frac{1}{2}}$ to $\pi r^{\frac{1}{2}}$. But even the maximum value of the dominant term can be changed by a change in the argument of the zeros.

Consider

$$F_1(z) = z \left[\prod_1^{\infty} \left(1 + \frac{z}{n^2}\right) \prod_1^{\infty} \left(1 + \frac{z}{n^2}\right) \right],$$

whose zeros have the same moduli as those of $F(z)$.

The maximum value of the dominant term of $\log |F_1(z)|$ is $2\pi r^{\frac{1}{2}}$, which is different from the corresponding maximum $\sqrt{2}\pi r^{\frac{1}{2}}$ for $F(z)$.

We have chosen a function of order less than unity because of the exponential factors which may occur in the product form of a function of order greater than or equal to unity. In the case of functions of order unity, a more striking example is afforded by the pair of integral functions* $\sin \pi z$ and $z^{-1}[\Gamma(z)]^2$. The moduli of the zeros of these two functions are the same in each case, but the maximum values of the dominant terms of their logarithms are respectively πr and $2r \log r$.

It is possible† to construct functions of finite‡ and infinite (including transfinite) orders for which $\log F(z)$ has a dominant term independent of the arguments of z and of the zeros, but these are exceptional cases.

It is clear, then, that functions of zero order behave at infinity in a manner quite different from that of functions of finite order, and that they deserve to be studied by special methods, not applicable, in general, to the theory of finite order.

In the general theorem of § 4, we have no means of determining the precise positions of the circles $|z| = r_s$. We may, however, expect that, in the case of functions of standard type, the results of this theorem will hold for circles in a part of the plane which can be determined.

In the following articles we shall undertake the approximation to functions of standard type.

* This example was given to me by one of the referees. It is due to Borel.

† This is an example of a theory which I propose to develop in another paper.

‡ Mr. Hardy has given examples of functions of finite order with the above property, *Proc. London. Math. Soc.*, Ser. 2, Vol. 2, 1904, pp. 332-339.

6. We proceed to establish the following theorem for functions of standard type:—

Let $F(z) = \prod_1^{\infty} (1+z/a_s)$ be any function of standard type, so that $\phi(s)$ increases with s after some finite value N of s .

We define the region R of the z plane as follows:—Assign any positive number k , which may be chosen as large as we please.

Let C_s be the circle $|z+a_s| = |a_s| \exp[-\chi(s)]$, where $\chi(s)$ is the greater of the two numbers k and $k[s/\phi(s)]$. Then R is the region of the plane which is exterior to all the circles C_s .

Then, when z is restricted to lie in the region R , a constant K exists such that

$$\log F(z) = \bar{P} + Q,$$

where

$$\bar{P} = n \log r - \sum_{s=1}^n \log a_s,$$

n having its usual meaning, and where

$$|Q| < \text{the greater of the numbers } K \text{ and } K[n/\phi(n)].$$

As for $\bar{P}$, if any positive number $\lambda < 1$ be assigned, then, when r is sufficiently great,

$$\bar{P} > \lambda n \phi(n).$$

Further, if the circle $|z| = r$ do not cut any of the circles C_s , then, when r is sufficiently great,

$$m(r) > [M(r)]^{1-\epsilon},$$

where ϵ is any assigned positive number.

We have (§ 3)

$$\log F(z) = \bar{P} + \bar{R} + \bar{S} + \log \left| 1 + \frac{z}{a_{n+1}} \right| + \log \left| 1 + \frac{a_n}{z} \right| + i\gamma_z,$$

where $|\gamma_z| < \pi$. We shall show that $\bar{P}$ is of order not less than that of $n\phi(n)$, while

$$\left| \bar{R} + \bar{S} + \log \left| 1 + \frac{z}{a_{n+1}} \right| + \log \left| 1 + \frac{a_n}{z} \right| + i\gamma_z \right|$$

is of order not greater than that of the greater of K and $K[n/\phi(n)]$, and we prove the latter result by considering separately the order of each of the five terms $\bar{R}$, $\bar{S}$, ...

First let us prove the result concerning $\bar{P}$.

$$\begin{aligned}
 \text{We have } \bar{P} = \Re P &= \Re \left\{ \log \prod_1^n \frac{z}{a_s} \right\} = n \log r - \sum_1^n \log a_s \\
 &> n \log a_n - \sum_1^n \log a_s \\
 &> n \phi(n) \log n - \log a_1 - \sum_2^n \phi(s) \log s.
 \end{aligned}$$

Let N be the value of s after which $\phi(s)$ increases, and let

$$\sum_1^N \log a_s = C,$$

where C is a finite constant. Then

$$\bar{P} > n \phi(n) \log n - C - \sum_{N+1}^n \phi(s) \log s > n \phi(n) \log n - \phi(n) \sum_{N+1}^n \log s - C.$$

$$\text{Now } \sum_{N+1}^n \log s \leq \sum_1^n \log s \leq \log(n!) \leq n \log n - n + \log \sqrt{2\pi} n + \epsilon_n,$$

where ϵ_n vanishes when n becomes infinite (Stirling's formula). Therefore

$$\begin{aligned}
 \bar{P} &> n \phi(n) \log n - \phi(n) \left\{ n \log n - n + \frac{1}{2} \log(2\pi) + \frac{1}{2} \log n + \epsilon_n \right\} - C \\
 &> \phi(n) \left(n - \frac{1}{2} \log n - \frac{1}{2} \log(2\pi) - C - \epsilon_n \right).
 \end{aligned}$$

Now, when n is sufficiently large, *i.e.*, when r is sufficiently large,

$$\frac{1}{2} \log n + \frac{1}{2} \log(2\pi) + C + \epsilon_n < (1 - \lambda) n,$$

where λ is any assigned number < 1 .

Therefore, when r is sufficiently large,

$$\bar{P} > \lambda n \phi(n), \quad (1)$$

where λ is any assigned number < 1 .

We notice, in passing, an upper limit for $\bar{P}$,

$$\bar{P} < n \log r < n \log a_{n+1} < n \phi(n+1) \log(n+1). \quad (2)$$

Next consider the term $\bar{R}$.

We notice once for all the two following inequalities.

If $|x| < 1$, then $\Re \log(1+x) < |x|$, algebraically,

and $|\Re \log(1+x)| \leq -\log(1-|x|)$.

For the first $\Re \log(1+x) = \log|1+x| \leq \log(1+|x|) \leq |x|$.

For the second $|\Re \log(1+x)| = \left| \Re \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots \right) \right|$

$$\leq |x| + \frac{|x|^2}{2} + \dots \leq -\log(1-|x|).$$

We shall frequently have occasion to use these inequalities in the course of the present paper, and shall hereafter use them without further remark.

We have
$$\bar{R} = \sum_{s=2}^{\infty} \Re \log \left(1 + \frac{z}{a_{n+s}} \right).$$

We divide this sum into two parts as follows.

Let λ^* be a constant which we shall choose later to be greater than a certain finite number.

Let η be the greater of the numbers λ/n , $\lambda/\phi(n)$. Then, by choosing λ sufficiently large, we can ensure that ηn is greater than any assigned number, and, since $\phi(n)$ tends to infinity with n , by choosing n sufficiently large we can make η as small as we please. We suppose for the present that n is so large that $\eta < 1$.

We divide the sum $\bar{R}$ into

$$\sigma_1 = \sum_{s=2}^{[\eta n]} \log |1+z/a_{n+s}|, \quad \text{and} \quad \sigma_2 = \sum_{[\eta n]+1}^{\infty} \log |1+z/a_{n+s}|.$$

We shall prove successively that σ_1 and σ_2 are of order not greater than the greater of $Kn/\phi(n)$ and K .

First consider σ_1 . We have

$$\begin{aligned} |\sigma_1| &\leq - \sum_{s=2}^{[\eta n]} \log \left(1 - \frac{r}{a_{n+s}} \right) \leq - \sum_2^{[\eta n]} \log \left(1 - \frac{a_{n+1}}{a_{n+s}} \right) \\ &\leq - \sum_2^{[\eta n]} \log \left[1 - \frac{(n+1)^{\phi(n+1)}}{(n+s)^{\phi(n+s)}} \right] \\ &\leq - \sum_2^{[\eta n]} \log \left[1 - \left(\frac{n+1}{n+s} \right)^{\phi(n+1)} \right] \\ &\quad [\text{since } \phi(n+s) \geq \phi(n+1)] \\ &\leq - \sum_2^{[\eta n]} \log \left[1 - \left(\frac{n+1}{n+s} \right)^{\lambda/\eta} \right] \quad (3) \\ &\quad [\text{since } \lambda/\eta = \text{the lesser of } n \text{ and } \phi(n) \leq \phi(n+1)]. \end{aligned}$$

* This λ is not the λ appearing in (1).

$$\begin{aligned}
 \text{Now} \quad \log \left[\left(\frac{n+s}{n+1} \right)^{\lambda/\eta} \right] &= -\frac{\lambda}{\eta} \log \frac{n+1}{n+s} = -\frac{\lambda}{\eta} \log \left(1 - \frac{s-1}{n+s} \right) \\
 &> \frac{\lambda}{\eta} \frac{s-1}{n+s} \\
 &> \frac{\lambda}{\eta} \frac{s-1}{n+\eta n} > \frac{\lambda}{2\eta} \frac{s-1}{n} \quad (\text{since } \eta < 1).
 \end{aligned}$$

$$\text{Hence} \quad 1/\left[1 - \left(\frac{n+1}{n+s}\right)^{\lambda/\eta}\right] < 1/\left[1 - \exp\left(-\frac{\lambda}{2\eta} \frac{s-1}{n}\right)\right]. \quad (4)$$

Now, when $x < \frac{1}{2}$,

$$e^{-x} < 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots < 1 - x + \frac{x}{2} \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots \right) < 1 - \frac{x}{2}.$$

$$\text{Hence, when} \quad \eta < \frac{1}{4}\lambda, \quad (5)$$

$$\text{so that} \quad \frac{\lambda}{2\eta} \frac{s-1}{n} < \frac{\lambda}{2\eta} \frac{n}{n} < \frac{1}{2},$$

$$1 - \exp\left[-\frac{\lambda}{2\eta} \frac{s-1}{n}\right] > 1 - \frac{1}{2} \frac{\lambda}{2\eta} \frac{s-1}{n},$$

and, from (4),

$$\begin{aligned}
 1/\left[1 - \left(\frac{n+1}{n+s}\right)^{\lambda/\eta}\right] &< 1/\left[\frac{1}{2} \frac{\lambda}{2\eta} \frac{s-1}{n}\right] < \frac{4\eta}{\lambda} \frac{n}{s-1} \\
 &< \frac{4([\eta n] + 1)}{\lambda(s-1)} < \frac{8[\eta n]}{\lambda(s-1)},
 \end{aligned}$$

provided $\lambda > 2$, so that $[\eta n] \geq \eta n - 1 \geq \lambda - 1 > 1$. Then, from (3),

$$\begin{aligned}
 |\sigma_1| &\leq \sum_2^{[\eta n]} \log \left(\frac{8[\eta n]}{\lambda(s-1)} \right) \leq \sum_2^{[\eta n]} \log \left(\frac{16[\eta n]}{\lambda s} \right) \quad (\text{since } s \geq 2) \\
 &< [\eta n] \log \left(\frac{16[\eta n]}{\lambda} \right) - \log \Gamma(1 + [\eta n]).
 \end{aligned}$$

Using the asymptotic formula for $\log \Gamma(x+1)$ (Stirling's formula), we obtain

$$\begin{aligned}
 |\sigma_1| &< [\eta n] \log \left(\frac{16}{\lambda} \right) + [\eta n] \log ([\eta n]) - [\eta n] \log [\eta n] + [\eta n] \\
 &\quad - \frac{1}{2} \log [\eta n] + A(n),
 \end{aligned}$$

where $A(n)$ is finite for all values of n ,

$$\begin{aligned} &< [\eta n] \left(\log \frac{16}{\lambda} + 1 \right) + A(n) \quad (\text{since } [\eta n] > 1) \\ &< \eta n \left(\log \frac{16}{\lambda} + 1 \right) + A(n). \end{aligned}$$

Let $\frac{1}{2}K_1$ be the greater of (1^0) , $\lambda \left(\log \frac{16}{\lambda} + 1 \right)$, and (2^0) the maximum value of $|A(n)|$. Then, if $\eta = \lambda/n$,

$$|\sigma_1| < \frac{1}{2}K_1 + \frac{1}{2}K_1 < K_1.$$

If $\eta = \lambda/\phi(n)$, so that $\phi(n) > n$,

$$|\sigma_1| < \frac{1}{2}K_1 \frac{n}{\phi(n)} + \frac{1}{2}K_1 < K_1 \frac{n}{\phi(n)}.$$

Hence, in any case, $|\sigma_1| < \text{the greater of } K_1 \text{ and } K_1 \frac{n}{\phi(n)}$. (6)

Now consider σ_2 . When $s \geq [\eta n] + 1 \geq \eta n$,

$$\begin{aligned} \frac{r}{a_{n+s}} &\leq \frac{a_{n+1}}{a_{n+s}} \leq \frac{(n+1)^{\phi(n+1)}}{(n+s)^{\phi(n+s)}} \leq \left(\frac{n+1}{n+s} \right)^{\phi(n+1)} \leq \left(\frac{n+1}{n+s} \right)^{\phi(n)} \\ &< \left[\frac{n+1}{n+1+\eta(n+1)-2} \right]^{\phi(n)} < \left\{ 1 / \left[1 + \eta - \frac{2}{n+1} \right] \right\}^{\phi(n)}. \end{aligned}$$

Choose $\lambda > 4$; then $\frac{2}{n+1} < \frac{1}{2}$, $\frac{4}{n} < \frac{1}{2}\eta$, and

$$\begin{aligned} \frac{r}{a_{n+s}} &< (1 + \tfrac{1}{2}\eta)^{-\phi(n)} < \left[1 + \tfrac{1}{2} \frac{\lambda}{\phi(n)} \right]^{-\phi(n)} < \exp \left[-\phi(n) \log \left\{ 1 + \tfrac{1}{2} \frac{\lambda}{\phi(n)} \right\} \right] \\ &< \exp \left[-\phi(n) \tfrac{1}{4} \frac{\lambda}{\phi(n)} \right] \quad [\text{if } \phi(n) > \lambda]^* \\ &< \exp(-\tfrac{1}{4}\lambda). \end{aligned}$$

Now, when $0 < x < 1$, $\frac{-\log(1-x)}{x} = 1 + \frac{x}{2} + \frac{x^2}{3} + \dots$ increases as x increases.

Hence, when $s > \eta n$,

$$-\log \left(1 - \frac{r}{a_{n+s}} \right) / \frac{r}{a_{n+s}} < \frac{e^{-\frac{1}{4}\lambda}}{\log(1 - e^{-\frac{1}{4}\lambda})} < \text{a finite } l.$$

* If $0 < x < \frac{1}{2}$, $\log(1+x) = x - \frac{x^2}{2} + \dots > x \left[1 - \frac{1}{2} \left(\frac{1}{2} + \frac{1}{2^2} + \dots \right) \right] > \frac{1}{2}x$.

$$\begin{aligned}
\text{Then } |\sigma_2| &\leq - \sum_{[\eta n]+1}^{\infty} \log \left(1 - \frac{r}{\alpha_{n+s}} \right) < l \cdot \sum_{[\eta n]+1}^{\infty} \frac{r}{\alpha_{n+s}} \\
&< l \alpha_{n+1} \sum_{\frac{1}{2}}^{\infty} \frac{1}{\alpha_{n+s}} < l \alpha_{n+1} \sum_{\frac{1}{2}}^{\infty} (n+s)^{-\phi(n+s)} \\
&< l \alpha_{n+1} \sum_{\frac{1}{2}}^{\infty} (n+s)^{-\phi(n+1)} < l(n+1)^{-\phi(n+1)} \int_1^{\infty} (n+x)^{-\phi(n+1)} dx \\
&< l(n+1)^{-\phi(n+1)} \frac{(n+1)^{1-\phi(n+1)}}{\phi(n+1)-1} \\
&< l \frac{n+1}{\phi(n+1)-1} < \frac{4ln}{\phi(n+1)},
\end{aligned}$$

when

$$\phi(n+1) > 2.$$

Hence

$$|\sigma_2| < \frac{4ln}{\phi(n)}. \quad (7)$$

Next let us consider the term $|\bar{S}|$.

The analysis is somewhat similar to the above, and I shall abbreviate it where this is possible.

$$\text{We divide the sum } \bar{S} = \sum_1^{n-1} \log |1 + a_s/z|,$$

$$\text{which may be written } \bar{S} = \sum_1^{n-1} \log |1 + a_{n-s}/z|,$$

into two parts,

$$\sigma_3 = \sum_1^{[\eta n]} \log |1 + a_{n-s}/z|, \quad \text{and} \quad \sigma_4 = \sum_{[\eta n]+1}^{n-1} \log |1 + a_{n-s}/z|,$$

where η is the number defined above.

Consider σ_3 . We have

$$\begin{aligned}
|\sigma_3| &\leq - \sum_1^{[\eta n]} \log \left[1 - \frac{a_{n-s}}{r} \right] \leq \sum_1^{[\eta n]} \log \left[1 / \left(1 - \frac{a_{n-s}}{\alpha_n} \right) \right] \\
&< \sum_1^{[\eta n]} \log \left[1 / \left(1 - \frac{(n-s)^{\phi(n-s)}}{n^{\phi(n)}} \right) \right] \\
&< \sum_1^{[\eta n]} \log \left[1 / \left(1 - \frac{(n-s)^{\phi(n)}}{n^{\phi(n)}} \right) \right] \quad [\text{since } \phi(n) > \phi(n-s)] \\
&< \sum_1^{[\eta n]} \log \left[1 / \left\{ 1 - \left(\frac{n-s}{n} \right)^{\lambda/\eta} \right\} \right] \quad [\text{since } \phi(n) \geq \lambda/\eta].
\end{aligned}$$

$$\text{Now } \log \left[\left(\frac{n}{n-s} \right)^{\lambda/\eta} \right] = - \frac{\lambda}{\eta} \log \left(1 - \frac{s}{n} \right) > \frac{\lambda s}{\eta n} > \frac{\lambda s}{2[\eta n]},$$

when $\lambda > 2$ and $\eta n > 2$.

Hence, reasoning as in the case above, we have

$$1/\left[1-\left(\frac{n-s}{n}\right)^{\lambda/\eta}\right] < \frac{8[\eta n]}{\lambda s} < \frac{16[\eta n]}{\lambda s}.$$

Then
$$|\sigma_3| < \sum_1^{[\eta n]} \log \left[\frac{16[\eta n]}{\lambda(s-1)} \right].$$

This is the same inequality as we obtained for $|\sigma_1|$.

Hence, by repeating the reasoning for that case, we have

$$|\sigma_3| < \text{the greater of } K_1 \text{ and } K_1 n/\phi(n). \quad (8)$$

Now consider σ_4 . We have

$$\sigma_4 = \sum_{s=[\eta n]+1}^{n-N-1} \log |1+a_{n-1}/z| + \sum_{s=1}^N \log |1+a_s/z|,$$

where N is the least integer such that $\phi(s+1) \geq \phi(s)$ when $s \geq N$.

If n be so large that $r \geq a_n > 2a_N$,

$$\left| \sum_1^N \log |1+a_s/z| \right| < \sum_1^N \left| \log \frac{1}{2} \right| < N \log 2. \quad (9)$$

Again, when $s \geq [\eta n] + 1 \geq \eta n$,

$$\begin{aligned} \frac{a_{n-s}}{r} &\leq \frac{a_{n-s}}{a_n} \leq \frac{(n-s)^{\phi(n-s)}}{n^{\phi(n)}} \leq \left(\frac{n-s}{n}\right)^{\phi(n)} \leq \left(\frac{n-\eta n}{n}\right)^{\phi(n)} \leq (1-\eta)^{\phi(n)} \\ &\leq (1-\eta)^{\lambda/\eta} \leq \exp \left[\frac{\lambda}{\eta} \log (1-\eta) \right] < \exp (-\lambda). \end{aligned}$$

Hence, by reasoning similar to that used in the case of σ_2 ,

$$-\log \left(1 - \frac{a_{n-s}}{r} \right) < l' \frac{a_{n-s}}{r},$$

where l' is a finite constant. Then

$$\begin{aligned} \left[\sum_{[\eta n]+1}^{n-N-1} \log \left| 1 + \frac{a_{n-s}}{z} \right| \right] &\leq - \sum_{[\eta n]+1}^{n-N-1} \log \left(1 - \frac{a_{n-s}}{r} \right) < \frac{l'}{r} \sum_{[\eta n]+1}^{n-N-1} a_{n-s} \\ &< \frac{l'}{a_n} \sum_1^{n-N-1} a_{n-s} < l' n^{-\phi(n)} \sum_1^{n-N-1} (n-s)^{\phi(n-s)} \\ &< l' n^{-\phi(n)} \sum_1^{n-N-1} (n-s)^{\phi(n)} \\ &< l' n^{-\phi(n)} \int_0^n (n-x)^{\phi(n)} dx \\ &< l' n^{-\phi(n)} \frac{n^{\phi(n)+1}}{\phi(n)+1} < l' \frac{n}{\phi(n)+1} < l' \frac{n}{\phi(n)}. \quad (10) \end{aligned}$$

Adding (9) and (10), if $\frac{1}{2}K_2$ be the greater of $N \log 2$ and l' , we see that

$$|\sigma_4| < \text{the greater of } K_2 \text{ and } K_2 n / \phi(n). \quad (11)$$

Next consider the term

$$T_1 = \log |1 + z/a_{n+1}|.$$

$|\log |1 + z/a_{n+1}||$ is a maximum for a given n when $|z + a_{n+1}|$ is a maximum or minimum.

Now, by the definitions of the number n , $|z + a_{n+1}|$ has its maximum when $z = a_{n+1}$. We then have

$$|T_1| = \log 2.$$

Again, by the definition of the region R , $|z + a_{n+1}|$ has its minimum value when z lies on the circumference of the circle C_{n+1} , and this value is $\exp[-\chi(n+1)]$.

Then $|T_1| = \chi(n+1) = \text{the greater of } k \text{ and } k(n+1)/\phi(n+1)$

$$< \text{the greater of } 2k \text{ and } 2k/\phi(n). \quad (12)$$

We proceed similarly with $T_2 = \log |1 + a_n/z|$. $|T_2|$ is a maximum when $|1/z + 1/a_n|$ is a maximum or a minimum.

The first case occurs when $z = a_n$, and we then have $|T_2| = \log 2$.

The second case occurs when z is the point on the circumference of C_n which is at the greatest distance from the origin.

In this case, since

$$\left| \frac{z + a_n}{a_n} \right| = \exp[-\chi(n)] \leq e^{-k},$$

so that

$$|z| \leq |a_n|(1 + e^{-k}),$$

$$\begin{aligned} |T_2| &= \left| \left[\log |1 + a_n/z| \right] \right| \leq \left| \log \frac{\exp[-\chi(n)]}{(1 + e^{-k})} \right| \\ &\leq \chi(n) + \log(1 + e^{-k}) \\ &\leq \text{the greater of } 2k + \log(1 + e^{-k}) \text{ and } [2k + \log(1 + e^{-k})]n/\phi(n). \end{aligned} \quad (13)$$

$$\text{Finally, } |\gamma_z| \leq \pi < \text{the greater of } 2\pi \text{ and } 2\pi n/\phi(n). \quad (14)$$

Now, adding the inequalities (6), (7), (8), (11), (12), (13), and (14), we have, if

$$K = K_1 + 4l + K_1 + K_2 + 2k + [2k + \log(1 + e^{-k})],$$

$$\begin{aligned} |\bar{R} + \bar{S} + \log |1 + z/a_{n+1}| + \log |1 + a_n/z| + |\gamma_z| \\ \leq |\sigma_1| + |\sigma_2| + |\sigma_3| + |\sigma_4| + |T_1| + |T_2| + |\gamma_z| \\ < \text{the greater of } K \text{ and } K n / \phi(n). \end{aligned}$$

Then the first part of the theorem is proved.

To prove the second part we recall that, if λ be any assigned positive number < 1 , then, when n is sufficiently large, $\bar{P} > \lambda n \phi(n)$.

Thus on any circle $|z| = r$ which does not cut any circle C_s , *i.e.*, which lies entirely within the region R ,

$$\log F(z) = \bar{P} + Q,$$

where $|Q| < \text{the greater of } K \text{ and } Kn/\phi(n)$.

Now, when n , or r , is sufficiently large,

$$\frac{K}{\lambda n \phi(n)} < \frac{1}{4}\epsilon, \quad \frac{Kn}{\phi(n)} / [\lambda n \phi(n)] < \frac{1}{4}\epsilon.$$

It follows at once that, when r is sufficiently large,

$$\log m(r) > (1 - \epsilon) \log M(r),$$

and therefore $m(r) > [M(r)]^{1-\epsilon}$.

7. The formula $\log F(z) = \bar{P} + Q$ of the last article may be used to approximate to the logarithm of any function of standard type.

In practice it is necessary to approximate to the finite sum $\sum_1^n \log a_s$

which occurs in $\bar{P}$, in terms of n , which is supposed large, to approximate to n as a function of r , and to effect the substitution of n in terms of r .

Both of the above approximations should be carried just so far, when this is possible, that the final approximation for $\bar{P}$ has an error of the order of $n/\phi(n)$, or a finite error, according as we have the inequality $|Q| < Kn/\phi(n)$ or $|Q| < K$.

The order of the terms in r thus obtained ranges from that of K or $Kn/\phi(n)$ as a lower limit (which is not attained), to the order of $\bar{P}$, which varies with different functions from $n\phi(n)$ to $n \log n \phi(n)$ [cf. the inequalities (1), (2), of § 6].

Thus, provided we can effect the approximations spoken of above, we can find some of the higher terms in the asymptotic expression of $\log F(z)$.

It may be remarked that in the case of some functions the expression Q is actually of order $n/\phi(n)$, so that the inequality for Q which constitutes the theorem of the last article is the most precise general inequality which can be obtained on our lines. In the case of certain functions for which $\phi(n)$ is of less order than n , so that $|Q| < Kn/\phi(n)$, Q is of slightly less order than $n/\phi(n)$. The difference, however, is usually irrelevant in practice, so far as finding more terms of $\log F(z)$ is concerned.

Since $\bar{P}$ is independent of the arguments of z and the zeros, the terms in r which we obtain are similarly independent, so that our method gives the same approximation for all functions whose constant term in the Taylor series is 1, and whose zeros have the same sequence of moduli.

The region R , however, to which z is confined, varies with the arguments of the a 's. It contains, however, the region R' which is the part of the plane exterior to every annulus A_s , whose boundaries are the circles with centre at the origin, touching the circle C_s internally and externally.

The region R' is independent of the arguments of the a 's. Hence, when z is confined to R' , a change in the arguments of the zeros can only affect terms of $\log F(z)$ whose order is not greater than the greater of K and $K n/\phi(n)$.

It is clear, from the alternative inequalities which occur in § 6, that there is a change in the behaviour of $\log F(z)$ when $\phi(n)$ is approximately n .

When $\phi(n)$ is of order not less than $n/\log n$, we can obtain a better approximation than that afforded by the theorem of § 6, and we shall in the following three articles consider the approximation to classes of functions of this type. The analysis in these cases is fortunately far simpler than that of § 6.

The theorem of § 6, then, is more particularly appropriate to functions for which $\phi(n)$ is of order less than $n/\log n$.

It is necessary, however, for the cases when $\phi(n)$, although not decreasing, is of irregular growth. It is possible,* *e.g.*, for $\phi(n)$, while being an increasing function, to be of order $n^{\frac{1}{2}}$ for an infinite set of values of n , and to be of order n^2 for another infinite set of values. In this case we should have the inequality $|Q| < K$ for certain ranges of z [*i.e.*, those ranges for which $\phi(n)$ is of order n^2], and for other ranges the inequality $|Q| < K n^{\frac{1}{2}}$.

We shall conclude this article with some remarks on the regions R and R' .

We suppose $\phi(s)$ is of less order than $s/\log s$, and that it is of regular growth (in a rough sense).

The radius ρ_s of the circle C_s is $a_s \exp[-ks/\phi(s)]$.

* Borel, *Leçons sur les Fonctions entières*. Borel constructs a function which we may call $f(x)$, which is of order comparable with e^x for an infinity of values of x as large as we please, and of order comparable with e^{x^2} for another similar range of values. If we take

$$\phi(n) = \frac{1}{n^{\frac{1}{2}}} \log f(n^{\frac{3}{2}}),$$

we have the case above.

The distance of $-a_s$ from the nearest other zero (usually)

$$\geq a_s - a_{s-1} \geq a_s \left[1 - \left(\frac{s-1}{s} \right)^{\phi(s)} \right] \geq a_s \frac{\phi(s)}{s} \quad (\text{approximately}).$$

Then the ratio of ρ_s to the distance from $-a_s$ to the nearest other zero $< \frac{s}{\phi(s)} \exp \left(-k \frac{s}{\phi(s)} \right)$ (approximately).

This tends to zero with $1/s$, and is very small if $\phi(s)$ increases slowly with s .

Thus the region R covers by far the greater part of the plane, and since the circles C_s do not intersect when s is large, as is evident from the above, the point z can move in the region R from any point in R to infinity.

The region R' covers the greater part of the plane, but consists of disconnected annuli.

8. THEOREM.—Let $F(z)$ be a function such that after a certain value N of s ,

$$\frac{\alpha_s}{\alpha_{s+1}} \leq \beta < 1; \tag{1}$$

but no other restriction is placed upon $\phi(s)$ than that implied by (1).

Assign any positive number k , which may be as large as we please.

Let R_1 be the region exterior to every circle C_s ,

$$|z + a_s| = e^{-k} |a_s|.$$

Then, when z is confined to the region R_1 , there is a number K such that

$$\log F(z) = \bar{P} + Q,$$

where

$$|Q| < K.$$

We notice firstly that the class of functions defined by (1) includes functions other than of standard type, and secondly that this theorem gives a better approximation than the theorem of § 6, for functions for which $\phi(n)$ is of order not lower than $n/\log n$, and lower than n .

We have

$$\log F(z) = \bar{P} + \bar{R} + \bar{S} + \log \left| 1 + \frac{z}{a_{n+1}} \right| + \log \left| 1 + \frac{a_n}{z} \right| + \gamma_z.$$

2 c 2

Consider $\bar{R}$. We have

$$\begin{aligned} |\bar{R}| &= \left| \left[\sum_{s=2}^{\infty} \log \left| 1 + \frac{z}{a_{n+s}} \right| \right] \right| \leq - \sum_{s=2}^{\infty} \log \left(1 - \frac{r}{a_{n+s}} \right) \\ &\leq - \sum_{s=2}^{\infty} \log \left(1 - \frac{a_{n+1}}{a_{n+s}} \right) \\ &\leq - \sum_{s=2}^{\infty} \log(1 - \beta^{s-1}) \quad (\text{when } n > N). \quad (2) \end{aligned}$$

Now $\sum \beta^{s-1}$ is convergent. Therefore $\Pi(1 - \beta^{s-1})$ is finite and not zero, and therefore $-\sum_{s=2}^{\infty} \log(1 - \beta^{s-1})$ is finite and $< K_1$.

$$\text{Hence} \quad |\bar{R}| < K_1. \quad (3)$$

$$\begin{aligned} \text{Again,} \quad |\bar{S}| &= \left| \left[\sum_{s=1}^{n-1} \log \left(1 + \frac{a_s}{z} \right) \right] \right| \leq - \sum_{s=1}^{n-1} \log \left(1 - \frac{a_s}{r} \right) \\ &\leq - \sum_{s=1}^{n-1} \log \left(1 - \frac{a_s}{a_n} \right) \\ &\leq - \sum_{s=1}^N \log \left(1 - \frac{a_s}{a_n} \right) - \sum_{s=N+1}^{n-1} \log \left(1 - \frac{a_s}{a_n} \right). \end{aligned}$$

Choose r so large that $r \geq a_n > 2a_N$.

Then, if $s < N$,

$$1 - \frac{a_s}{a_n} > \frac{1}{2}, \quad \text{and} \quad - \sum_{s=1}^N \log \left(1 - \frac{a_s}{a_n} \right) < N \log 2.$$

Then

$$\begin{aligned} |\bar{S}| &\leq N \log 2 - \sum_{s=N+1}^{n-1} \log(1 - \beta^{n-s}) \leq N \log 2 - \sum_{s=1}^{\infty} \log(1 - \beta^s) \\ &< \text{a finite constant } K_2. \end{aligned} \quad (4)$$

We prove, by reasoning as in § 6, that

$$|T_1| = |\log(1 + z/a_{n+1})|, \quad \text{and} \quad |T_2| = |\log(1 + a_n/z)|,$$

are less than finite constants H_1 and H_2 . (5)

$$\text{Finally,} \quad |\iota\gamma_z| \leq \pi. \quad (6)$$

From (3), (4), (5), (6), we have, if $K_3 = K_1 + K_2 + H_1 + H_2 + \pi$,

$$|Q| = |\bar{R} + \bar{S} + T_1 + T_2 + \iota\gamma_z| < K_3.$$

We have so far assumed that n is greater than some finite value, N' suppose.

Let K_4 be the maximum value of Q for all values of z belonging to R , and such that $n < N'$. Let K be greater than K_3 and K_4 .

Then $|Q| < K$, and the theorem is proved.

Taking the exponential of each side of the formula

$$\log F(z) = \bar{P} + Q,$$

we have

$$F(z) = C(z) e^{\bar{P}},$$

where

$$e^K > |C(z)| > e^{-K}.$$

Let R'_1 be the region derived from R_1 in the same manner that R' was derived from R . Then R'_1 is independent of the arguments of the zeros.

Hence, if $F(z) = \Pi(1+z/a_s)$, $F_1(z) = \Pi(1+z/a'_s)$ be two functions of the class defined in the theorem, for which the sequence of the moduli of the zeros is the same, then, if $|z| = r$ lies within R'_1 ,

$$e^{2K} > |F_1(z)/F(z)| > e^{-2K}.$$

Thus $F(z)$ and $F_1(z)$ only differ by a finite factor in the region R'_1 .

If we can express $\bar{P}$ in terms of r , with a finite error, we can find all the large terms in the asymptotic expression for $\log F(z)$.

The ratio of ρ_s , the radius of C_s , to the least possible distance d_s from $-a_s$ to the nearest other zero is greater than in the case considered in § 7.

We have

$$a_s \geq \beta^{-s}.$$

Then (usually) $d_s = a_s - a_{s-1} \geq (1/\beta - 1) \beta^{-s}$,

$$\rho_s = e^{-k} a_s, \quad \rho_s/d_s \leq e^{-k}/(e^{1/\beta} - 1),$$

and is finite, though by choosing k sufficiently large we can make it as large as we please. We can thus ensure that the circles C do not intersect, and that R'_1 exists.

9. THEOREM.—Let $F(z) = \prod_{s=1}^{\infty} (1+z/a_s)$ be a function such that

$$\lim_{s \rightarrow \infty} a_s/a_{s+1} = 0,$$

but no further restriction is placed upon $\phi(s)$.

Let $R_2(\eta)$ be the region of the z -plane which consists of all the annuli $\eta a_{s+1} > |z| > a_s/\eta$, where s is such that $\eta a_{s+1} > a'_s/\eta$, and where η is a positive number < 1 .

Then, if any positive number ϵ be assigned, there is a corresponding

non-zero positive number η_ϵ , such that when z is confined to the region R_{η_ϵ} , and when moreover $|z| > K_\epsilon$, where K_ϵ is a positive number depending on ϵ , we have

$$\log F(z) = P + Q,$$

where

$$P = n \log z - \sum_{s=1}^n \log a_s,$$

and where

$$|Q| < \epsilon.$$

The class of functions considered is evidently contained in the class considered in the last article.

It is evident, however, since we propose in this case to obtain the inequality $|Q| < \epsilon$, that we must take count of the imaginary part of $\log F(z)$, so that $\bar{P}$ will no longer serve as an approximation with the accuracy which we require.

Again, $\log(1+z/a_{n+1})$ is not small when $|z/a_{n+1}|$ is nearly unity, so that the theorem requires a new kind of region R .

We have

$$\log F(z) = P + R + S + \log(1+z/a_{n+1}) + \log(1+a_n/z),$$

where the logarithms in R , S , and in the last two terms are supposed to have their principal values.

Let ϵ_1 , which for the present we suppose less than $\frac{1}{2}$, be a positive number, which we shall presently choose suitably.

Then there exists a ν such that when $s > \nu$,

$$\frac{a_s}{a_{s+1}} < \epsilon_1^2. \quad (1)$$

Then, if z be any point of the region $R(\epsilon_1)$, there is an n such that

$$r/a_{n+1} < \epsilon_1, \quad a_n/r < \epsilon_1, \quad (2)$$

and such that n has its usual meaning.

Now, if $|x| < \frac{1}{2}$, and if $\log(1+x)$ has its principal value,

$$|\log(1+x)| < 2|x|. *$$

$$\text{Moreover} \quad \left| \frac{z}{a_{n+s}} \right| = \frac{r}{a_{n+1}} \frac{a_{n+1}}{a_{n+2}} \dots \frac{a_{n+s-1}}{a_{n+s}} < \epsilon_1^{2s-1} < \frac{1}{2},$$

provided $n > \nu$.

$$\text{Hence} \quad |\log(1+z/a_{n+s})| < 2\epsilon_1^{2s-1} \quad (s = 1, 2, \dots),$$

* $|\log(1+x)| = \left| x - \frac{x^2}{2} + \dots \right| \leq |x| + \frac{|x|^2}{2} + \dots < |x| + |x|^2 \dots < \frac{|x|}{1-|x|} < 2|x|.$

and therefore

$$\left| \log \left(1 + \frac{z}{a_{n+1}} \right) + R \right| \leq \sum_{s=1}^{\infty} \left| \log \left(1 + \frac{z}{a_{n+s}} \right) \right|$$

$$< 2 \sum_{s=1}^{\infty} \epsilon_1^{2s-1} < \frac{2\epsilon_1}{1-\epsilon_1^2} < \frac{3}{2}\epsilon_1. \quad (3)$$

Again, if $n > \nu + 1$,

$$\left| \log \left(1 + \frac{a_n}{z} \right) + S \right| \leq \sum_{s=1}^n \left| \log \left(1 + \frac{a_s}{z} \right) \right|$$

$$\leq \sum_{s=1}^{\nu} \left| \log \left(1 + \frac{a_s}{z} \right) \right| + \sum_{s=\nu+1}^n \left| \log \left(1 + \frac{a_s}{z} \right) \right|. \quad (4)$$

When $s > \nu$, $\left| \frac{a_s}{z} \right| = \frac{a_s}{a_{s+1}} \dots \frac{a_{n-1}}{a_n} \frac{a_n}{r} < \epsilon_1^{2(n-s)} \epsilon_1 < \frac{1}{2}.$

When $s \leq \nu$, $\left| \frac{a_s}{z} \right| < \epsilon_1^{2(n-N)} \epsilon_1 < \frac{1}{2}.$

Then, from (4),

$$\left| \log \left(1 + \frac{a_n}{z} \right) + S \right| < 2 \left[\nu \epsilon_1^{2(n-N)+1} + \sum_{s=\nu+1}^n \epsilon_1^{2(n-s)+1} \right]$$

$$< \epsilon_1 \left[2\nu + \sum_{s=0}^{\infty} \epsilon_1^{2s} \right] < \epsilon_1 \left[2\nu + \frac{1}{1-\epsilon_1^2} \right] < \epsilon_1 \left[2\nu + \frac{3}{2} \right]. \quad (5)$$

Hence, from (3) and (5),

$$|Q| = \left| R + S + \log \left(1 + \frac{z}{a_{n+1}} \right) + \log \left(1 + \frac{a_n}{z} \right) \right| < \epsilon_1 \left[\frac{3}{2} + 2\nu + \frac{3}{2} \right],$$

when $n > \nu + 1$, or when $|z| > \text{a finite } K'_{\epsilon_1}$ depending on ϵ_1 . Choose

$$\epsilon_1 = \epsilon / (3 + 2\nu), \quad K_{\epsilon} = K'_{\epsilon_1}, \quad \eta_{\epsilon} = \epsilon_1.$$

Then, when z is confined to $R(\eta_{\epsilon})$ and $|z| > K_{\epsilon}$, we have $|Q| < \epsilon$, which proves the theorem.

Since $\text{Lt } a_s/a_{s+1} = 0$, when s is sufficiently large, we have $\eta_{\epsilon} a_{s+1} > a_s/\eta_{\epsilon}$, so that the region $R_z(\eta_{\epsilon})$ exists.

The breadth of the annulus which passes between the points a_s and a_{s+1} is large compared with a_s , but small compared with a_{s+1} .

Taking the exponential of each side of $\log F(z) = P + Q$, we have, for large values of z within $R_z(\eta_{\epsilon})$,

$$F(z) = [1 + \epsilon(z)] \frac{z^n}{a_1 a_2 \dots a_n},$$

where

$$|\epsilon(z)| < \exp(\epsilon) - 1 < 2\epsilon.$$

If we can approximate to P as a function of z , with an error which tends to zero with $1/r$, this approximation gives all the large* and finite terms of $\log F(z)$.

10. THEOREM. — Let $F(z) = \prod_{s=1}^{\infty} (1 + z/a_s)$ be a function, such that $\chi(s) = \phi(s)/s$ is an increasing function after some finite value N of s , and such that

$$\lim_{s=\infty} \chi(s) = \infty.$$

We define the region R_3 as follows.

Assign a positive number k such that $0 < k < \frac{1}{2}$, and where k may be taken as small as we please.

Then R_3 is the region of the plane which consists of all the annuli

$$(s+k)^{\phi(s)} < |z| < (s+1-k)^{\phi(s+1)}.$$

Then when z is confined to the region R_3 , and when $r >$ some finite constant H ,

$$\log F(z) = P + Q,$$

where

$$|Q| < K \exp[-\frac{1}{2}k\chi(n)],$$

K being a finite constant.

Since z lies in R_3 , there is an integer n , such that $r = (n+\theta)^{\phi(n)}$ ($\theta > k$) and $r = (n+\theta')^{\phi(n+1)}$ ($\theta' < 1-k$). Moreover, n evidently has its proper meaning.

We suppose r so large that $n > N$. Then

$$\phi(n+s) \geq \frac{n+s}{n+s-1} \phi(n+s-1) > \phi(n+s-1).$$

We have

$$\begin{aligned} \frac{r}{a_{n+1}} &= \left(\frac{n+\theta'}{n+1}\right)^{\phi(n+1)} < \left(\frac{n+1-k}{n+1}\right)^{\phi(n+1)} < \left[\left(1-\frac{k}{n+1}\right)^{n+1}\right]^{\lceil \phi(n+1) \rceil / (n+1)} \\ &< [e^{-k}]^{\chi(n+1)} < e^{-k\chi(n)} < e^{-\frac{1}{2}k\chi(n)}. \end{aligned} \quad (1)$$

* I shall use the phrase "large terms" to mean "terms tending to infinity with $|z|$," "small terms" to mean "terms tending to zero with $1/|z|$," and "finite terms" to mean "terms whose moduli lie between fixed finite limits."

† We have $\log\left(1 - \frac{k}{n+1}\right)^{n+1} = (n+1) \log\left(1 - \frac{k}{n+1}\right) < -k.$

Similarly

$$\frac{\alpha_n}{r} = \left(\frac{n}{n+\theta}\right)^{\phi(n)} < \left(\frac{n}{n+k}\right)^{\phi(n)} < \left[\left(1+\frac{k}{n}\right)^n\right]^{-\chi(n)} < (e^{\frac{1}{2}k})^{-\chi(n)*} < e^{-\frac{1}{2}k\chi(n)}. \quad (2)$$

Let N_1 be the integer such that $\chi(N_1+s) > k$ for all values of s , and N' the greater of N and N_1 .

Then, when $s > N'$,

$$\frac{\alpha_s}{\alpha_{s+1}} < \frac{s^{\phi(s)}}{(s+1)^{\phi(s+1)}} < \left(\frac{s}{s+1}\right)^{\phi(s)} < \left[\left(1+\frac{1}{s}\right)^s\right]^{-\chi(s)} < (e^{\frac{1}{2}})^{-k} < e^{-\frac{1}{2}k}. \quad (3)$$

Suppose $n > N'$, and, further, so large that $e^{-\frac{1}{2}k\chi(n)} < \frac{1}{2}$. Then, from (1), (2), and (3), if $\lambda = \exp(-\frac{1}{2}k)$,

$$\frac{r}{\alpha_{n+s}} < \lambda^{\chi(n)+s} < \frac{1}{2};$$

and, when $s > N'$,
$$\frac{\alpha_s}{r} < \lambda^{\chi(n)+(n-s)} < \frac{1}{2}.$$

Also when $s \leq N'$,
$$\frac{\alpha^s}{r} < \frac{\alpha_{N'}}{r} < \lambda^{\chi(n)+(n-N')}.$$

Hence
$$\left|\log\left(1+\frac{z}{\alpha_{n+s}}\right)\right| < 2\left|\frac{r}{\alpha_{n+s}}\right| < 2\lambda^{\chi(n)+s};$$

when $s > N'$,
$$\left|\log\left(1+\frac{\alpha_s}{z}\right)\right| < 2\left|\frac{\alpha_s}{z}\right| < 2\lambda^{\chi(n)+(n-s)};$$

and, when $s \leq N'$,
$$\left|\log\left(1+\frac{\alpha_s}{z}\right)\right| < 2\left|\frac{\alpha_s}{z}\right| < 2\lambda^{\chi(n)+(n-N')}.$$

Then $|Q| = \left|\left\{\log\left(1+\frac{z}{\alpha_{n+1}}\right)+R\right\}+\left\{\log\left(1+\frac{\alpha_n}{z}\right)+S\right\}\right|$

$$\leq \sum_{s=1}^{\infty} \left|\log\left(1+\frac{z}{\alpha_{n+s}}\right)\right| + \sum_{s=1}^n \left|\log\left(1+\frac{\alpha_s}{z}\right)\right|$$

$$< 2\lambda^{\chi(n)} \left[\sum_{s=1}^{\infty} \lambda^s + \sum_{s=1}^{n-N'} \lambda^s + N' \lambda^{n-N'} \right]$$

$$< 2 \left[\frac{2}{1-\lambda} + N \lambda^{-N} \right] \lambda^{\chi(n)} < K e^{-\frac{1}{2}k\chi(n)},$$

where
$$K = 2 \left[\frac{2}{1-\lambda} + N' \lambda^{-N'} \right].$$

* We have $\log\left(1+\frac{k}{n}\right)^n > n \log\left(1+\frac{k}{n}\right) > n\left(\frac{1}{2}\frac{k}{n}\right)$, since $\frac{k}{n} < \frac{1}{2n} < \frac{1}{2}$.

We have supposed $n > N'$: this is equivalent to supposing r greater than a finite constant.

Thus the theorem is proved.

It is possible, by finding closer inequalities than (1) and (2), to show that

$$|Q| < K' \exp[-k(1-\epsilon)\chi(n)]$$

when r is sufficiently large, where ϵ is as large as we please. The difference is unimportant in practice.

If we can calculate P in terms of z , we have an expression which represents $\log F(z)$ asymptotically with an error of order $e^{-\chi(n)}$.

It may happen that P can be asymptotically expanded as in a divergent series of descending powers of some function of r , and that the error committed by stopping at any particular term is always of greater order than $e^{-\chi(n)}$. In this case, the asymptotic expansion for P obtains for $\log F(z)$. Such an example occurs in § 15.

11. It will have been observed that in §§ 6, 8, 9, 10 the terms

$$\left| \log \left(1 + \frac{z}{a_{n+1}} \right) \right|, \quad \left| \log \left(1 + \frac{a_n}{z} \right) \right|$$

satisfy an inequality precisely similar to that satisfied by $|Q|$; indeed, the various regions R and R' are defined so that this may be the case.

It is easily seen that in the case of any function $F(z)$, which belongs to one of the classes considered in the above articles, we have, in the cases of §§ 6 and 8,

$$\log F(z) = \bar{P} + \log \left| 1 + \frac{z}{a_{n+1}} \right| + \log \left| 1 + \frac{a_n}{z} \right| + Q_1,$$

where Q_1 satisfies the same inequality* as Q , and where z may have any value whatever; and, in the cases of §§ 9 and 10,

$$\log F(z) = P + \log \left(1 + \frac{z}{a_{n+1}} \right) + \log \left(1 + \frac{a_n}{z} \right) + Q_1,$$

where Q_1 satisfies the same inequalities as Q , and z may have any value (subject only, in the case of § 9, to the condition that $|z|$ is greater than a certain value).

These formulæ enable us to study the behaviour of $F(z)$ near its zeros

* That is, of course, when K is suitably rechosen.

and, in the cases of the functions of §§ 9 and 10, near the circles with their centres at the origin, and passing through the zeros: *e.g.*, if $F(z)$ be one of the functions of § 8, we have, near the zero $-a_n$,

$$\log F(z) = \bar{P} + \log \left(1 + \frac{a_n}{z} \right) + Q_1,$$

where $|Q_1| < \text{a finite } K$.

12. We shall now apply our general methods and results to the investigation of some particular functions.

The simplest function of zero order is

$$F(z) = \prod_{s=1}^{\infty} \left(1 + \frac{z}{e^{s\omega}} \right) \quad (\Re \omega > 0).$$

$F(z)$ evidently belongs to the class of functions considered in § 8. This function has been considered by M. Mellin,* Dr. Barnes,† and Mr. Hardy.‡

M. Mellin obtains a result, which, written in our notation, is as follows :—

$$\begin{aligned} \log F(z) + \log \left[\left(1 + \frac{1}{z} \right) F \left(\frac{1}{z} \right) \right] \\ = \frac{(\log z)^2}{2\omega} - \frac{1}{2} \log z + \left(\frac{\pi^2}{6\omega} + \frac{\omega}{12} \right) - \sum_{m=1}^{\infty} \frac{z^{2m\pi i/\omega} + z^{-2m\pi i/\omega}}{2m \sinh \left(\frac{2m\pi^2}{\omega} \right)}. \end{aligned} \quad (1)$$

$$\text{When } |z| > 1, \quad \log \left[\left(1 + \frac{1}{z} \right) F \left(\frac{1}{z} \right) \right] = - \sum_{s=1}^{\infty} \frac{(-)^s z^{-s}}{s(1 - e^{-s\omega})},$$

and we have

$$\begin{aligned} \log F(z) \\ = \frac{(\log z)^2}{2\omega} - \frac{1}{2} \log z + \left(\frac{\pi^2}{6\omega} + \frac{\omega}{12} \right) - \left\{ \sum_{m=1}^{\infty} \frac{z^{2m\pi i/\omega} + z^{-2m\pi i/\omega}}{2m \sinh \frac{2m\pi^2}{\omega}} \right\} + \sum_{s=1}^{\infty} \frac{(-)^s z^{-s}}{s(1 - e^{-s\omega})}. \end{aligned} \quad (2)$$

When ω is real, the term in crooked brackets

$$= \sum_{m=1}^{\infty} \frac{\cos \left(\frac{2m\pi \log z}{\omega} \right)}{2m \sinh \frac{2m\pi^2}{\omega}},$$

and is finite.

* *Acta Soc., Fenn.*, t. xxix.
† *Phil. Trans.*, Ser. A, Vol. 199; *Camb. Phil. Trans.*, Vol. xix., pp. 333-335 and 433-435.
‡ *Quarterly Journal of Mathematics*, 1905.

Hence, when ω is real, (2) affords a complete (convergent) asymptotic expansion.

It happens in the case of this function (with ω complex), that our methods enable us to find a complete asymptotic expansion.

The case is quite exceptional, but it exemplifies to some extent the general procedure.

Let $\omega = \omega_1 + i\omega_2$. n is the first integer for which $|z^{-1}e^{(n+1)\omega}| > 1$, or for which $(n+1)\omega_1 > \log r$.

$$\text{Let} \quad (n+1)\omega_1 = \log r + \beta\omega_1, \quad \text{where } 0 < \beta < 1. \quad (4)$$

$$\text{Then} \quad (n+1)\omega = \log r + \beta\omega_1 + (n+1)\omega_2 i = \log z + a\omega_1,$$

where

$$a = \beta + \frac{(n+1)\omega_2 - \theta}{\omega_1} i \quad (\theta = \arg z)$$

and

$$z = e^{(n+1)\omega - a\omega_1}. \quad (5)$$

One value of $\log F(z)$

$$= \left\{ n \log z - \sum_1^n s\omega \right\} + \sum_1^\infty \log \left(1 + \frac{z}{e^{(n+s)\omega}} \right) + \sum_1^n \log \left(1 + \frac{e^{s\omega}}{z} \right). \quad (6)$$

$$\text{Now} \quad \sum_1^\infty \log \left(1 + \frac{z}{e^{(n+s)\omega}} \right) = \sum_1^\infty \log \left(1 + \frac{e^{\omega - a\omega_1}}{e^{s\omega}} \right) = \log F(e^{\omega - a\omega_1}), \quad (7)$$

$$\begin{aligned} \sum_1^n \log \left(1 + \frac{e^{s\omega}}{z} \right) &= \sum_1^n \log \left(1 + \frac{e^{a\omega_1}}{e^{(n-s+1)\omega}} \right) = \sum_1^n \log \left(1 + \frac{e^{a\omega_1}}{e^{s\omega}} \right) \\ &= \sum_1^\infty \log \left(1 + \frac{e^{a\omega_1}}{e^{s\omega}} \right) - \sum_{n+1}^\infty \log \left(1 + \frac{e^{a\omega_1}}{e^{s\omega}} \right) \\ &= \log F(e^{a\omega_1}) - \sum_1^\infty \log \left(1 + \frac{1}{e^{(n+1)\omega - a\omega_1} e^{(s-1)\omega}} \right) \\ &= \log F(e^{a\omega_1}) - \sum_0^\infty \log \left(1 + \frac{1}{ze^{s\omega}} \right). \end{aligned} \quad (8)$$

$$P = \left\{ n \log z - \sum_1^n s\omega \right\} = n \log z - \frac{1}{2}\omega n(n+1)$$

$$= \frac{\log z}{\omega} (\log z + a\omega_1 - \omega) - \frac{1}{2\omega} (\log z + a\omega_1) (\log z + a\omega_1 - \omega)$$

[on substituting for n from (5)]

$$= \frac{1}{2\omega} (\log z)^2 - \frac{1}{2} \log z - \frac{a\omega_1}{2\omega} (a\omega_1 - \omega). \quad (9)$$

We have for α ,

$$\begin{aligned}\alpha &= \beta - \frac{\theta_1}{\omega_1} + \frac{\omega_2 \iota}{\omega_1^2} \omega_1 (n+1) = \beta - \frac{\theta_1}{\omega_1} + \frac{\omega_2 \iota}{\omega_1^2} (\log r + \beta \omega_1) \\ &= \frac{\omega_2}{\omega_1^2} \log z + \beta - \frac{\theta_1}{\omega_1} + \frac{\omega_2 \iota}{\omega_1} \beta + \frac{\omega_2}{\omega_1^2} \theta,\end{aligned}$$

$$\text{or} \quad \alpha = \frac{\omega_2}{\omega_1^2} \log z + \frac{\omega \beta}{\omega_1} - \frac{\omega}{\omega_1^2} \theta, \quad (10)$$

where $0 < \beta < 1$.

Adding (6), (7), and (8), we have

$$\begin{aligned}\log F(z) &= \left[\frac{1}{2\omega} (\log z)^2 - \frac{1}{2} \log z - \frac{a\omega_1}{2\omega} (a\omega_1 - \omega) \right] \\ &\quad + \log F(e^{a\omega_1}) + \log F(e^{\omega - a\omega_1}) + \sum_1^{\infty} \frac{(-)^{s-1}}{s(1 - e^{-s\omega})} \frac{1}{z^s}, \quad (11)\end{aligned}$$

on substituting for $\sum_0^{\infty} \log(1 + z^{-1} e^{-s\omega})$ its expansion in descending powers of z . α is given as a function of $\log z_1$, and of β and θ (which are finite) by equation (10).

Now $R\alpha = \beta$. Hence $|e^{a\omega_1}|$ and $|e^{\omega - a\omega_1}|$ are equal to $e^{\beta\omega_1}$ and $e^{(1-\beta)\omega_1}$ respectively. If $0 < \beta < 1$, these are both $< e^{\omega_1} < |\text{the first zero of } F(z)|$.

If, then, $0 < \beta < 1$, both limits being excluded, $F(e^{a\omega_1})$ and $F(e^{\omega - a\omega_1})$ are finite both ways. Then their logarithms are finite.

Thus in equation (11) the terms in square brackets give the large terms, and the other terms are finite or small.

Substituting for α from (10), we obtain, from (11),

$$\begin{aligned}\log F(z) &= \frac{\omega_1^2 + \omega_2^2}{2\omega_1^2 \omega} (\log z)^2 - \frac{1}{2\omega_1} \left\{ \omega_1 + \omega_2(2\beta - 1) + \frac{2\theta\omega_2}{\omega_1} \right\} (\log z) \\ &\quad + \left[\frac{1}{2} \left(\beta - \frac{\iota\theta}{\omega_1} \right)^2 - \frac{1}{2\omega} \left(\beta - \frac{\iota\theta}{\omega_1} \right)^2 + \log F(e^{a\omega_1}) + \log F(e^{\omega - a\omega_1}) \right] \\ &\quad + \sum_1^{\infty} \frac{(-)^{s-1}}{s(1 - e^{-s\omega})} \frac{1}{z^s}, \quad (12)\end{aligned}$$

where the terms in square brackets are the finite terms.

When ω is real, *i.e.*, $\omega_2 = 0$, this expression simplifies considerably.

Put $\omega = \omega_1 = 1$, $\omega_2 = 0$.

$\log F(z)$

$$\begin{aligned}&= \frac{1}{2} (\log z)^2 - \frac{1}{2} \log z + \left[\frac{1}{2} (\beta - \iota\theta) - \frac{1}{2} (\beta - \iota\theta)^2 + \log F(e^{\beta - \iota\theta}) + \log F(e^{(1-\beta) - \iota\theta}) \right] \\ &\quad + \sum_1^{\infty} \frac{(-)^{s-1}}{s(1 - e^{-s})} \frac{1}{z^s}.\end{aligned} \quad (13)$$

Comparing this with (3), we see that the large and the small terms agree.

We have chosen a different determination of the logarithm of $F(z)$ from that of M. Mellin.

M. Mellin's $\log F(z)$ is equal to $\sum_1^{\infty} \log(1+ze^{-s\omega})$, each logarithm on the right-hand side having its principal value (*i.e.*, having the modulus of its imaginary part less than π).

Our determination is that implied by equation (6.) Thus the finite terms in (3) and (13) may differ by some multiple of $2\pi i$. We may notice that it was quite possible, *a priori*, that the large terms should differ in the same way.

We have expressed $\log F(z)$, in the general case of ω complex, in terms of $\log z$ rather than in terms of $\log r$, in order to exhibit the correspondence between our form and M. Mellin's.

Written in this way, the large terms have imaginary parts which do not really count as large terms.

Let us find the expression in terms of $\log r$.

We have

$$P = n \log z - \frac{1}{2} \omega n(n+1) \\ = \frac{1}{\omega_1} [\log r - (1-\beta)\omega_1] (\log r + i\theta) - \frac{\omega_1 + i\omega_2}{2\omega_1^2} (\log r + \beta\omega_1) (\log r + \beta\omega_1 - \omega_1),$$

which reduces to

$$P = \frac{1}{2\omega_1} (\log r)^2 - \frac{1}{2} \log r \\ + \left[\frac{\beta(1-\beta)\omega_1}{2} + \frac{i}{2\omega_1^2} \{ [\log r + (\beta-1)\omega_1] (2\theta\omega_1 - \beta\omega_1\omega_2 - \omega_2 \log r) \} \right], \quad (15)$$

where the terms in square brackets count as finite. The remaining terms of $\log F(z)$ may be written as before, and we have

$$\log F(z) = P + \log \{ F(e^{a\omega_1}) F(e^{\omega - a\omega_1}) \} + \sum_1^{\infty} \frac{(-)^{s-1}}{s(1-e^{-s\omega})z^s}, \quad (16)$$

where P is given by equation (15).

Thus we see that the large terms of $\log F(z)$ are $(\log r)^2/(2\omega_1) - \frac{1}{2} \log r$.

Finally, suppose $|z|$ tends either to $|e^{n\omega}|$ or to $|e^{(n+1)\omega}|$. Then β tends to 1 or 0. But, if z tends to a point on the circle $|z| = |e^{n\omega}|$ other than the zero $-e^{n\omega}$, the right-hand sides of (12) and (15) are continuous, and so are their left-hand sides. Hence the equations (12) and (15) are valid when $\beta = 1$ or 0, provided z be not actually at the corresponding zero $-e^{n\omega}$ or $-e^{(n+1)\omega}$.

13. The comparison of the equation (11) of the last article with M. Mellin's result (1), leads to certain other formulæ. We have

$$\left(\frac{\pi^2}{6\omega} + \frac{\omega}{12}\right) - \sum_{m=1}^{\infty} \frac{z^{2m\pi\iota/\omega} + z^{-2m\pi\iota/\omega}}{2m \sinh\left(\frac{2m\pi^2}{\omega}\right)} \\ = \log[F(e^{a\omega_1}) F(e^{\omega-a\omega_1})] - \frac{a\omega_1}{2\omega} (a\omega_1 - \omega), \quad (1)$$

where the $2k\pi\iota$ is supposed absorbed into the logarithm, and where $z^{2m\pi\iota/\omega}$ is interpreted to mean $\exp(2m\pi\iota\omega^{-1} \log z)$, where

$$|\arg z| < \pi, \quad \log z = \log |z| + \iota \arg z.$$

Let $(1+1/z) F(z) F(1/z) = \phi(z), \quad \phi(e^{\iota\theta}) = \psi(\theta);$

so that $\phi(1/z) = z\phi(z).$

Retaining the usual signification for n , we have

$$\alpha = \beta + \frac{(n+1)\omega_2 - \theta}{\omega_1} \iota,$$

$$e^{\omega-a\omega_1} = \exp[(1-\beta)\omega_1 - (\theta - n\omega_2)\iota],$$

$$e^{a\omega_1} = e^{\omega} \exp[-(1-\beta)\omega_1 + (\theta - n\omega_2)\iota].$$

Then

$$F(e^{a\omega_1}) F(e^{\omega-a\omega_1}) = \phi[e^{(1-\beta)\omega_1 - (\theta - n\omega_2)\iota}] = e^{-(1-\beta)\omega_1 + (\theta - n\omega_2)\iota} \phi[e^{-(1-\beta)\omega_1 + (\theta - n\omega_2)\iota}].$$

Substituting in the last term of the right-hand side of (1) the value of α in terms of $\log z$, i.e., $\omega_2/\omega_1^2 \log z + [\omega(\beta\omega_1 - \iota\theta)]/\omega_1^2$, and using (2), we obtain, after some reduction,

$$\sum_{m=1}^{\infty} \frac{z^{2m\pi\iota/\omega} + z^{-2m\pi\iota/\omega}}{2m \sinh\left(\frac{2m\pi^2}{\omega}\right)} \\ = -\frac{\omega_2^2}{2\omega\omega_1^2} (\log z)^2 + \frac{\iota\omega_2}{2\omega_1^2} \{(2\beta+1)\omega_1 - 2\iota\theta\} \log z \\ + \left[\frac{\omega}{2\omega_1^2} (\beta\omega_1 - \iota\theta) [(\beta-1)\omega_1 - \iota\theta] - (1-\beta)\omega \right. \\ \left. - \log \phi\{e^{-(1-\beta)\omega_1 + (\theta - n\omega_2)\iota}\} + \left(\frac{\pi^2}{6\omega} + \frac{\omega}{12}\right) \right], \quad (3)$$

where the term in square brackets is finite, and n and β are determined by $0 < \beta < 1$, $\log |z| = \omega_1(n+1-\beta)$.

The equation (3) gives an asymptotic expression for the left-hand side. We notice that the real part of the right-hand side tends to $-\infty$ as $|z|$ tends to ∞ .

Next suppose that ω is real and $|z| = e^{(n+1)\omega}$. Then $\omega = \omega_1$, $\omega_2 = 0$, $\beta = 1$. The formula (3) is still valid, provided $|\theta| < \pi$.

We have

$$z^{2m\pi i/\omega} = \exp \left\{ \frac{2m\pi i}{\omega} \log z \right\} = \exp \left[\frac{2m\pi i}{\omega} \{ (n+1)\omega + i\theta \} \right] = \exp \left(-\frac{2m\pi\theta}{\omega} \right),$$

$\frac{1}{2}(z^{2m\pi i/\omega} + z^{-2m\pi i/\omega}) = \cosh \left(\frac{2m\pi\theta}{\omega} \right)$, and $\phi \{ e^{-(1-\beta)\omega_1 + (\theta - n\omega_2)i} \}$ becomes $\phi(e^{i\theta})$ or $\psi(\theta)$. Thus (3) becomes

$$\sum_{n=1}^{\infty} \frac{\cosh \left(\frac{2m\pi\theta}{\omega} \right)}{n \sinh \left(\frac{2m\pi}{\omega} \right)} = -\frac{\theta^2}{2\omega} + \left(\frac{\pi^2}{6\omega} + \frac{\omega}{12} \right) - \left[\log \psi(\theta) + \frac{i\theta}{2} \right]. \quad (4)$$

$$\begin{aligned} \text{Now } e^{\frac{1}{2}i\theta} \psi(\theta) &= e^{\frac{1}{2}i\theta} (1 + e^{-i\theta}) \prod_1^{\infty} \left\{ \left(1 + \frac{e^{i\theta}}{e^{s\omega}} \right) \left(1 + \frac{e^{-i\theta}}{e^{s\omega}} \right) \right\} \\ &= 2 \cos \frac{1}{2}\theta \prod_1^{\infty} (1 + 2q^{2s} \cos \theta + q^{4s}), \end{aligned}$$

on putting $q = e^{-\frac{1}{2}\omega}$.

In the notation of MM. Tannery and Molk,*

$$\begin{aligned} q_0 &= \prod_1^{\infty} (1 - q^{2s}), \\ \mathfrak{S}_2(x) &= 2q_0 q^{\frac{1}{2}} \cos \pi x \prod_1^{\infty} \{ 1 + 2q^{2s} \cos 2\pi x + q^{4s} \}. \end{aligned}$$

$$\text{Then } \mathfrak{S}_2(x) = q_0 q^{\frac{1}{2}} e^{i\pi x} \psi(2\pi x), \quad \log q_0 = - \sum_{s=1}^{\infty} \frac{q^{2s}}{s(1-q^{2s})}.$$

Then, from (4), we obtain

$$\log \mathfrak{S}_2(x) = \left(\frac{\pi^2}{6\omega} - \frac{\omega}{24} \right) - \sum_{s=1}^{\infty} \frac{q^{2s}}{s(1-q^{2s})} - \frac{2\pi^2 x^2}{\omega} - \sum_{m=1}^{\infty} \frac{\cosh \left(\frac{4m\pi^2 x}{\omega} \right)}{\sinh \left(\frac{2m\pi^2}{\omega} \right)}, \quad (5)$$

where $\omega = 2 \log(1/q)$.

This expansion is valid when x and q are real, $0 < q < 1$, $|x| < \frac{1}{2}$. The series involving x is highly convergent when ω is small, i.e., when q is nearly unity.

When $|x| < \frac{1}{2}$ and q is nearly unity, i.e., ω is small, we have

$$\log [\mathfrak{S}_2(x)/q_0] = \left(\frac{\pi^2}{12} - \pi^2 x^2 \right) \frac{1}{\log(1/q)} + \epsilon(q),$$

where $\epsilon(q)$ tends to 0 as q tends to 1.

* *Éléments de la Théorie des Fonctions elliptiques*, t. II.

14. In the case of functions $F(z)$, such that $\lim_{s \rightarrow \infty} a_s/a_{s+1} = e^{-\omega}$, $\Re \omega > 0$, we can express the finite term of $\log F(z)$ by means of the function $\phi(x)$ defined in the last article.

THEOREM.—Let $F(z) = \prod_{s=1}^{\infty} (1+z/a_s)$, and let $\lim_{s \rightarrow \infty} a_s/a_{s+1} = e^{-\omega}$, and suppose z is confined to the region R .

Then, when any positive ϵ is assigned, there is a constant K_ϵ , such that, when $|z| > K_\epsilon$, we have

$$\log F(z) = P + \log \phi(z/a_n) + Q',$$

where $|Q'| < \epsilon$.

Let η be a small positive number, which we shall presently choose suitably. Then there is a number N , such that, when $s > N$,

$$\frac{a_s}{a_{s+1}} = e^{-\omega} (1 + \eta_s),$$

where $|\eta_s| < \eta$.

Suppose r so large that $n > N$, and let $\omega = \omega_1 + i\omega_2$. Then, if $s \geq 1$,

$$\begin{aligned} \left| \left(1 + \frac{z}{a_n} e^{-s\omega} \right) / \left(1 + \frac{z}{a_{n+s}} \right) - 1 \right| &= \left| \frac{\left(e^{-s\omega} \frac{z}{a_n} \right) \left(1 - \frac{a_n e^{s\omega}}{a_{n+s}} \right)}{1 + z/a_{n+s}} \right| \\ &\leq \left[e^{-s\omega_1} \left| \frac{a_{n+1}}{a_n} \right| \right] \left| \frac{1 - (1 + \eta_n)(1 + \eta_{n+1}) \dots (1 + \eta_{n+s-1})}{e^{-k}} \right| \\ &< e^{-s\omega_1} e^{\omega_1} (1 + \eta) \left| \frac{1 - (1 + \eta)^s}{e^{-k}} \right|. \end{aligned} \quad (1)$$

Now $(1 + \eta)[(1 + \eta)^s - 1] = (1 + \eta)[1 + (1 + \eta) + \dots + (1 + \eta)^{s-1}](1 + \eta - 1)$
 $< s\eta(1 + \eta)^s < s\eta \exp[s \log(1 + \eta)] < s\eta \exp(s\eta)$.

Choose $\eta < \frac{1}{2}\omega_1$. Then, from (1),

$$\left| \left(1 + \frac{z}{a_n} e^{-s\omega} \right) / \left(1 + \frac{z}{a_{n+s}} \right) - 1 \right| < \eta e^{k + \omega_1 s} e^{-\frac{1}{2}s\omega_1}. \quad (2)$$

Now $\lim_{s \rightarrow \infty} s e^{-\frac{1}{2}s\omega_1} = 0$, so that $s e^{-\frac{1}{2}s\omega_1}$ has a maximum h .

Choose η so that $\eta \cdot h e^{k + \omega_1} < \frac{1}{2}$.

Then, from (2),

$$\left| \log \left(1 + \frac{z}{a_n} e^{-s\omega} \right) - \log \left(1 + \frac{z}{a_{n+s}} \right) \right| < \left| \log (1 - \eta e^{k+\omega_1} s e^{-\frac{1}{2}s\omega_1}) \right| \\ < 2\eta e^{k+\omega_1} s e^{-\frac{1}{2}s\omega_1};$$

and therefore

$$\left| \sum_{s=1}^{\infty} \log \left(1 + \frac{z}{a_{n+s}} \right) - \sum_{s=1}^{\infty} \log \left(1 + \frac{z}{a_n} \right) \right| < 2\eta e^{k+\omega_1} \sum_{s=1}^{\infty} s e^{-\frac{1}{2}s\omega_1} < K'\eta, \quad (3)$$

since both series on the left-hand side are uniformly convergent, and since $\sum s e^{-\frac{1}{2}s\omega_1}$ is convergent.

Again, we can show in a similar manner that

$$\left| \sum_{s=0}^{n-N} \log \left(1 + \frac{a_{n-s}}{z} \right) - \sum_{s=0}^{n-N} \log \left(1 + \frac{a_n}{z} e^{-s\omega} \right) \right| < K''\eta. \quad (4)$$

Now choose n so large that

$$\left| \frac{a_N}{z} \right| < \eta < \frac{1}{2}, \quad e^{-(n-N)\omega_1} < \eta < \frac{1}{2}.$$

Then, when $s \geq n-N$,

$$\left| \log \left(1 + \frac{a_{n-s}}{z} \right) \right| \leq \left| \log \left(1 - \frac{a_N}{r} \right) \right| < 2 \frac{a_N}{r} < 2\eta;$$

and, similarly,
$$\left| \log \left(1 + \frac{a_n}{z} e^{-s\omega} \right) \right| < 2\eta.$$

Hence

$$\left| \sum_{s=n-N+1}^{n-1} \log \left(1 + \frac{a_{n-s}}{z} \right) - \sum_{s=n-N+1}^{n-1} \log \left(1 + \frac{a_n}{z} e^{-s\omega} \right) \right| \\ < \sum_{s=n-N+1}^{n-1} (2\eta + 2\eta) < 4N \cdot \eta. \quad (5)$$

Finally, when $s > n$,

$$\log \left(1 + \frac{a_n}{z} e^{-s\omega} \right) < 2 \left| \frac{a_n}{z} e^{-s\omega} \right| < 2e^{-n\omega_1} e^{-(s-n)\omega_1},$$

so that
$$\left| \sum_{s=n}^{\infty} \log \left(1 + \frac{a_n}{z} e^{-s\omega} \right) \right| < 2e^{-(n-1)\omega_1} \sum_{s=n+1}^{\infty} e^{-(s-n+1)\omega_1} < \frac{2e^{-n\omega_1}}{1-e^{-\omega_1}}.$$

Choose n so large that this last expression is less than η . Then

$$\left| \sum_{s=n}^{\infty} [0] - \sum_{s=n}^{\infty} \log \left(1 + \frac{a_n}{z} e^{-s\omega} \right) \right| < \eta. \quad (6)$$

Adding (3), (4), (5), and (6), we have

$$\begin{aligned} & \left| \left[\sum_{s=1}^{\infty} \log \left(1 + \frac{z}{a_{n+s}} \right) + \sum_{s=0}^{n-1} \log \left(1 + \frac{a_{n-s}}{z} \right) \right] \right. \\ & \quad \left. - \left[\sum_{s=1}^{\infty} \log \left(1 + \frac{z}{a_n} e^{-s\omega} \right) + \sum_{s=0}^{\infty} \left(1 + \frac{a_n}{z} e^{-s\omega} \right) \right] \right| \\ & < (K' + K'' + 4N + 1) \eta < K \eta, \end{aligned}$$

or
$$| [\log F(z) - P] - [\log \phi(z/a_n)] | < K \eta.$$

Choose $\eta = \epsilon/K$. Then, when n is greater than some finite number depending on η , or when r is greater than some number K_ϵ ,

$$\log F(z) = P + \log \phi(z/a_n) + Q',$$

where
$$| Q' | < \epsilon.$$

As an example, consider

$$F(z) = \prod_{s=1}^{\infty} \left[1 + \frac{z}{(a+sv)^k e^{s\omega}} \right],$$

where k is real, $\Re \omega > 0$, and a and v are any complex numbers.

We evidently have $\lim_{s \rightarrow \infty} a_s/a_{s+1} = e^{-\omega}$, so that the formula of the theorem holds in this case.

n is determined by

$$\log |[a + (n+1)v]^k e^{(n+1)\omega}| > \log r \geq \log |(a+nv)^k e^{n\omega}|,$$

whence n is determined as a function of r by the relation

$$n\omega_1 + k \log |a + vn| - \log r = -a\omega_1, \quad (1)$$

where
$$0 \leq \alpha < 1 + \frac{k}{\omega_1} \left| \frac{va}{v+a} \right| \frac{1}{n}, \quad (2)$$

so that α is a function of r .

We approximate to P as a function of n , retaining terms of order not less than $1/n$, and to n as a function of r and α , to the same order of approximation.

Substituting from the second approximation in the first, we obtain

$$\begin{aligned} \log F(z) &= \left[\frac{1}{2\omega_1} (\log r)^2 - \frac{k}{\omega_1} \log r \log_2 r + (c - \tfrac{1}{2}) \log r + \frac{k^2}{2\omega_1} (\log_2 r)^2 - k(b+c) \log_2 r \right] \\ &\quad + [\omega_1 \{c^2 - c + a(1-a)\} + k \{ \log \Gamma(1+a/v) + 1 - \tfrac{1}{2} \log 2\pi \}] \\ &\quad - i \left\{ \frac{\omega_2}{2\omega_1^2} \left[(\log r)^2 - 2k \log r \log_2 r + \omega_1 (2c+1-2a) \log r + k^2 (\log_2 r) \right. \right. \\ &\quad \left. \left. + k [2k - \omega_1 (2c+1-2a)] (\log_2 r) \right. \right. \\ &\quad \left. \left. + \{ \omega_1^2 [c^2 + (1-2a)c + a(1-a)] - 2k(b-c+a) \} \right] \right. \\ &\quad \left. + (k\gamma - \theta) [\log r - k \log_2 r + \omega_1 (c-a)] + kb' \log_2 r \right\} \\ &\quad + \log [\phi(z/a_n)] + Q', \end{aligned}$$

where $|Q'| < \epsilon$, when r is sufficiently large, and where

$$\theta = \arg z, \quad \log_2 r = \log \log r, \quad a/v = b + ib', \quad \gamma = \arg v$$

and

$$c = (k/\omega_1) (\log \omega_1 - \log |v|).$$

The third term of the above expression is purely imaginary and counts as a finite term.

Hence, when z is not near a zero, so that $\phi(z/a_n)$ is finite, the first term gives the large terms.

It will be noticed that no terms in a occur in these large terms, which are expressed by means of the ordinary functions of analysis.

If ω is real, so that $\omega_2 = 0$, the above expression simplifies considerably, the greater part of the third term disappearing.

The comparative simplicity of the large terms is somewhat remarkable, as will be seen on working through the rather complicated analysis.

If we only wish to obtain the large terms, we have the following result:—

If z be confined to the region R defined in § 8, we have

$$\begin{aligned} \log F(z) &= \frac{1}{2\omega_1} (\log r)^2 - \frac{k}{\omega_1} \log r \log_2 r + (c - \tfrac{1}{2}) \log r + \frac{k}{2\omega_1^2} (\log_2 r)^2 \\ &\quad - k(b+c) \log_2 r + Q, \end{aligned}$$

where

$$|Q| < K.$$

We notice that the dominant term, $(\log r)^2/(2\omega_1)$, is independent of k . Thus, multiplying the n -th zero $-e^{n\omega}$ by n^k does not affect the dominant term of $\log F(z)$.

If $k = 0$, we have the function of the last article, and the large terms reduce, as they should, to $(\log r)^2/(2\omega_1) - \frac{1}{2} \log r$.

15. We shall work out one other particular case. Consider the function

$$F(z) = \prod_{s=1}^\infty \left[1 + \frac{z}{e^{(a+s\omega)^k}} \right],$$

k real and positive, $\Re \omega^k > 0$.

The two cases, $k > 1$, and $k < 1$, present very different characteristics, and are separated by the case $k = 1$, which has intermediate characteristics, as we shall explain later.

We shall consider in detail the case when $k > 1$. We have

$$\phi(s) = \Re \frac{(a+s\omega)^k}{\log s},$$

and is an increasing function of the same value of s . Also

$$\lim_{s=\infty} s^{-1} \phi(s) = \infty.$$

Hence $F(z)$ belongs to the class of functions considered in § 10. The number n is determined by the relation

$$| \exp [a + (n+1) \omega]^k | > r \geq | \exp (a+n\omega)^k |$$

or
$$\Re [a + (n+1) \omega]^k > \log r \geq \Re (a+n\omega)^k$$

or
$$\Re \left\{ \omega^k (n+1)^k \left[1 + \frac{ka}{\omega} \frac{1}{n+1} + \frac{A_n}{(n+1)^2} \right] \right\} > \log r > \Re \left\{ \omega^k n^k \left[1 + \frac{ka}{\omega} \frac{1}{n} + \frac{A'_n}{n} \right] \right\},$$

where A_n, A'_n are finite.

Put
$$\Re \omega^k = \gamma^k, \quad \Re \omega^{k-1} a = \gamma^{k-1} \gamma_1.$$

Then

$$\gamma^{(n+1)} \left[1 + \frac{\gamma_1}{\gamma} \frac{1}{n+1} + \frac{B_n}{(n+1)^2} \right] > (\log r)^{1/k} > \gamma^n \left[1 + \frac{\gamma_1}{\gamma} \frac{1}{n} + \frac{B'_n}{n^2} \right],$$

and
$$n = \frac{1}{\gamma} [(\log r)^{1/k} - \gamma_1 - a], \tag{1}$$

where a lies between $-c/n$ and $\gamma + c'/n$, for some (positive) values of c and c' . When z is confined to the region R_3 , we have (§ 10)

$|Q| < K \exp [-\frac{1}{2} k \phi(n)/n]$, or $|Q| < K \exp [-\lambda n^{k-1}/\log n]$,
for some non-zero value of λ . (2)

Again, we have
$$P = n \log z - \sum_1^n (a+s\omega)^k. \tag{3}$$

Now, when n is large, we have the asymptotic expansion*

$$\sum_1^n (a+s\omega)^k = \xi(-k|a+\omega) + \sum_{j=0}^l \frac{S'_j(a+\omega)}{j!} \left[\left(\frac{d}{dx} \right)^j \frac{x^{k+1}}{k+1} \right]_{x=n\omega} + J_l, \quad (4)$$

where l is any positive integer, and $|J_l n^{l-k-1}|$ tends to zero when n tends to infinity.

$$\text{Now} \quad \lim_{n=\infty} |Q n^p| \leq \lim_{n=\infty} |K e^{-\lambda n^{k-1}/\log n} n^p| = 0.$$

Hence, substituting from (2) in P , we may take l to be any integer we please, and we obtain

$$\begin{aligned} \log F(z) = n \log z - \frac{S'_0(a+\omega)}{k+1} (\omega n)^{k+1} - S'_1(a+\omega) (\omega n)^k \\ - \sum_{p=2}^l \frac{S'_p(a+\omega)}{p!} k(k-1) \dots (k-p+2) (\omega n)^{k-p+1} - \xi(-k|a+\omega) + K_l, \end{aligned} \quad (5)$$

where

$$K_l = -J_l + Q,$$

so that

$$\lim_{n=\infty} K_l n^{l-k-1} = 0.$$

Then

$$\lim_{r=\infty} |K_l (\log r)^{(l-k-1)/k}| = 0.$$

Thus, on substituting $n = (1/\gamma) [(\log r)^{1/k} - \gamma_1 - a]$, (5) provides an asymptotic expansion for $\log F(z)$ in descending powers of $[(\log r)^{1/k} - \gamma_1 - a]$.

On arranging the right-hand side in terms whose order in r decrease, and regarding pure imaginaries as finite terms, we obtain

$$\begin{aligned} \log F(z) = \frac{k}{(k+1)\gamma} (\log r)^{(k+1)/k} - (\gamma_1/\gamma + \tfrac{1}{2}) \log r \\ - \frac{k}{2\gamma} [\alpha(a-\gamma) - \gamma_1^2 + \gamma_2\gamma + \tfrac{1}{6}\gamma^2] (\log r)^{(k-1)/k} + \text{lower terms}, \end{aligned} \quad (6)$$

where

$$\gamma = [\Re \omega^k]^{1/k},$$

$$\gamma_1 = [\Re a \omega^{k-1}] / [\Re \omega^k]^{(k-1)/k}, \quad \gamma_2 = [\Re a^2 \omega^{k-2}] / [\Re \omega^k]^{(k-2)/k}.$$

The first two terms involve only known analytic functions.

We may give some idea of the nature of the region R_3 , by observing that the restriction of z to R_3 is equivalent to restricting a to lie between $A/\log n$ and $1-B/\log n$, where A and B are finite constants depending on k which can be made as small as we please by taking k sufficiently small.

* Barnes, *Proc. London Math. Soc.*, Ser. 2, Vol. 3, Part 4, p. 262. The above formula is obtained by changing a into $a + \omega$.

As the possible range of α is from $-c/n$ to $1+c'/n$, we see that the restricted range differs but slightly from the total possible range.

In the case when $k < 1$, $\phi(n)$ is of less order than n , and we cannot find all the large terms of $F(z)$. We restrict z to lie in the region R defined in § 6. Then we may retain terms of order greater than that of $n/\phi(n)$, or $n^{1-k} \log n$, or $(\log r)^{1/k-1} \log \log r$. Thus, when $1 > k > \frac{1}{2}$, so that $\log r > (\log r)^{1/k-1}$, we have

$$\log F(z) = \frac{k}{(k+1)\gamma} (\log r)^{(k+1)/k} - \left(\frac{\gamma_1}{\gamma} + \frac{1}{2} \right) \log r$$

$$+ \text{terms of order not greater than } (\log r)^{1/k-1} \log \log r,^* \quad (7)$$

and, when $k \leq \frac{1}{2}$,

$$\log F(z) = \frac{k}{(k+1)\gamma} (\log r)^{(k+1)/k}$$

$$+ \text{terms of order not greater than } (\log r)^{1/k-1} \log \log r. \quad (8)$$

The ratio ρ_s/α_s , where ρ_s is the radius of the circle C_s , is in this case of order $e^{-s^{1-k} \log s}$, while the ratio d_s/α_s , where d_s is the lower limit of the distance between $-\alpha_s$ and the nearest other zero, is of order $1/s^{1-k}$.

Hence ρ_s/d_s is of order $\exp(-s^{1-k} \log s)$, which is very small.

16. The two examples which we have worked out in detail suffice to show how, in practice, the value of P is approximately calculated in terms of r . In particular cases, various special artifices may be used to facilitate the approximation, such as replacing a sum by an integral in cases when the consequent error is of an order which we are to neglect.

We give the following summary of particular cases:—

Call, for convenience of reference, the case of the function $\prod_{s=1}^{\infty} (1+ze^{-s\omega})$, considered in § 12, I., the case $\prod_{s=1}^{\infty} \left[1 + \frac{z}{(\alpha+sv)^k e^{s\omega}} \right]$, II., the case $\prod_{s=1}^{\infty} \left[1 + \frac{z}{e^{(\alpha+sv)^k}} \right]$, with $k > 1$, III., and the case of the same form, with $k < 1$, IV.

* It can be shown, by a closer study of $\bar{R}$ and $\bar{S}$, that this error is actually of order $(\log r)^{1/k-1}$, except near the circles C_s .

$$\text{V.} \quad F(z) = \prod_{s=2}^{\infty} \left[1 + \frac{z}{\exp \{ (\log s)^{1+k} \}} \right],$$

k real and positive, z confined to the region R ;

$$\log F(z) = \exp [(\log r)^{1/(1+k)}] \left[\frac{1}{k+1} (\log r)^{k/(k+1)} - \frac{1}{(k+1)k} (\log r)^{(k-1)/(k+1)} + \dots \right. \\ \left. \pm \frac{1}{(k+1)k \dots (k-l+1)} (\log r)^{(k-l)/(k+1)} \right],$$

with an error of order $\exp [(\log r)^{1/(1+k)}] (\log r)^{-k/(k+1)}$, where l is the greatest integer less than $2k$, and where the series in square brackets stops at the term $\pm \frac{1}{(k+1)k \dots 1} (\log r)^{(k-k)/(k+1)}$ when k is an integer.

$$\text{VI.} \quad F(z) = \prod_{s=1}^{\infty} \left[1 + \frac{z}{e^{\omega e^{\sigma s}}} \right],$$

α real and positive, $\omega = \omega_1 + i\omega_2$, $\omega_1 > 0$;

$$\log F(z) = \frac{1}{\alpha} [\log_2 r - \log \omega_1 + \log (1+\alpha) - \alpha] \log z \\ - \frac{\omega}{e^{\alpha}-1} \frac{1+\alpha}{\omega_1} \log r + \frac{\omega e^{\alpha}}{e^{\alpha}-1} \\ + (\text{an expression of order } r^{-h} \text{ where } h > 0),$$

where $[1 - \{\log(1+\alpha)\}/\alpha]$ is the fractional part of $(\log \log r - \log \omega_1)/\alpha$, and z is supposed restricted to the part of the plane defined by

$$k < \alpha < (1-k)(e^{\alpha}-1),$$

where k is a constant as small as we please. The total possible range of α is from 0 to $e^{\alpha}-1$.

The constant h depends on k , and tends to zero with k .

17. We shall conclude by some remarks concerning certain general facts of which the above six cases afford illustration.

Of the six cases, IV. and V. are cases in which $\text{Lt } \alpha_{s+1}/\alpha_s = 1$, or α_s increases less rapidly than $e^{s\omega}$. In these two cases, and under the various restrictions placed upon z , we have been able to find, in terms of known analytic functions, an expression which proceeds to precisely that order of approximation which our theory allows as valid.

III. and VI. are cases in which $\text{Lt } \alpha_{s+1}/\alpha_s = \infty$, or in which α_s increases more rapidly than $e^{s\omega}$. These two cases are distinguished by the presence of terms of the expansion of $\log F(z)$ involving α , where α is

determined by means of the fractional part of some analytic function of r . Now such a function of r as a cannot possibly be expressed in terms of the ordinary known analytic functions, and I think it is obvious from the manner in which a enters into the expansions in cases III. and VI., that the whole expansion cannot possibly be expressed in terms of known analytic functions.*

It is easy to multiply instances of both types, and we are led to these conclusions: If α_s increases less rapidly than $e^{s\omega}$ (for all positive values of ω), and is expressed in terms of known analytic functions, then the terms in $\bar{P}$ which our theory permits us to retain as an approximation for $\log F(z)$, can be expressed in terms of known analytic functions of n and r , and, moreover, the problem of finding these terms is practically soluble, however complicated a function of s α_s may be. Further, provided we can invert the equation $\alpha_n = t$, into an equation of the form $n = \chi(t) + \tau$, where $\chi(t)$ is expressed in finite terms, and in terms of known functions of t , and where $|\tau|$ is finite for all values of n , then we can express the above expression (for $\bar{P}$ in terms of n) in terms of known functions of r .

If, on the other hand, α_s increases more rapidly than $e^{s\omega}$, then there will be *large* terms in the expansion for $\log F(z)$ which involve a number a analogous to the a 's in cases III. and VI., and consequently the complete expansion cannot be effected in terms of known functions of r .

Again, when $\text{Lt } \alpha_{s+1}/\alpha_s = \infty$, it is not always possible in practice to find all the terms of P which are admissible, even by the introduction of a number a , nor is it always possible even to find all the large terms of P . For example, if $\alpha_s = \exp[\exp(\exp s)]$, to determine all the large terms of P , we should have to sum the series $\sum_1^n \exp(e^s)$ with an error only finite, which is impossible by any known analysis.

It is found that the more nearly the rate of increase of α_s approaches that of $e^{s\omega}$ (while remaining greater than that rate) the smaller is the order of the highest term of $\log F(z)$ which involves a , in comparison with the order of the dominant term. In consideration of this fact, and of the behaviour of functions for which $\text{Lt } \alpha_{s+1}/\alpha_s = 1$, we might expect in the case when $\text{Lt } \alpha_{s+1}/\alpha_s = a$ a finite number greater than 1, which separates the two cases which we have been considering, all the large terms of $\log F(z)$ are sometimes expressible in terms of known functions, while the finite part is not so expressible. This we see to be the case in cases I. and II., and instances can be multiplied.

It will be noticed that our conclusions confirm those of Dr. Barnes†

* It is easy to see that the occurrence of a is not due to the fact that we have expressed the various terms of the expansion in terms of r , instead of in terms of z .

† Cf. § 1.

as to the impossibility of solving, by the method of contour integration, the problem of the complete asymptotic expansion of $\log F(z)$ when a_s is of high order in s .

We might expect, however, that when $\lim_{s \rightarrow \infty} a_s/a_{s+1} = 1$, this method might, in certain cases, be applied to obtain a complete expansion.

This is, as a matter of fact, the case, and it is possible to obtain such an expansion for the function IV. I shall reserve this theory for another paper.

It may be remarked that the property of requiring non-analytic functions for the complete asymptotic expansion, is not confined to functions of zero order, and, indeed, has no real connection with the order of the maximum modulus.

The same property belongs to certain functions* defined by a product form of the type $\Pi [1 + (z/a_s)^{f(s)}]$, where $f(s)$ becomes infinite with s , and $|a_1|, |a_2|, \dots$ tend in value to infinity, or to a finite limit.

These functions may be integral functions of any order—zero, finite, infinite, or transfinite. They may also be functions whose Taylor series has a finite radius of convergence.

The property, however, does appear to be intimately connected with the rapidity of the convergence of the product form. In the particular case of functions of zero order of standard type, the general condition for the property is equivalent to the condition that a_s should increase as rapidly or more rapidly than $e^{s\omega}$, and this condition, again, can be expressed in terms of the order of $|F(z)|$ *qua* function of $|z|$, but the question of *order* is really quite irrelevant.

* I hope, in another paper, to develop a theory of these functions, which is somewhat analogous to that of the present paper. It is easy, in this theory, to give a multitude of particular examples, which bear out the rule governing the property.

HARMONIC EXPANSIONS OF FUNCTIONS OF TWO VARIABLES

By A. C. DIXON.

[Received September 9th, 1906.—Read November 8th, 1906.—Received, in revised form, June 30th, 1907.]

THE object of this paper is to extend to functions of two variables the results of an earlier one* (*Proc. London Math. Soc.*, Ser. 2, Vol. 3, pp. 83–103), and so to establish the validity of such expansions in series of partial harmonics as have been proposed by Klein and Bôcher.†

Each of two real variables x, y has a range from 0 to 1, and a fairly general function of them is expanded in a double series, each term of which is the product of a function of x and a function of y ; these functions are respectively determined by differential equations, in which occur two parameters λ, μ , which have different values in the different terms of the series.

By means of Poincaré's extension of Cauchy's residue theorem an expression is given for the sum of a finite number of terms of the series as a multiple integral: the choice of a suitable field for the integration is a matter of some difficulty. The convergency is investigated by a discussion of the order of magnitude of the remainder, in which certain approximate solutions of the differential equations are used. This discussion is somewhat intricate, owing to difficulties which do not present themselves in the simpler case of functions of one variable.

The question of the expansibility of a given function in a series of multiples of given functions divides itself into two parts, one relating to the nature of the function to be expanded, and the other to what may be called the completeness of the proposed series. For instance, it is impossible to expand $\sin x$ ($0 < x < \pi$) in multiples of $\sin 2x, \sin 3x, \sin 4x, \dots$, because this series of functions is incomplete; but even the complete series $\sin x, \sin 2x, \dots$ is inadequate to express a function which

* Hereinafter referred to as I. There are important papers on the same subject by Stekloff in the *Comptes Rendus* for 1902, pp. 848–851, and Hilbert (*Gött. Nachrichten*, 1904).

† For references see § 32 (note) below.

is not a limit of continuous functions. It follows from the result of this paper that the double series discussed are equivalent in this kind of completeness to double Fourier series; it is, in fact, possible to expand in a series of the present kind the general term of a double Fourier series, and conversely.

The following is a list of the headings of paragraphs, and under each heading are indicated the symbols introduced in the corresponding paragraph and used afterwards:—

1. The Fundamental Differential Equations. ($x, y, \lambda, \mu, S, T, U, V, X, Y, \phi, \psi$.)
2. ϕ and its Derivatives. [$\phi(x, a), \phi', \phi'', \phi''', W$.]
3. Functions ϕ with different Parameters.
4. Reality of Solutions of $\Theta = 0, Z = 0$. ($\Theta, Z, E, E_1, \dots, G, G_1, \dots$)
5. Distinctness of Solutions of $\Theta = 0, Z = 0$. (H .)
6. A certain Double Integral. (Φ, Ψ .)
- 6a. Known Results relating to Double Contour Integrals. (L, M, Ω .)
- 6b. General Idea of the Proposed Application of these Results.
7. Approximations by means of Hyperbolic Functions when λ, μ are Great. ($P, Q, f, \delta, d\rho, d\sigma, k, ||, \text{"normal."}$)
8. Approximation by means of Riccati Functions. (ω, w .)
9. Approximate Value of ω .
10. Approximations to $\omega(a, 0)$ and its Derivatives, when a is Negative and Great. (A, α, B, β .)
11. The same when the Vectorial Angle of a is between $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$. ($\checkmark$)
12. The same for other Great Values of a . Values of AB, α, β .
13. The Function χ . (χ .)
14. Approximation to $\omega(b, a)$.
15. Application to ϕ .
16. Case when P Vanishes near a Terminus.
17. Case when P Vanishes more than Once. (Υ .)
18. Can the Terms in $\Sigma \Upsilon$ destroy each other? ($p, q, x_1, x_2, \dots$)
19. Another Method.
- 20, 21. Case of Terminal Short Sections. (s .)
22. Stationary Values of S/T . Riccati case. (ξ, η, ζ, ω .)
23. Hyperbolic Case.
24. Combination of the Two Cases.
25. Short Sections.
26. Approximate Values of Θ, Φ . (δ .)
27. Distribution of the Values of μ for which $\Theta = 0$. ($p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2, \tau, C, D, C', D'$.)
- 28-31. The Field of Integration (k, p, q, g, h) ($m_0, m_1, \dots, \mu_0, \mu_1, \dots, \alpha, \beta, \gamma, g', h', r$) (K) (g', Q).
32. Supplement to § 6b. (λ_{rs}, μ_{rs} .)
33. Expansion of an Arbitrary Analytical Function. (f, H, x_0, x_1, y_0, y_1 .)
34. The Remainder. A Lemma.
- 35-37. Discussion of the Remainder.
38. Statement of the Result.
39. Case when E_4 or $G_4 = 0$.
40. Double Solutions of the Equations $\Theta = 0, Z = 0$.
41. Form of the General Term in the Expansion.
42. Special Cases.
43. Validity at the Boundary.

The Fundamental Differential Equations.

1. Let x, y be two real variables, each of which may have any value from 0 to 1, inclusive, and let λ, μ be parameters.

Let ϕ denote a function of x and the parameters, satisfying the equation

$$\frac{d^2\phi}{dx^2} + (\lambda S - \mu T + X)\phi = 0, \quad (1)$$

where S, T, X are real analytical functions of x only.

Let ψ denote a function of y and the parameters, satisfying the equation

$$\frac{d^2\psi}{dy^2} + (-\lambda U + \mu V + Y)\psi = 0, \quad (2)$$

where U, V, Y are real analytical functions of y only.

Suppose also that, for all values of x, y , the sign of $SV - UT$ is constant, say positive. By a simple transformation of λ, μ we may make S, T, U, V all positive. For since $SV - UT$ is never zero, no value of S/T can be equal to any value of U/V , and one of these two fractions, say U/V , must be always finite. Let α be a real value which neither of the fractions takes: then $S - \alpha T, U - \alpha V$ are of constant signs. If these signs are opposite, let α be changed and let it travel through all the values of U/V to another value α' , which is never taken by either fraction; then $S - \alpha' T, U - \alpha' V$ will have the same constant sign. The values α, α' are only restricted by inequalities, and we may therefore take two values α, β , such that

$$S - \alpha T, S - \beta T, U - \alpha V, U - \beta V, \beta - \alpha$$

have the same sign. Then

$$\lambda S - \mu T = \lambda_1 S_1 - \mu_1 T_1, \quad -\lambda U + \mu V = -\lambda_1 U_1 + \mu_1 V_1,$$

$$\text{where} \quad S_1 = (S - \beta T)/(\beta - \alpha), \quad T_1 = (S - \alpha T)/(\beta - \alpha),$$

$$U_1 = (U - \beta V)/(\beta - \alpha), \quad V_1 = (U - \alpha V)/(\beta - \alpha),$$

$$\lambda_1 = \mu - \alpha\lambda, \quad \mu_1 = \mu - \beta\lambda,$$

$$S_1 V_1 - U_1 T_1 = (SV - UT)/(\beta - \alpha).$$

Hence, if λ_1, μ_1 are used as parameters instead of λ, μ the signs will be positive as desired unless $\beta - \alpha$ is negative; if $\beta - \alpha$ is negative we may take $-\mu_1, -\lambda_1$ as parameters.

It will be supposed that this transformation has been made, so that $S, T, U, V, SV - UT$ are all positive, and never zero.

ϕ and its Derivatives.

2. The notation $\phi(x, a)$ will be used for a function ϕ satisfying the equation (1), and also the initial conditions $\phi = 0$, $d\phi/dx = 1$, when $x = a$, $\phi'(x, a)$, $\phi''(x, a)$, ... will be used for $\partial/\partial x \phi(x, a)$, $\partial/\partial a \phi(x, a)$, ...

Here a may be any particular value of x , and if b is another value of x a second solution of (1) is $\phi(x, b)$, and therefore

$$\phi(x, a) \phi'(x, b) - \phi'(x, a) \phi(x, b)$$

is independent of x .

Putting $x = a, b$ in turn, we have

$$\phi(b, a) = -\phi(a, b). \quad (3)$$

Taking c to be a third value of x we have a third solution $\phi(x, c)$ which must be a linear combination of the other two, so that

$$A\phi(x, a) + B\phi(x, b) + C\phi(x, c) = 0,$$

where A, B, C do not depend on x . Their values are found by giving to x the values a, b , and thus we have

$$\phi(x, a) \phi(b, c) + \phi(x, b) \phi(c, a) + \phi(x, c) \phi(a, b) = 0.$$

From (3) it follows that

$$\phi'(b, a) = -\phi'(a, b),$$

so that

$$\phi''(a, a) = -1;$$

also

$$\phi''(b, a) = -\phi''(a, b),$$

so that

$$\phi''(a, a) = 0.$$

Hence $-\phi''(x, a)$ is the solution of (1) which satisfies the initial conditions $\phi = 1$, $d\phi/dx = 0$ when $x = a$. We have therefore

$$\phi(x, a) \phi''(x, a) - \phi'(x, a) \phi'(x, a) = 1.$$

A like notation, $\psi(y, a)$, ... will be used for the solutions of (2).

As an illustration take the equation

$$\frac{d^2\phi}{dx^2} + k^2\phi = 0$$

where k is constant. Here

$$\phi(x, a) = \frac{1}{k} \sin k(x-a), \quad \phi'(x, a) = \cos k(x-a),$$

$$\phi''(x, a) = -\cos k(x-a), \quad \phi'''(x, a) = k \sin k(x-a).$$

If ϕx denotes any solution of the equation (1), and we suppose x to follow a path beginning at a and ending at b , the final values ϕb , $\phi' b$ depend on the initial values ϕa , $\phi' a$, and are in fact derived from them by a homogeneous linear transformation whose matrix is

$$\begin{pmatrix} -\phi''(b, a) & \phi'(b, a) \\ -\phi'''(b, a) & \phi''(b, a) \end{pmatrix}.$$

Calling this $W(b, a)$, we have

$$\frac{d}{dx} W(x, a) = \begin{pmatrix} 0, & 1 \\ -\lambda S + \mu T - X, & 0 \end{pmatrix} W(x, a),$$

and the initial condition $W(a, a) = 1$.

If $W(b, a)$ and $W(c, b)$, the matrices corresponding to the paths ab , bc are known, the matrix for the whole path ac is given by the relation

$$W(c, a) = W(c, b) W(b, a),$$

which holds for the solutions of the equation

$$\frac{dW}{dx} = MW,$$

where W , M may be matrices of any, the same, order whose constituents are functions of x , and M is supposed known, W unknown.

If the equation is

$$\frac{dW}{dx} = WM,$$

the corresponding relation is $W(c, a) = W(b, a) W(c, b)$,

the initial condition being still $W(a, a) = 1$.

When W , M are single functions these relations are immediate consequences of the relation

$$\exp u \exp v = \exp(u + v),$$

and when they are matrices there is no difficulty in the proof.

Functions ϕ with Different Parameters.

3. In §§ 3-6 there is not much that is new in substance and the methods and results are therefore not given at great length.

It is known that $\phi(x, a)$, $\psi(y, a)$ and their derivatives are analytical functions of the parameters λ, μ , having no singularity except at infinity, that is, are integral functions. This is true even when S, T, U, V, X, Y are not analytical functions, but only integrable; the following method of proof enables us to find their expansions in powers of λ, μ , as well as other results of importance.

Suppose ϕ to become ϕ_1 when λ, μ are turned into λ_1, μ_1 , then we have

$$\phi(x, a) \phi_1''(x, b) - \phi_1(x, b) \phi''(x, a) = \{(\lambda - \lambda_1)S - (\mu - \mu_1)T\} \phi(x, a) \phi_1(x, b),$$

and therefore by integration between the limits c, u ,

$$\begin{aligned} & \phi(u, a) \phi_1'(u, b) - \phi_1(u, b) \phi'(u, a) - \phi(c, a) \phi_1'(c, b) + \phi_1(c, b) \phi'(c, a) \\ &= (\lambda - \lambda_1) \int_c^u S \phi(x, a) \phi_1(x, b) dx - (\mu - \mu_1) \int_c^u T \phi(x, a) \phi_1(x, b) dx. \end{aligned}$$

The substitution of 0, 1 for a, b, c, u in this gives an important series of relations, as in I., pp. 85, 86. Making $\lambda - \lambda_1$ small and $\mu = \mu_1$, we have

$$\begin{aligned} & \phi(u, a) \frac{\partial}{\partial \lambda} \phi'(u, b) - \phi'(u, a) \frac{\partial}{\partial \lambda} \phi(u, b) - \phi(c, a) \frac{\partial}{\partial \lambda} \phi'(c, b) + \phi'(c, a) \frac{\partial}{\partial \lambda} \phi(c, b) \\ &= - \int_c^u S \phi(x, a) \phi(x, b) dx; \end{aligned}$$

and on putting $u = a$, $c = b$,

$$\frac{\partial}{\partial \lambda} \phi(a, b) = - \int_b^a S \phi(a, x) \phi(x, b) dx,$$

from which the more general result could be derived after differentiation as to a, b . Similarly,

$$\frac{\partial}{\partial \mu} \phi(a, b) = \int_b^a T \phi(a, x) \phi(x, b) dx.$$

The higher derivatives may also be found; the general formula is

$$\frac{\partial^n}{\partial \lambda^{n-r} \partial \mu^r} \phi(x_0, x_{n+1}) = \text{the coefficient of } \kappa^{n-r} \text{ in}$$

$$(n-r)! r! \iiint \dots \prod_{p=1}^n (T_p - \kappa S_p) \prod_{p=0}^n \phi(x_p, x_{p+1}) \prod_{p=1}^n dx_p,$$

the integration being over all such values that $x_0, x_1, x_2, \dots, x_n, x_{n+1}$ lie in reverse order on the path chosen from x_{n+1} to x_0 : that is, we are to integrate first as to x_1 from x_2 to x_0 , then as to x_2 from x_3 to x_0 , and so on.

Reality of Solutions of $\Theta = 0$, $Z = 0$.

4. Now let $E, G, E_1, \dots, E_4, G_1, \dots, G_4$ be real constants, and write Θ, Z for

$$E_1 \phi(1, 0) + E_2 \phi'(1, 0) + E_3 \phi''(1, 0) + E_4 \phi'''(1, 0) - 2E,$$

$$G_1 \psi(1, 0) + G_2 \psi'(1, 0) + G_3 \psi''(1, 0) + G_4 \psi'''(1, 0) - 2G,$$

respectively, so that Θ, Z are integral functions of λ, μ . It is to be proved that the values of λ, μ for which $\Theta = 0, Z = 0$ are real and generally distinct, if the coefficients $E_1, G_1, \dots$ satisfy certain inequalities. With the notation of § 3, we have (as in I., pp. 87, 103)

$$\Theta \{E_3 \phi_1(1, 0) + E_4 \phi'_1(1, 0)\} - \Theta_1 \{E_3 \phi(1, 0) + E_4 \phi'(1, 0)\}$$

$$= \int_0^1 [\{E_3 \phi(x, 1) + E_4 \phi'(x, 1) + E \phi(x, 0)\} \\ \times \{E_3 \phi_1(x, 1) + E_4 \phi'_1(x, 1) + E \phi_1(x, 0)\} \\ + (E_1 E_4 - E_2 E_3 - E^2) \phi(x, 0) \phi_1(x, 0)] \{(\lambda - \lambda_1) S - (\mu - \mu_1) T\} dx. \quad (5)$$

Suppose that $\Theta = 0$ for values of λ, μ that are not real, and let λ_1, μ_1 be conjugate imaginaries to λ, μ ; then Θ_1 is conjugate to Θ and must vanish, so that the left-hand side of (5) is zero. Also $\phi(x, 0)$ and $\phi_1(x, 0), \dots$, are pairs of conjugates, and thus if $E_1 E_4 - E_2 E_3 - E^2$ is positive the expression in square brackets on the right in (5) is always positive. Since the whole integral vanishes the purely imaginary factor

$(\lambda - \lambda_1)S - (\mu - \mu_1)T$ cannot be of constant sign, but $(\lambda - \lambda_1)/(\mu - \mu_1)$ must be a mean among the values of T/S .

If $Z = 0$ for the same values of λ , μ , and if $G_1G_4 - G_2G_3 - G^2$ is positive, it follows in like manner that $(\lambda - \lambda_1)/(\mu - \mu_1)$ must be a mean among the values of V/U . Hence Θ , Z cannot both vanish except for real values of λ , μ , since the greatest value of T/S is less than the least of V/U .

The same conclusion follows if $E_1E_4 - E_2E_3 - E^2 = 0$; see I., p. 87. We shall assume that $E_1E_4 - E_2E_3 - E^2$ and $G_1G_4 - G_2G_3 - G^2$ are positive or zero so that the equations $\Theta = 0$, $Z = 0$ cannot both be satisfied by imaginary values of λ , μ .

Distinctness of Solutions of $\Theta = 0$, $Z = 0$.

5. The values of λ , μ given by the equations $\Theta = 0$, $Z = 0$ are generally distinct as well as real. If they were not, the Jacobian $\partial(\Theta, Z)/\partial(\lambda, \mu)$ would vanish as well as Θ , Z . But we have

$$\frac{\partial \Theta}{\partial \lambda} = - \int_0^1 \{ E_1 \phi(1, x) \phi(x, 0) + E_2 \phi'(1, x) \phi(x, 0) + E_3 \phi(1, x) \phi'(x, 0) + E_4 \phi'(1, x) \phi'(x, 0) \} S dx,$$

with like expressions for the other derivatives* and hence

$$\begin{aligned} \frac{\partial(\Theta, Z)}{\partial(\lambda, \mu)} &= \int_0^1 \int_0^1 \{ E_1 \phi(1, x) \phi(x, 0) + E_2 \phi'(1, x) \phi(x, 0) + E_3 \phi(1, x) \phi'(x, 0) + E_4 \phi'(1, x) \phi'(x, 0) \} \\ &\quad \times \{ G_1 \psi(1, y) \psi(y, 0) + G_2 \psi'(1, y) \psi(y, 0) + G_3 \psi(1, y) \psi'(y, 0) + G_4 \psi'(1, y) \psi'(y, 0) \} (SV - UT) dy dx. \end{aligned}$$

* A more general formula is

$$\Theta \delta \theta - \theta \delta \Theta = - \int_0^1 \{ A [\phi(x, 0)]^2 + 2B \phi(x, 0) \phi'(x, 0) + C [\phi'(x, 0)]^2 \} (S \delta \lambda - T \delta \mu) dx,$$

where

$$\theta = e_1 \phi(1, 0) + e_2 \phi'(1, 0) + e_3 \phi(1, 0) + e_4 \phi'(1, 0) - 2e,$$

$$A = (12) - (13) \phi'^2 - (14) \phi' \phi'' - (23) \phi' \phi'' - (24) \phi'^2 + (10) 2\phi' + (20) 2\phi'',$$

$$B = (13) \phi \phi' + (14) \phi \phi'' + (23) \phi' \phi' + (24) \phi' \phi'' - (10) \phi - (20) \phi' + (30) \phi' + (40) \phi'',$$

$$C = (34) - (13) \phi^2 - (14) \phi \phi' - (23) \phi \phi' - (24) \phi'^2 - (30) 2\phi - (40) 2\phi',$$

$$(12) = E_1 e_2 - e_1 E_2, \quad (30) = E_3 e - e_3 E, \quad \dots,$$

$$\phi, \phi', \phi'', \phi''' = \phi(1, 0), \phi'(1, 0), \phi''(1, 0), \phi'''(1, 0) \text{ respectively.}$$

The quadratic in $\phi(x, 0)$ and $\phi'(x, 0)$ has for its discriminant

$$AC - B^2 = (E_1 E_4 - E_2 E_3 - E^2) \theta^2 - (E_1 e_4 + E_4 e_1 - E_2 e_3 - E_3 e_2 - 2Ee) \Theta \theta + (e_1 e_4 - e_2 e_3 - e^2) \Theta^2,$$

and is therefore of constant sign if

$$(E_1 + \kappa e_1)(E_4 + \kappa e_4) - (E_2 + \kappa e_2)(E_3 + \kappa e_3) - (E + \kappa e)^2$$

is never negative for any real value of κ .

In this expression the subject of integration has a constant sign when Θ, Z vanish, for while $SV - UT$ is always positive, the first factor is

$$\begin{aligned} & \{E_1 \phi'(1, 0) + E_2 \phi''(1, 0)\} [\phi(x, 0)]^2 \\ & + \{-E_1 \phi(1, 0) - E_2 \phi'(1, 0) + E_3 \phi''(1, 0) + E_4 \phi'''(1, 0)\} \phi(x, 0) \phi'(x, 0) \\ & + \{-E_3 \phi(1, 0) - E_4 \phi'(1, 0)\} [\phi'(x, 0)]^2, \text{ say } H, \end{aligned}$$

a quadratic in $\phi(x, 0), \phi'(x, 0)$ whose discriminant is

$$4(E_1 E_4 - E_2 E_3) - (\Theta + 2E)^2,$$

so that H is of constant sign when $\Theta = 0$; the second factor can be similarly expressed.

Hence the Jacobian can only vanish when one factor or the other vanishes for all values of x, y , so that we have either

$$E_1 = E\phi''(1, 0), \quad E_2 = -E\phi'(1, 0), \quad E_3 = -E\phi'(1, 0), \quad E_4 = E\phi(1, 0),$$

$$E_1 E_4 - E_2 E_3 - E^2 = 0, \quad E \neq 0, \quad \partial\Theta/\partial\lambda = 0 = \partial\Theta/\partial\mu,$$

$$\text{or else} \quad G_1 = G\psi''(1, 0), \quad \dots, \quad \partial Z/\partial\lambda = 0 = \partial Z/\partial\mu.$$

(Compare I., p. 88.)

Thus, if λ, μ are taken as Cartesian coordinates, the two curves $\Theta = 0, Z = 0$ have their intersections all distinct unless one passes through a node on the other. Neither curve can have a triple point, for

$$\begin{aligned} \frac{\partial^2 \Theta}{\partial \lambda^2} &= 2 \int_0^1 \int_x^1 \{E_1 \phi(1, x_1) \phi(x_1, x) \phi(x, 0) + E_2 \phi'(1, x_1) \phi(x_1, x) \phi(x, 0) \\ &+ E_3 \phi(1, x_1) \phi(x_1, x) \phi'(x, 0) + E_4 \phi'(1, x_1) \phi(x_1, x) \phi'(x, 0)\} SS_1 dx_1 dx, \end{aligned}$$

$$\text{which reduces to} \quad -2E \int_0^1 \int_x^1 [\phi(x_1, x)]^2 SS_1 dx_1 dx$$

$$\text{or} \quad -E \int_0^1 \int_0^1 [\phi(x_1, x)]^2 SS_1 dx_1 dx,$$

$$\text{when} \quad E_1 = E\phi''(1, 0), \quad E_2 = -E\phi'(1, 0), \quad \dots,$$

that is, when the curve $\Theta = 0$ has a double point.

$$\text{Similarly} \quad \frac{\partial^2 \Theta}{\partial \mu^2} = -E \int_0^1 \int_0^1 [\phi(x_1, x)]^2 TT_1 dx_1 dx,$$

$$\frac{\partial^2 \Theta}{\partial \lambda \partial \mu} = E \int_0^1 \int_0^1 [\phi(x_1, x)]^2 ST_1 dx_1 dx.$$

These second derivatives cannot vanish, so that there can be no triple points. In the case of a double point

$$\alpha^2 \frac{\partial^2 \Theta}{\partial \lambda^2} + 2\alpha\beta \frac{\partial^2 \Theta}{\partial \lambda \partial \mu} + \beta^2 \frac{\partial^2 \Theta}{\partial \mu^2} \\ = -E \int_0^1 \int_0^1 [\phi(x_1, x)]^2 (aS - \beta T)(aS_1 - \beta T_1) dx_1 dx,$$

which cannot vanish if $aS - \beta T$ is of constant sign.

Hence on each branch $d\mu/d\lambda$, if real, must be a mean among the values of S/T . Also $d\mu/d\lambda$ must be real, for, if not, the equation $\Theta = 0$ could be satisfied by giving λ a consecutive real value and μ an imaginary one, which we know to be impossible, since the ratio of the imaginary parts of λ, μ is a mean among the values of T/S , and T, S are always positive.* For the same reason if the values of $d\mu/d\lambda$ are equal, the point must be a tacnode and not a cusp.

A tacnode is not impossible, nor is any order of contact between real branches of the curve $\Theta = 0$ impossible, as may best be seen by the example of the differential equation

$$\phi'' + (\lambda - \mu) \phi = 0,$$

when $E_1 = 0 = E_4, \quad E_2 = -E_3 = E = 1.$

We then have $\phi(1, 0) = (\lambda - \mu)^{-\frac{1}{2}} \sin(\lambda - \mu)^{\frac{1}{2}},$

$$\phi'(1, 0) = -\phi'(1, 0) = \cos(\lambda - \mu)^{\frac{1}{2}},$$

$$\Theta = \phi'(1, 0) - \phi'(1, 0) - 2 = 2 \cos(\lambda - \mu)^{\frac{1}{2}} - 2,$$

so that the curve $\Theta = 0$ consists of the straight line $\lambda = \mu$ counted once,

* The following is another proof. Let

$$\alpha = \int_0^1 T[\phi(x, 0)]^2 dx, \quad \beta = \int_0^1 S[\phi(x, 0)]^2 dx,$$

$$\text{then } \alpha^2 \frac{\partial^2 \Theta}{\partial \lambda^2} + 2\alpha\beta \frac{\partial^2 \Theta}{\partial \lambda \partial \mu} + \beta^2 \frac{\partial^2 \Theta}{\partial \mu^2} \\ = -E \int_0^1 \int_0^1 \int_0^1 [\{\phi(x, x_1) \phi(x_2, 0) \phi(x_3, 0)\}^2 - \{\phi(x, x_2) \phi(x_1, 0) \phi(x_3, 0)\}^2 \\ - \{\phi(x_1, x_3) \phi(x_2, 0) \phi(x, 0)\}^2 + \{\phi(x_2, x_3) \phi(x, 0) \phi(x_1, 0)\}^2] \\ \times SS_1 T_2 T_3 dx dx_1 dx_2 dx_3 \\ = 2E \int_0^1 \int_0^1 \int_0^1 \int_0^1 \phi(x, 0) \phi(x_1, 0) \phi(x_2, 0) \phi(x_3, 0) \phi(x, x_3) \phi(x_1, x_2) SS_1 T_2 T_3 dx dx_1 dx_2 dx_3 \\ = 2E \left\{ \int_0^1 \int_0^1 \phi(x, 0) \phi(x_1, 0) \phi(x, x_1) ST_1 dx dx_1 \right\}^2,$$

a quantity which is of opposite sign to the coefficient of α^2 in the quadratic form, or else is zero. Hence the factors of the form are real.

and the lines $\lambda = \mu + 4n^2\pi^2$, each counted twice, where $n = 1, 2, 3, 4, \dots$; any point on a repeated branch must be reckoned as a point of contact of two branches.

At any ordinary point of the curve $\Theta = 0$, the value of $d\mu/d\lambda$ is also a mean among the values of S/T , for $\alpha \frac{\partial \Theta}{\partial \lambda} + \beta \frac{\partial \Theta}{\partial \mu}$ is the integral of the product when $-\alpha S + \beta T$ is multiplied by the quadratic H in $\phi(x, 0)$, $\phi'(x, 0)$, which has been proved to be of constant sign. Similarly on the curve $Z = 0$, $d\mu/d\lambda$ is always a mean among the values of U/V , and thus the curves cut at a finite angle whether at an ordinary or a double point.

A certain Double Integral.

6. The main question we have to discuss relates to the double integral

$$\iint \frac{\Phi(x, t) \Psi(y, u)}{\Theta Z} d\mu d\lambda,$$

$$\begin{aligned} \text{where } \Phi(x, t) &= E_1 \phi(1, t) \phi(x, 0) + E_2 \phi'(1, t) \phi(x, 0) \\ &\quad + E_3 \phi(1, t) \phi'(x, 0) + E_4 \phi'(1, t) \phi'(x, 0) + E \phi(t, x), \\ \Psi(y, u) &= G_1 \psi(1, u) \psi(y, 0) + G_2 \psi'(1, u) \psi(y, 0) \\ &\quad + G_3 \psi(1, u) \psi'(y, 0) + G_4 \psi'(1, u) \psi'(y, 0) + G \psi(u, y). \end{aligned}$$

$$\text{Since } \phi(1, t) \phi(x, 0) - \phi(1, x) \phi(t, 0) = \phi(1, 0) \phi(x, t),$$

it will be seen that

$$\Phi(x, t) - \Phi(t, x) = \Theta \phi(x, t),$$

and that when Θ vanishes Φ is a symmetric function of x, t ; similarly

$$\Psi(y, u) - \Psi(u, y) = Z \psi(y, u),$$

so that Ψ is a symmetric function when $Z = 0$.*

* The following are properties of Φ, Θ given here for reference.

$$\frac{\partial \Theta}{\partial \lambda} = - \int_0^1 S \Phi(x, x) dx, \quad \frac{\partial \Theta}{\partial \mu} = \int_0^1 T \Phi(x, x) dx,$$

$$\frac{\partial}{\partial \lambda} \Phi(x, t) = - \int_t^1 S_1 \Phi(x, x_1) \phi(x_1, t) dx_1 - \int_0^x S_1 \Phi(x_1, t) \phi(x, x_1) dx_1 + E \int_0^1 S_1 \phi(x_1, x) \phi(x_1, t) dx_1,$$

and so for $\partial \Phi / \partial \mu$. Where there is a node on the curve $\Theta = 0$, $\Phi = 0$ for all values of x, t , and

$$\delta \Phi(x, t) = E \int_0^1 (S_1 \delta \lambda - T_1 \delta \mu) \phi(x_1, x) \phi(x_1, t) dx_1,$$

from which could be found the values already given (§ 5) for the second derivatives of Θ .

Since Φ , Ψ , Θ , Z are integral functions of λ , μ , the value of the double integral, when taken over a closed region of two dimensions, will depend on the situation of that region in relation to the values of λ , μ for which ΘZ vanishes, and will be unaltered by continuous deformation of that region so long as it contains no point where ΘZ vanishes. (Picard and Simart, *Fonctions algébriques de deux variables indépendantes*, Vol. I., ch. iii.)

Known Results relating to Double Contour Integrals.

6a. Suppose that in the second or outer integration, with respect to λ , the path* is a closed contour L , and that to each point of L there corresponds a certain closed path of integration M , in the μ -plane: as λ moves round L , M may vary continuously, but its final position must coincide with its initial one, and in no position may it pass through any of the points where Θ vanishes, say μ' , or the points where Z vanishes, say μ'' .

For certain values of λ there will be coincidences among the points μ' , μ'' , and if L encloses none of these values of λ , the value of the double integral will be zero. If such a point λ_r is enclosed, it may be (1) a branch point for μ' (or μ''), or (2) correspond to a double point on the curve $\Theta = 0$ (or $Z = 0$), or (3) be such that Θ , Z vanish for a common value of μ , say μ_r , when $\lambda = \lambda_r$. These three alternatives do not exclude each other.

In case (1) the effect will depend on the position of the value μ' (or μ''), affected by the branch point, in regard to the contour M . In the initial position M will, in general, enclose some of the points μ' , μ'' , and exclude others. As λ travels round L the points μ' , μ'' describe paths which, singly or together, form one or more closed circuits. The circuits described by the enclosed points will be separate from those described by the excluded points, since the curve M separates the two sets of points in all positions. (This does not hinder an excluded point from taking a position which has formerly been occupied by an enclosed point, since the value of λ has changed.)

Now, if L encloses a branch point which affects only enclosed points, or only excluded points, the resulting contribution to the double integral is zero. If, on the other hand, L contains branch points which permute enclosed with excluded points, one or more of the circuits described by

* Each of the complex variables λ , μ is of course represented by a point in a separate plane, as in Argand's diagram.

the enclosed points may be irreducible, and thence will arise a contribution equal to a period of a certain single integral of the Abelian type.

As an illustration of the effect of branch points take

$$\iint (\lambda^4 + \mu^2 - 1)^{-1} d\mu d\lambda,$$

where the problem is somewhat simplified, but not essentially changed, by the absence of any second factor in the denominator of the subject of integration. Here there are two values of μ' , namely $\pm(1 - \lambda^4)^{\frac{1}{2}}$, and they are permuted by a circuit round any one of the branch points in the λ -plane, $\pm 1, \pm i$.

Hence, if L consists of a simple contour enclosing one of these branch points, M must enclose both or neither of the values μ' ; otherwise its final position could not be the same as its initial one, and the region defined would not be closed. If, however, L encircles two branch points, or the same one twice, M may be taken a simple contour enclosing one of the μ' points, say $(1 - \lambda^4)^{\frac{1}{2}}$.

The integration with respect to μ now gives the result

$$2i\pi \int_L \frac{1}{2\mu'} d\lambda \quad \text{or} \quad i\pi \int_L (1 - \lambda^4)^{-\frac{1}{2}} d\lambda,$$

and this, when L encircles two branch points, is one of the periods of an Abelian integral associated with the equation

$$\lambda^4 + \mu^2 - 1 = 0.$$

In case (3) when Θ, Z vanish together, the result may be stated in the same way, the Abelian integral being of the third kind, and the period that associated with a contour encircling its parametric points. The value of the contribution is, however, capable of being directly expressed in a simple form.

Suppose λ to leave the path L and travel inwards to λ_r , the point under discussion, without passing through any position in which there is a coincidence among the points μ', μ'' . As λ travels we can deform M continuously, so that M does not pass through a μ' or a μ'' until λ actually reaches λ_r . Then, if at λ_r an enclosed μ' comes into coincidence with an excluded μ'' or conversely, it ceases to be possible to make M pass between them, and there will be a contribution to the value of the double integral of

$$\pm (2i\pi)^2 \Omega(\lambda_r, \mu_r),$$

where

$$\Omega = \Phi(x, t) \Psi(y, u) \left/ \frac{\partial(\Theta, Z)}{\partial(\lambda, \mu)} \right.$$

(in general, the numerator of the subject of integration divided by the Jacobian of the factors of the denominator). The lower sign is to be taken when μ'' is excluded, the upper when it is enclosed.

When the coincidence is between an enclosed μ' and an enclosed μ'' , or between an excluded μ' and an excluded μ'' , there is no contribution, and the coincidence does not hinder us from drawing M in the way prescribed.

Case (2) is essentially the same as case (3), for, when the curve $\Theta = 0$ has a node at (λ_r, μ_r) , Θ may be expressed as the product of two functions, each without singularity in the domain of (λ_r, μ_r) , and each vanishing at that point to the first order only.

General Idea of the Proposed Application of these Results.

6b. The values of λ for which a μ' coincides with a μ'' form an infinite series, and it will be our object so to choose the region of integration that the value of the double integral will be expressed by the corresponding series $\Sigma\Omega$, multiplied by $(2i\pi)^2$. We must avoid the possibility of contributions from branch points as in case (1), and take in all possible contributions under case (3).

The general idea will be to have the μ'' points, where Z vanishes, enclosed by M , and the μ' points, where Θ vanishes, excluded. If there is a node on $Z = 0$, so that two μ'' points coincide, this will make no difference in the form of the result, since the coincidence will be between two enclosed points; and similarly if the node is on $\Theta = 0$. For the present we shall suppose the intersections of the two curves to be simple, so as to avoid complications that fall beyond the scope of the account that has been given.

It will be noticed that the terms of the series $\Sigma\Omega$ are symmetric functions of x, t as well as of y, u .

In order to decide on a suitable region of integration, we must study the distribution of the values of μ for which Θ, Z vanish severally when λ has a fixed value, and more particularly when that value is great and not real. With a view to this we find approximate values for ϕ, ψ and their derivatives.

As an illustration of the use to be made of the double integral here given, it may be pointed out that the Fourier expansion in a series of sines depends on the series

$$\sum_{n=1}^{\infty} \sin n\pi x \sin n\pi t,$$

which is the series of periods of the integral

$$-\frac{1}{4i\pi} \int \frac{\sin(x\sqrt{\lambda}) \sin\{(1-t)\sqrt{\lambda}\}}{\sqrt{\lambda} \sin\sqrt{\lambda}} d\lambda.$$

Here, again, the subject of integration is unsymmetrical, but the residues are symmetrical, as functions of x and t . The subject of integration, moreover, is very small when λ is great and complex or negative, provided $x < t$. The object of my former paper (I.) was to generalize this expression and apply it to the series of Liouville and Sturm for functions of one variable. It is now to be further extended to functions of two variables.

Approximations by means of Hyperbolic Functions when λ, μ are Great.

7. If we write $P(x)$ for $-\lambda S + \mu T - X$, and suppose λ, μ to be great, the following formulæ are, in general, approximately true.*

$$\left. \begin{aligned} \phi(b, a) &= \{P(a)P(b)\}^{-\frac{1}{2}} \sinh \int_a^b P^{\frac{1}{2}} dx \\ \phi'(b, a) &= \{P(b)/P(a)\}^{\frac{1}{2}} \cosh \int_a^b P^{\frac{1}{2}} dx \\ \phi''(b, a) &= -\{P(a)/P(b)\}^{\frac{1}{2}} \cosh \int_a^b P^{\frac{1}{2}} dx \\ \phi'''(b, a) &= -\{P(a)P(b)\}^{\frac{1}{2}} \sinh \int_a^b P^{\frac{1}{2}} dx \end{aligned} \right\} \quad (6)$$

In proving these formulæ and discussing the restrictions under which they hold, we shall use the following lemma, which is closely related to the formulæ of § 3.

Let $f(x, a)$ denote the function which satisfies the equation

$$\frac{d^2 f}{dx^2} = Qf$$

and the further conditions that when $x = a$, $f = 0$, and $df/dx = 1$. Let Q be changed to $Q + \delta Q$, and suppose δf to denote the consequent increase in f . Then

$$\delta f(b, a) = \int_a^b f(b, x) \{f(x, a) + \delta f(x, a)\} \delta Q dx \quad (7)$$

$$= \int_a^b \{f(b, x) + \delta f(b, x)\} f(x, a) \delta Q dx. \quad (8)$$

For

$$f''(x, b) = Qf(x, b)$$

and

$$f''(x, a) + \delta f''(x, a) = (Q + \delta Q) \{f(x, a) + \delta f(x, a)\}.$$

* Accurately

$$\phi(b, a) = (\sigma' a \sigma' b)^{-\frac{1}{2}} \sinh(\sigma b - \sigma a),$$

where σx is a function defined by the equation

$$(\sigma' x)^2 - \frac{1}{2} \{\sigma x, x\} = P(x).$$

If we put y for $(\sigma' x)^{-\frac{1}{2}}$, this equation becomes

$$y'' + 1/y^3 = y P(x),$$

which, with the slightly more general $y'' + e/y^3 = y P(x)$,

may be added to the list of forms associated with the linear equation of the second order, and which, like the linear equation, is such that, if one solution is known, all the rest can be deduced.

If ϕ_1, ϕ_2 are two solutions of (1) and are such that $\phi_1' \phi_2 - \phi_2' \phi_1 = 1$, then the most general value of y is $(A\phi_1^2 + 2B\phi_1\phi_2 + C\phi_2^2)^{\frac{1}{2}}$, where A, B, C are constants such that $B^2 - AC = e$: a value of σx is $\tanh^{-1} \phi_1/\phi_2$.

Hence

$$\begin{aligned} f'(x, b) \{f(x, a) + \delta f(x, a)\} - f(x, b) \{f'(x, a) + \delta f'(x, a)\} \\ = -\delta Q f(x, b) \{f(x, a) + \delta f(x, a)\}. \end{aligned}$$

Hence

$$\begin{aligned} \int_a^b f(b, x) \{f(x, a) + \delta f(x, a)\} \delta Q dx \\ = \int_{x=a}^{x=b} [f'(x, b) \{f(x, a) + \delta f(x, a)\} - f(x, b) \{f'(x, a) + \delta f'(x, a)\}] \\ = [f(b, a) + \delta f(b, a)] - [-f(a, b)] = \delta f(b, a), \end{aligned}$$

since

$$f(a, a) = f(b, b) = 0$$

and

$$f'(a, a) = f'(b, b) = 1 = f'(a, a) + \delta f'(a, a);$$

so that

$$\delta f'(a, a) = 0.$$

Interchanging a, b in the result, we have the second form (8) for $\delta f(b, a)$.

Now, in order that $f(b, a)$ may, for all values of a, b , have the value

$$\{P(a)P(b)\}^{-\frac{1}{2}} \sinh \int_a^b P^{\frac{1}{2}} dx,$$

proposed above as an approximation to $\phi(b, a)$, we must put

$$Q = P + \frac{5}{16}(P'/P)^2 - \frac{1}{4}P''/P,$$

since this is then the value of $f''(x, a)/f(x, a)$, as may readily be verified.

Then, if

$$\delta Q = -\frac{5}{16}(P'/P)^2 + \frac{1}{4}P''/P,$$

we have

$$Q + \delta Q = P \quad \text{and} \quad f + \delta f = \phi.$$

The error committed when $f(b, a)$ is taken as the value of $\phi(b, a)$ is $\delta f(b, a)$, and an estimate of its magnitude can be formed by the use of the Lemma (7) as follows.

If α, β are real quantities,

$$\sinh(\alpha + i\beta) \sinh(\alpha - i\beta) = \sinh^2 \alpha + \sin^2 \beta = \cosh^2 \alpha - \cos^2 \beta.$$

Hence

$$|\sinh(\alpha + i\beta)| \geq |\sinh \alpha| \quad \text{and} \quad \leq \cosh \alpha.$$

Thus, if the element $P^{\frac{1}{2}} dx$ is separated into a real part $d\rho$ and a purely imaginary part $i d\sigma$,

$$\left| \sinh \int_a^b P^{\frac{1}{2}} dx \right| \leq \cosh \int_a^b d\rho,$$

and this again does not exceed $\cosh \int_a^b |d\rho|$, where the integral contains the same elements as $\int_a^b d\rho$ but all taken positively, and is less than

$\cosh \int_a^b |d\rho|$ unless $d\rho$ is of constant sign, since the hyperbolic cosine increases continually with its argument.

Suppose η to be the greatest absolute value of

$$\delta f(x_1, x_0) / \{P(x_1) P(x_0)\}^{-\frac{1}{2}} \cosh \int_{x_0}^{x_1} |d\rho|$$

for all positions of x_1, x_0 on the path from a to b .

We have

$$|f(x_1, x_0)| = \left| \{P(x_1) P(x_0)\}^{-\frac{1}{2}} \sinh \int_{x_0}^{x_1} P^{\frac{1}{2}} dx \right| \leq |P(x_1) P(x_0)|^{-\frac{1}{2}} \cosh \int_{x_0}^{x_1} |d\rho|,$$

and therefore

$$|\phi(x_1, x_0)| = |f(x_1, x_0) + \delta f(x_1, x_0)| \leq (1 + \eta) |P(x_1) P(x_0)|^{-\frac{1}{2}} \cosh \int_{x_0}^{x_1} |d\rho|,$$

But, by (7),

$$\begin{aligned} |\delta f(b, a)| &= \left| \int_a^b f(b, x) \phi(x, a) \delta Q dx \right| \\ &\leq \int_a^b |f(b, x)| |\phi(x, a)| |\delta Q| |dx| \\ &\leq \int_a^b |P(b) P(x) P(x) P(a)|^{-\frac{1}{2}} \cosh \int_x^b |d\rho| (1 + \eta) \cosh \int_a^x |d\rho| |\delta Q| |dx|. \end{aligned}$$

Now
$$\int_x^b |d\rho| + \int_a^x |d\rho| = \int_a^b |d\rho|,$$

and therefore
$$\cosh \int_x^b |d\rho| \cosh \int_a^x |d\rho| < \cosh \int_a^b |d\rho|.$$

Let M be the greatest value of $|P^{-\frac{1}{2}} \delta Q|$ and h the length of the path of integration, that is, $\int_a^b |dx|$. Then we have

$$|\delta f(b, a)| < (1 + \eta) |P(b) P(a)|^{-\frac{1}{2}} \cosh \int_a^b |d\rho| Mh,$$

and the same is clearly true if intermediate values x_1, x_0 are put in the places of b, a . Thus we must have

$$\eta < (1 + \eta) Mh \quad \text{or} \quad \eta < Mh / (1 - Mh)$$

if $Mh < 1$. Now a, b are, in our case, between 0 and 1; so that h is finite, and therefore η will be small if M is small. In applying the result, we shall be able to suppose λ, μ to have finite ratios to k^2 , where k is an integer that is increased without limit. If two quantities A, B , depending on λ, μ , are such that A/B and B/A do not, in absolute magnitude,

increase beyond a fixed finite limit when k is increased without limit, we shall say that A, B are of the same order of magnitude, or that B is of the same order as A , or that B is comparable with A , or symbolically $A \parallel B$.

Thus P, P', P'' are, in general, of the same order of magnitude as k^2 , but for special values of x one or more of them may fall below that order, that is, the ratio P/k^2 , for instance, may diminish without limit as k increases, and it will do so, for example, if λ, μ depend on k in such a way that λ/μ constantly $= T/S$ for the value of x in question.

Since

$$\delta Q = -\frac{5}{16}(P'/P)^2 + \frac{1}{4}P''/P$$

and M is the greatest value of $|P^{-\frac{1}{2}}\delta Q|$ it follows that $M \parallel 1/k$ if $P \parallel k^2$ throughout the path of integration.

Hence, under the same restriction, η will $\parallel 1/k$ and the error $\delta f(b, a)$ will tend to vanish in comparison with $f(b, a)$ as k increases (1) if the argument $\int_a^b d\rho$ is not less than $\int_a^b |d\rho|$, that is, if the sign of $d\rho$ is constant along the path,* and also (2) if the argument $\int_a^b |d\rho|$ does not increase beyond a finite limit when k is increased, in which case $\int_a^b d\rho$ is finite also, and the factor $\sinh \int_a^b P^{\frac{1}{2}} dx$ will be finite.

We are *not* generally entitled to use $f(b, a)$ as an approximation for $\phi(b, a)$

(1) if for some value of x on the path ab , P/k^2 tends to zero as k increases: for then δQ may increase without limit with k ;

(2) if $d\rho$ is not of constant sign on the path ab , and $\int_a^b |d\rho|$ increases indefinitely with k , for then the possible error

$$\eta \{P(b)P(a)\}^{-\frac{1}{2}} \cosh \int_a^b |d\rho|$$

may much exceed the estimated value

$$\{P(b)P(a)\}^{-\frac{1}{2}} \sinh \int_a^b P^{\frac{1}{2}} dx.$$

The last exception (2) can easily be avoided in the application we have

* Since x is a real variable, this implies that the real part of $P^{\frac{1}{2}}$ is of constant sign. It may be remarked, once for all, that, if u is positive, the real part of $(u+iv)^{\frac{1}{2}}$ lies between $\sqrt{u}$ and $(u^2+v^2)^{\frac{1}{4}}$, and the lateral part between $\frac{1}{2}v/\sqrt{u}$ and $\frac{1}{2}v(u^2+v^2)^{-\frac{1}{4}}$, and that, if u is negative, the lateral part lies between $\sqrt{u}$ and $i(u^2+v^2)^{\frac{1}{4}}$, the real part between $\frac{1}{2}v/\sqrt{-u}$ and $\frac{1}{2}v(u^2+v^2)^{-\frac{1}{4}}$.

Neither part of $(u+iv)^{\frac{1}{2}}$ can vanish or change sign unless v does so.

When v is small as compared to u , and u is positive, the real part of $(u+iv)^{\frac{1}{2}}$ differs from $\sqrt{u}$ by a quantity $< (u^2+v^2)^{\frac{1}{4}} - \sqrt{u}$ and therefore $\parallel v^2/u^{\frac{3}{2}}$ at most.

to make, but the restriction implied by (1) is a difficulty that must be removed.

Under the same restrictions, the derivatives of $f(b, a)$ furnish approximate values for those of $\phi(b, a)$. For instance,

$$\delta f'(b, a) = \phi'(b, a) - f'(b, a),$$

and by differentiating (7) with respect to b , we have

$$\delta f'(b, a) = \int_a^b f'(b, x) \{f(x, a) + \delta f(x, a)\} \delta Q dx.$$

But

$$f'(b, a) = \{P(b)/P(a)\}^{\frac{1}{2}} \cosh \int_a^b P^{\frac{1}{2}} dx - \frac{1}{4} P'(b) [P(a)]^{-\frac{1}{2}} [P(b)]^{-\frac{3}{2}} \sinh \int_a^b P^{\frac{1}{2}} dx,$$

$$\text{and } |f'(b, a)| \gtrsim |P(b)/P(a)|^{\frac{1}{2}} \cosh \int_a^b |d\rho| \{1 + \frac{1}{4} |P'(b) [P(b)]^{-\frac{3}{2}}|\}.$$

Thus

$$|\delta f'(b, a)| \gtrsim \int_a^b |P(b)/P(x)|^{\frac{1}{2}} \cosh \int_x^b |d\rho| \{1 + \frac{1}{4} |P'(b) [P(b)]^{-\frac{3}{2}}|\} \\ \times (1 + \eta) |P(x)P(a)|^{-\frac{1}{2}} \cosh \int_a^x |d\rho| |\delta Q| |dx|.$$

$$\text{or } |P(b)/P(a)|^{\frac{1}{2}} (1 + \eta) \{1 + \frac{1}{4} |P'(b) [P(b)]^{-\frac{3}{2}}|\} \int_a^b |P^{-\frac{1}{2}} \delta Q| \cosh \int_a^b |d\rho| |dx|$$

$$\text{or } |P(b)/P(a)|^{\frac{1}{2}} \{1 + \frac{1}{4} |P'(b) [P(b)]^{-\frac{3}{2}}|\} M\eta (1 + \eta) \cosh \int_a^b |d\rho|.$$

Under the restrictions assumed, $P'(b) [P(b)]^{-\frac{3}{2}}$, M , η are $\ll 1/k$, so that $\delta f'(b, a) \parallel$

$$k^{-1} |P(b)/P(a)|^{\frac{1}{2}} \cosh \int_a^b |d\rho|,$$

that is, $1/k$ of the estimated value $f'(b, a)$.

Moreover, the second term in the value of $f'(b, a)$ is only $\parallel \delta f'(b, a)$, and thus we have, as stated in (6),

$$\phi'(b, a) = \{P(b)/P(a)\}^{\frac{1}{2}} \cosh \int_a^b P^{\frac{1}{2}} dx$$

$$\text{with an error } \parallel k^{-1} |P(b)/P(a)|^{\frac{1}{2}} \cosh \int_a^b |d\rho|.$$

The same method applies to $\phi'(b, a)$ and $\phi''(b, a)$.

We shall apply the term "normal" to approximations in which the error is of the order of $1/k$ of the estimated value or $1/k^2$ of the derivative of the estimated value. The reference to the derivative is meant to save misunderstanding in the neighbourhood of a point where the function

vanishes.* The values given under (6) are normal approximations under the restrictions assumed; they are, however, valid approximations under less narrow restrictions. The relative magnitude of the error, η , at most $\parallel$ the greatest value of $\delta Q/P^{\frac{3}{2}}$, that is,

$$P''/4P^{\frac{3}{2}} - 5P'^2/16P^{\frac{5}{2}} \quad \text{or} \quad -P^{-\frac{1}{2}} d^2/dx^2(P^{-\frac{1}{2}}),$$

and, since P' , P'' do not exceed the order of k^2 , the error will be relatively small if $P^{\frac{3}{2}}$ does not fall as low as the order k^4 , that is, if P is always great in comparison with $k^{\frac{4}{3}}$. Also, when $d\rho$ is not of constant sign, the approximation is valid if $M \exp \int_a^b |d\rho|$ is small, which will happen if the real part of P is always negative and $\parallel k^2$, and the lateral part small in comparison with $k \log k$.

The results of §§ 8-14 are not wanted till § 22, and the reader is advised to skip them for the time being. They relate to exceptional cases in which the formulæ of § 7 do not apply.

Approximation by means of Riccati Functions.

8. We must now seek for approximate values of ϕ and its derivatives when there is a value of x on the path of integration at which P/k^2 diminishes without limit when k increases. The case that is of importance is when the lateral part of P is of a lower order than k^2 but of constant sign, while the real part is generally $\parallel k^2$ but occasionally zero. Thus the path of integration passes near a value of x for which $P=0$,† and in such a neighbourhood the values (6) could not well be approximations to $\phi(b, a)$, ..., since they are not one-valued functions of b , on account of the surd factor, whereas $\phi(b, a)$ and its derivatives are one-valued functions of b .

When the zero of P is simple and isolated‡ we may use, instead of the hyperbolic sine of the former approximation, the function ω which satisfies the equation

$$\frac{d^2\omega}{dx^2} = x\omega.$$

This equation is included in Riccati's, and the functions satisfying it are as definite as the sine and cosine, though not so familiar. The notation $\omega(b, a)$, $\omega'(b, a)$, ..., will be used as in former cases.

* For instance, if the approximate value were $\cos kx$, with a possible error of $1/k$, the approximation would be normal even when $x = \pi/2k$, although for that value the error would exceed the estimated value.

† If we write $P_1, \iota P_2$ for the parts of P which are real and purely imaginary when x is real, and suppose $P_1 = 0$ when $x = a$, then $P = 0$ for a value of x which is approximately

$$a - \iota P_2(a)/P'_1(a)$$

and which tends to coincide with a when k is made great, since P_2 is, by supposition, of a lower order than k^2 .

‡ That is, when no other zero tends to coincide with it as k increases.

Now, if, in (1), we change the independent variable from x to w , a known function of x , the equation becomes

$$\frac{d^2}{dw^2} (\phi \sqrt{w'}) = \phi \sqrt{w'} [P + \frac{1}{2} \{w, x\}] / w'^2,$$

where $\{w, x\}$ is the Schwarzian derivative.

We choose w so that $P/w'^2 = w$, that is, we take

$$\frac{2}{3} w^{\frac{3}{2}} = \int P^{\frac{1}{2}} dx,$$

fixing the lower limit in this integral at the (complex) value of x for which P vanishes. Then w is an analytical function of x for values within a certain domain which includes the point where P vanishes and which does not diminish indefinitely as k increases. When x is in the domain at a finite distance from the zero of P , $w \parallel k^{\frac{2}{3}}$. The domain includes the real value of x for which the real part of P vanishes, since the zero of P tends to that value as a limit when k increases.

We propose to prove that an approximate value of $\phi(x, x_0)$ is

$$(w'w'_0)^{-\frac{1}{2}} \omega(w, w_0),$$

and therefore substitute this expression for $f(x, x_0)$ in the Lemma (7). We must then have

$$Q = f''/f = P - \frac{1}{2} \{w, x\}, \quad \delta Q = \frac{1}{2} \{w, x\} = w'''/2w' - 3w''^2/4w'^2, \quad Q + \delta Q = P;$$

so that $f + \delta f = \phi$.

In the domain δQ is finite, however great k may be, since we may exclude from the domain values of x for which w' tends to zero as k increases. To ascertain the degree of approximation, we must find the order of magnitude of $f(b, x)f(x, a)/f(b, a)$ by a discussion of the function ω .

Approximate Value of ω .

9. Now, for great values of the argument, we may approximate to ω by means of the functions χ which satisfy the equation

$$\frac{d^2 \chi}{dx^2} = \left(x + \frac{5}{16x^2}\right) \chi$$

and which are known, since we have, in fact,

$$\chi(b, a) = a^{-\frac{1}{2}} b^{-\frac{1}{2}} \sinh \frac{2}{3} (b^{\frac{3}{2}} - a^{\frac{3}{2}}).$$

In the Lemma (7) we may put χ for f , ω for $f + \delta f$, and $-\frac{5}{16x^2}$ for δQ . Thus

$$\omega(b, a) - \chi(b, a) = -\frac{5}{16} \int_a^b \chi(b, x) \omega(x, a) \frac{dx}{x^2}.$$

Suppose now that, as x travels from a to b , the real part of $x^{\frac{3}{2}}$ does not ever decrease; then the real part of $x_1^{\frac{3}{2}} - x_0^{\frac{3}{2}}$ is positive or zero, and

$$|\sinh \frac{2}{3} (x_1^{\frac{3}{2}} - x_0^{\frac{3}{2}})| \gtrsim |\exp \frac{2}{3} (x_1^{\frac{3}{2}} - x_0^{\frac{3}{2}})|$$

if x_0, x_1 are two values on the path ab in the order stated.

Suppose κ to be the greatest value of

$$|\omega(x_1, x_0) - \chi(x_1, x_0)| / |x_1^{-\frac{1}{3}} x_0^{-\frac{1}{3}} \exp \frac{2}{3} (x_1^{\frac{2}{3}} - x_0^{\frac{2}{3}})|$$

for all such positions of x_0, x_1 on the path.

Then we have, as in § 7,

$$|\omega(b, a) - \chi(b, a)| \gtrsim \frac{5}{16} |a^{-\frac{1}{3}} b^{-\frac{1}{3}} \exp \frac{2}{3} (b^{\frac{2}{3}} - a^{\frac{2}{3}})| (1 + \kappa) \int_a^b |x^{-\frac{1}{3}} dx|,$$

and thus

$$\kappa \gtrsim \frac{5}{16} (1 + \kappa) \int_a^b |x^{-\frac{1}{3}} dx|.$$

Now $\int_a^b |x^{-\frac{1}{3}} dx| = \frac{2}{3}$ of the length of the path described by the point representing $x^{-\frac{1}{3}}$, and this will be of no higher order of magnitude than the greatest value of $|x^{-\frac{1}{3}}|$ on the path. Hence κ is of the same order also when x is great all along the path ab .

We conclude, therefore, that, if a, b can be joined by a path along which x is always great, and the real part of $x^{\frac{2}{3}}$ does not ever decrease, the difference between $a^{\frac{1}{3}} b^{\frac{1}{3}} \omega(b, a)$ and $a^{\frac{1}{3}} b^{\frac{1}{3}} \chi(b, a)$, that is, $\sinh \frac{2}{3} (b^{\frac{2}{3}} - a^{\frac{2}{3}})$, $|| \exp \frac{2}{3} (b^{\frac{2}{3}} - a^{\frac{2}{3}})$ multiplied by the greatest value of $x^{-\frac{1}{3}}$ on the path.

As in § 7, it follows that

$$\begin{aligned} a^{\frac{1}{3}} b^{-\frac{1}{3}} \omega'(b, a) - \cosh \frac{2}{3} (b^{\frac{2}{3}} - a^{\frac{2}{3}}), & \quad a^{-\frac{1}{3}} b^{\frac{1}{3}} \omega'(b, a) + \cosh \frac{2}{3} (b^{\frac{2}{3}} - a^{\frac{2}{3}}), \\ a^{-\frac{1}{3}} b^{-\frac{1}{3}} \omega''(b, a) + \sinh \frac{2}{3} (b^{\frac{2}{3}} - a^{\frac{2}{3}}) & \end{aligned}$$

are also of the same order of magnitude.

Approximations to $\omega(a, 0)$ and its Derivatives when a is Negative and Great.

10. The approximations here given do not apply immediately to $\omega(a, 0)$ and its derivatives. In order to deal with these we notice first that they can be expanded directly from the differential equation in series of ascending powers of a , which are absolutely convergent for all finite values of a . Thus

$$\begin{aligned} \omega(a, 0) &= a + \frac{a^4}{3 \cdot 4} + \frac{a^7}{3 \cdot 4 \cdot 6 \cdot 7} + \frac{a^{10}}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} + \dots, \\ \omega'(a, 0) &= 1 + \frac{a^3}{3} + \frac{a^6}{3 \cdot 4 \cdot 6} + \dots, \\ -\omega''(a, 0) &= 1 + \frac{a^3}{2 \cdot 3} + \frac{a^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{a^9}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} + \dots, \\ -\omega'''(a, 0) &= \frac{a^2}{2} + \frac{a^5}{2 \cdot 3 \cdot 5} + \dots \end{aligned}$$

When $a = -1$, in particular, these are all numerically less than 1.

Again, if $\frac{d^2 \omega}{dx^2} = x\omega$,

$$\frac{d}{dx} \left\{ \omega^2 - \frac{1}{x} \left(\frac{d\omega}{dx} \right)^2 \right\} = \frac{1}{x^2} \left(\frac{d\omega}{dx} \right)^2,$$

a positive quantity; so that, if x is negative, $\omega^2 - \omega'^2/x$ decreases as $-x$ increases. Hence, when $a < -1$,

$$\{\omega(a, 0)\}^2 - \{\omega'(a, 0)\}^2/a < 2, \quad \{\omega'(a, 0)\}^2 - \{\omega(a, 0)\}^2/a < 2,$$

since each of these < 2 when $a = -1$.

Thus, since a is negative,

$$\omega(a, 0) < \sqrt{2}, \quad \omega'(a, 0) < \sqrt{2}, \quad \omega'(a, 0) < \sqrt{(-2a)}, \quad \omega''(a, 0) < \sqrt{(-2a)}.$$

Now the approximate values found for $\omega(b, a)$ and its derivatives may be used when a, b are both negative, since, on the negative part of the real axis, the real part of $x^{\frac{2}{3}}$ is always 0. Also, if a is numerically less than b , the error $\parallel a^{-\frac{2}{3}}$, since $\frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}})$, the argument of the exponential factor, is a pure imaginary.

We have then, as in § 2,

$$\omega(b, 0) = -\omega(a, 0) \omega'(b, a) + \omega'(a, 0) \omega(b, a)$$

accurately, and approximately

$$b^{\frac{1}{3}} \omega(b, 0) = a^{\frac{1}{3}} \omega(a, 0) \cosh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) + a^{-\frac{1}{3}} \omega'(a, 0) \sinh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}). \quad (10)$$

Since $\omega(a, 0) < \sqrt{2}$, $\omega'(a, 0) < \sqrt{(-2a)}$, and the hyperbolic factor is subject to a possible error $\parallel a^{-\frac{2}{3}}$, the whole expression is subject to an error $\parallel a^{-\frac{2}{3}}$ at most.

Also, since $\omega'(b, 0) = -\omega(a, 0) \omega''(b, a) + \omega''(a, 0) \omega'(b, a)$, we have

$$b^{-\frac{1}{3}} \omega'(b, 0) = a^{\frac{1}{3}} \omega(a, 0) \sinh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) + a^{-\frac{1}{3}} \omega'(a, 0) \cosh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}), \quad (11)$$

with a possible error of the same order again. From (10), (11) we have

$$b^{\frac{1}{3}} \omega(b, 0) \cosh \frac{2}{3}b^{\frac{2}{3}} - b^{-\frac{1}{3}} \omega'(b, 0) \sinh \frac{2}{3}b^{\frac{2}{3}} = a^{\frac{1}{3}} \omega(a, 0) \cosh \frac{2}{3}a^{\frac{2}{3}} - a^{-\frac{1}{3}} \omega'(a, 0) \sinh \frac{2}{3}a^{\frac{2}{3}},$$

with a possible error $\parallel a^{-\frac{2}{3}}$ at most. This is true when a, b are real and negative, and b is numerically greater than a .

Hence, as a travels along the real axis towards $-\infty$,

$$a^{\frac{1}{3}} \omega(a, 0) \cosh \frac{2}{3}a^{\frac{2}{3}} - a^{-\frac{1}{3}} \omega'(a, 0) \sinh \frac{2}{3}a^{\frac{2}{3}}$$

tends to a finite limiting value, say A_1 .

In the same way

$$b^{\frac{1}{3}} \omega(b, 0) \sinh \frac{2}{3}b^{\frac{2}{3}} - b^{-\frac{1}{3}} \omega'(b, 0) \cosh \frac{2}{3}b^{\frac{2}{3}} = a^{\frac{1}{3}} \omega(a, 0) \sinh \frac{2}{3}a^{\frac{2}{3}} - a^{-\frac{1}{3}} \omega'(a, 0) \cosh \frac{2}{3}a^{\frac{2}{3}},$$

with a possible error $\parallel a^{-\frac{2}{3}}$, and the last expression also tends to a finite limit, say A_2 , as a travels towards $-\infty$ on the real axis.

Hence

$$a^{\frac{1}{3}} \omega(a, 0) = A_1 \cosh \frac{2}{3}a^{\frac{2}{3}} - A_2 \sinh \frac{2}{3}a^{\frac{2}{3}},$$

and

$$a^{-\frac{1}{3}} \omega'(a, 0) = A_1 \sinh \frac{2}{3}a^{\frac{2}{3}} - A_2 \cosh \frac{2}{3}a^{\frac{2}{3}},$$

with an error in each case $\parallel a^{-\frac{2}{3}}$ at most. In forming this estimate of the error, we used the inequalities $\omega(a, 0) < \sqrt{2}$, $\omega'(a, 0) < \sqrt{(-2a)}$; but, as it now appears that $a^{\frac{1}{3}} \omega(a, 0)$ and $a^{-\frac{1}{3}} \omega'(a, 0)$ do not exceed certain finite limits, however great $-a$ may be, we may say that the errors are $\parallel a^{-\frac{2}{3}}$ at most.

Again $\omega(a, 0)$, $\omega'(a, 0)$, $\cosh \frac{2}{3}a^{\frac{2}{3}}$, $a^{\frac{1}{3}} \sinh \frac{2}{3}a^{\frac{2}{3}}$ are real, and thus $A_1 a^{-\frac{1}{3}}$, $A_2 a^{\frac{1}{3}}$ are real. We may therefore put

$$A_1 = A \sinh a, \quad A_2 = -A \cosh a,$$

where a is a pure imaginary and A a complex quantity whose amplitude (vectorial angle) is $-\frac{1}{4}\pi$ or $\frac{3}{4}\pi$, that of a being π .

$$\text{Thus } \omega(a, 0) = Aa^{-\frac{1}{3}} \sinh(\frac{2}{3}a^{\frac{2}{3}} + a), \quad \omega'(a, 0) = Aa^{\frac{1}{3}} \cosh(\frac{2}{3}a^{\frac{2}{3}} + a)$$

nearly. In the same way, starting with the accurate equation,

$$\omega'(b, 0) = -\omega'(a, 0) \omega'(b, a) + \omega''(a, 0) \omega(b, a),$$

we may prove the approximate formulæ

$$a^{\frac{1}{3}} \omega'(a, 0) = B \sinh(\frac{2}{3}a^{\frac{2}{3}} + \beta), \quad a^{-\frac{1}{3}} \omega''(a, 0) = B \cosh(\frac{2}{3}a^{\frac{2}{3}} + \beta),$$

where the error $\parallel a^{-\frac{1}{3}}$ in each case, β is a pure imaginary, and B a complex quantity of amplitude $-\frac{1}{4}\pi$ or $\frac{3}{4}\pi$.

The same when the Vectorial Angle of a is between $\frac{1}{3}\pi$ and $\frac{5}{3}\pi$.

11. These expressions have been found on the supposition that a is real, negative, and great. They may be extended by the use of the formulæ for $\omega(b, a)$ and its derivatives. Since the vectorial angle of a is π , that of $a^{\frac{2}{3}}$ is $\frac{2}{3}\pi$; so that $a^{\frac{2}{3}}$ lies on the lower part of the imaginary axis. If, then, along the path ab the real part of $x^{\frac{2}{3}}$ constantly increases, $b^{\frac{2}{3}}$ must lie in the fourth or fifth quadrant, and the vectorial angle of b must lie between π and $\frac{5}{3}\pi$. For such a value of b we may use the approximate values of $\omega(b, a)$, ..., but the smallness of the error depends on our being able to keep the path ab well away from the origin, since the order of the possible error is that of the least value of $|x^{-\frac{2}{3}}|$ on the path. Hence the approximation would not be valid if the vectorial angle of b were actually $\frac{5}{3}\pi$.

The approximate formulæ give

$$\begin{aligned} \omega(b, 0) &= -\omega(a, 0) \omega'(b, a) + \omega'(a, 0) \omega(b, a) \\ &= Aa^{-\frac{1}{3}} \sinh(\frac{2}{3}a^{\frac{2}{3}} + a) a^{\frac{1}{3}} b^{-\frac{1}{3}} \cosh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) \\ &\quad + Aa^{\frac{1}{3}} \cosh(\frac{2}{3}a^{\frac{2}{3}} + a) a^{-\frac{1}{3}} b^{-\frac{1}{3}} \sinh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) \\ &= Ab^{-\frac{1}{3}} \sinh(\frac{2}{3}b^{\frac{2}{3}} + a), \end{aligned}$$

with a possible error $\parallel x_0^{-\frac{1}{3}} b^{-\frac{1}{3}} \cosh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}})$, where by x_0 we are to understand the least absolute value taken by x on the path ab . Now a may be as great numerically as we please, and, if we restrict b to have an amplitude between π and $\frac{4}{3}\pi$, $b^{\frac{2}{3}}$ will lie in the fourth quadrant, and the path ab may be so drawn that $|x|$ is never less than $|b|$, so that we have $x_0 = |b|$.

Hence, as $a^{\frac{2}{3}}$ is a pure imaginary, the possible error in the value of $\omega(b, 0)$

$$\parallel b^{-\frac{1}{3}} \exp \frac{2}{3}b^{\frac{2}{3}}.$$

Similarly $\omega'(b, 0) = Ab^{\frac{1}{3}} \sinh(\frac{2}{3}b^{\frac{2}{3}} + \alpha)$,

with a possible error $\parallel b^{-\frac{1}{3}} \exp \frac{2}{3}b^{\frac{2}{3}}$.

To the same respective degrees of approximation

$$\omega(b, 0) = Bb^{-\frac{1}{3}} \sinh(\frac{2}{3}b^{\frac{2}{3}} + \beta), \quad \omega^{\wedge}(b, 0) = Bb^{\frac{1}{3}} \cosh(\frac{2}{3}b^{\frac{2}{3}} + \beta).$$

The amplitude of b has been restricted to values greater than π and less than $\frac{4}{3}\pi$, but it is easily seen that it may be greater than $\frac{4}{3}\pi$ so long as it does not approach $\frac{5}{3}\pi$. Also, if it is less than π but greater than $\frac{1}{3}\pi$, a path ab can be drawn along which one of the two values of $b^{\frac{2}{3}}$ has an increasing real part, and the expressions given still hold. We may say that the errors are $\parallel$

$$b^{-\frac{2}{3}} \exp \frac{2}{3} \sqrt[3]{b^3} \quad \text{and} \quad b^{-\frac{1}{3}} \exp \frac{2}{3} \sqrt[3]{b^3},$$

using $\sqrt[3]{b^3}$ for that value which has a positive real part, while we keep the fractional power, such as $b^{\frac{2}{3}}$, to denote the value whose amplitude is $\frac{2}{3}$ of that of b .

The same for other great values of a . Values of AB, α, β .

12. It is now easy to find approximate values for $\omega(a, 0)$ and its derivatives when the vectorial angle of a takes any value whatever, as, for instance, a value between $\pm \frac{1}{3}\pi$.

The expressions for $\omega(a, 0)$, ... as series shew that

$$\omega(ae^{\frac{2}{3}i\pi}, 0) = e^{\frac{2}{3}i\pi} \omega(a, 0),$$

$$\omega'(ae^{\frac{2}{3}i\pi}, 0) = \omega'(a, 0),$$

$$\omega^{\wedge}(ae^{\frac{2}{3}i\pi}, 0) = \omega^{\wedge}(a, 0),$$

$$\omega^{\wedge\wedge}(ae^{\frac{2}{3}i\pi}, 0) = e^{-\frac{2}{3}i\pi} \omega^{\wedge\wedge}(a, 0).$$

When the amplitude of a is between 0 and $\frac{1}{3}\pi$, that of $ae^{\frac{2}{3}i\pi}$ is between $\frac{2}{3}\pi$ and π , and we may therefore use the known formulæ for $\omega(ae^{\frac{2}{3}i\pi}, 0)$.

Thus

$$\begin{aligned} \omega(a, 0) &= e^{-\frac{2}{3}i\pi} \omega(ae^{\frac{2}{3}i\pi}, 0) = e^{-\frac{2}{3}i\pi} Aa^{-\frac{1}{3}} e^{-\frac{1}{3}i\pi} \sinh(-\frac{2}{3}a^{\frac{2}{3}} + \alpha) \\ &= Ae^{-\frac{5}{6}i\pi} a^{-\frac{1}{3}} \sinh(-\frac{2}{3}a^{\frac{2}{3}} + \alpha) \end{aligned}$$

approximately. So also

$$\omega'(a, 0) = Ae^{\frac{1}{6}i\pi} a^{\frac{1}{3}} \cosh(-\frac{2}{3}a^{\frac{2}{3}} + \alpha), \quad \omega^{\wedge}(a, 0) = Be^{-\frac{1}{3}i\pi} a^{-\frac{1}{3}} \sinh(-\frac{2}{3}a^{\frac{2}{3}} + \beta),$$

$$\omega^{\wedge\wedge}(a, 0) = Be^{\frac{1}{6}i\pi} a^{\frac{1}{3}} \cosh(-\frac{2}{3}a^{\frac{2}{3}} + \beta),$$

nearly, the possible errors being as before.

The values of α, β can now be found from the consideration that $\omega(a, 0)$ and its derivatives are real when a is real and positive, as follows at once from the expansions in powers of a . As a increases, $\sinh(-\frac{2}{3}a^{\frac{2}{3}} + \alpha)$ tends to equality with $-\frac{1}{2} \exp(\frac{2}{3}a^{\frac{2}{3}} - \alpha)$, so that its vectorial angle is π in the limit, and $Ae^{-\frac{5}{6}i\pi - \alpha}$ must be real. Since the vectorial angle of A is $-\frac{1}{3}\pi$, we must have

$$-\frac{1}{3}\pi - \frac{5}{6}\pi - \alpha = \text{a multiple of } i\pi,$$

so that we may take $\alpha = -\frac{1}{12}i\pi$, say. In the same way we may take $\beta = -\frac{5}{12}i\pi$.

Again $1 = \omega(a, 0) \omega^{\wedge}(a, 0) - \omega'(a, 0) \omega^{\vee}(a, 0) = AB \sinh(a - \beta)$,

so that $AB = \operatorname{cosech} \frac{1}{3}\iota\pi = -2\iota/\sqrt{3}$.

Since the amplitude of A is $-\frac{1}{4}\pi$ or $\frac{3}{4}\pi$ and that of AB is $-\frac{1}{2}\pi$, it follows that B/A is positive.

When the amplitude of a is between 0 and $-\frac{1}{3}\pi$ similar expressions can be found. The same expressions may, in fact, be used for amplitudes between $\pm\frac{1}{3}\pi$, *exclusive*; we apparently get different results by deriving the functions of a from those of $ae^{\frac{2}{3}\iota\pi}$ and $ae^{-\frac{2}{3}\iota\pi}$, but the difference disappears if we use the formulæ

$$-\sinh(-\frac{2}{3}a^{\frac{3}{2}} + a) = \cosh(-\frac{2}{3}a^{\frac{3}{2}} + a) = \frac{1}{2} \exp(\frac{2}{3}a^{\frac{3}{2}} - a),$$

which hold approximately when a is great and has an amplitude between $\pm\frac{1}{3}\pi$, *exclusive*.

The Function χ .

13. As a tends to $+\infty$, $\omega(a, 0)/\omega^{\vee}(a, 0)$ tends to the value $-A/B$. Let $\chi(a)$ be written for $\omega(a, 0) + A/B \omega^{\vee}(a, 0)$; then χ, χ' cannot vanish except for real negative values of the argument, though they tend to zero when the argument is positive and increasing. For we have

$$\frac{d}{dx} \frac{\omega^{\vee}(x, 0)}{\omega(x, 0)} = \{\omega(x, 0)\}^{-2} = \frac{4}{A^2 \iota} x^{\frac{1}{2}} e^{-\frac{2}{3}x^{\frac{3}{2}}}$$

nearly, and, by integration between a and $+\infty$,

$$\frac{\omega^{\vee}(a, 0)}{\omega(a, 0)} + \frac{B}{A} = \frac{2\iota}{A^2} e^{-\frac{2}{3}a^{\frac{3}{2}}}$$

if a is a great quantity of amplitude between $\pm\frac{1}{3}\pi$. Thus

$$\chi(a) = \frac{A}{B} \omega(a, 0) \frac{2\iota}{A^2} e^{-\frac{2}{3}a^{\frac{3}{2}}},$$

which tends to zero as the real part of $a^{\frac{3}{2}}$ increases, being $\parallel a^{-\frac{1}{2}} e^{-\frac{2}{3}a^{\frac{3}{2}}}$.

Similarly

$$\frac{d}{dx} \frac{\omega^{\wedge}(x, 0)}{\omega'(x, 0)} = -x \{\omega'(x, 0)\}^{-2},$$

from which it follows that $\chi'(a)$ also tends to zero.

As x increases from 0, $\omega^{\vee}(x, 0)/\omega(x, 0)$ continually increases, beginning at $-\infty$, while $\omega^{\wedge}(x, 0)/\omega'(x, 0)$ continually decreases, beginning at 0. Thus $\chi(x), \chi'(x)$ can never vanish for positive values of x . Moreover,

$$\chi(x + iy) \chi'(x - iy) - \chi(x - iy) \chi'(x + iy) = 2iy \int_x^{\infty} \chi(x + iy) \chi(x - iy) dx,$$

from which it follows in the usual way that $\chi(x + iy)$ or $\chi'(x + iy)$ can only vanish when $y = 0$. It is therefore proved that χ, χ' can only vanish for real negative values of the argument.

Approximation to $\omega(b, a)$.

14. We can now find approximate values for $\omega(b, a)$ and its derivatives whatever the amplitudes of a, b , provided the absolute values of a, b are great. Take, for instance, the amplitude of b to be small and that of a nearly equal to π : to fix the ideas, let these amplitudes be between $\pm \frac{1}{6}\pi$ and $\pi \pm \frac{1}{6}\pi$ respectively. These limits are amply wide enough for the application that is to be made of the results; if they were not, somewhat wider ones could be taken without affecting the argument.

Now the real part of $b^{\frac{2}{3}}$ is great, and thus

$$\sinh\left(\frac{2}{3}b^{\frac{2}{3}}-a\right), \quad \cosh\left(\frac{2}{3}b^{\frac{2}{3}}-a\right), \quad \frac{1}{2}\exp\left(\frac{2}{3}b^{\frac{2}{3}}-a\right)$$

are practically the same, the differences being inconsiderable even in comparison with $b^{-\frac{1}{3}}$. The same is true when the argument is $\frac{2}{3}b^{\frac{2}{3}}-\beta$.

Also, we have

$$\omega(b, a) = \omega(a, 0) \omega'(b, 0) - \omega(b, 0) \omega'(a, 0),$$

the approximate value of which is

$$Aa^{-\frac{1}{3}} \sinh\left(\frac{2}{3}a^{\frac{2}{3}}+a\right) \frac{1}{2}Be^{\frac{5}{6}\pi} b^{-\frac{1}{3}} \exp\left(\frac{2}{3}b^{\frac{2}{3}}-\beta\right)$$

$$-\frac{1}{2}Ae^{\frac{5}{6}\pi} b^{-\frac{1}{3}} \exp\left(\frac{2}{3}b^{\frac{2}{3}}-a\right) Ba^{-\frac{1}{3}} \sinh\left(\frac{2}{3}a^{\frac{2}{3}}+\beta\right)$$

or

$$\frac{1}{2}ABa^{-\frac{1}{3}}b^{-\frac{1}{3}} \left\{ \sinh\left(\frac{2}{3}a^{\frac{2}{3}}-\frac{\pi}{12}\right) \exp\left(\frac{2}{3}b^{\frac{2}{3}}+\frac{5}{4}\pi\right) - \sinh\left(\frac{2}{3}a^{\frac{2}{3}}-\frac{5\pi}{12}\right) \exp\left(\frac{2}{3}b^{\frac{2}{3}}+\frac{\pi}{4}\right) \right\}$$

or, on simplifying and substituting the value of AB ,

$$a^{-\frac{1}{3}}b^{-\frac{1}{3}} \exp\left(\frac{2}{3}b^{\frac{2}{3}}+\frac{3}{4}\pi\right) \sinh\left(\frac{2}{3}a^{\frac{2}{3}}-\frac{\pi}{4}\right).$$

For this, again, we may write

$$a^{-\frac{1}{3}}b^{-\frac{1}{3}} \left\{ \sinh\frac{2}{3}(b^{\frac{2}{3}}+a^{\frac{2}{3}}) + \sinh\frac{2}{3}(b^{\frac{2}{3}}-a^{\frac{2}{3}}) \right\},$$

so long as the real parts of $b^{\frac{2}{3}} \pm a^{\frac{2}{3}}$ are both positive and great; this is really no restriction, for, when a is negative, $a^{\frac{2}{3}}$ is purely imaginary and the condition is certainly fulfilled. When the amplitude of a is below π , the real part of $a^{\frac{2}{3}}$ is negative, and in $b^{\frac{2}{3}}+a^{\frac{2}{3}}$ it is possible that the real part may not be great or may even be negative; but, if this happened, the first term $\sinh\frac{2}{3}(b^{\frac{2}{3}}+a^{\frac{2}{3}})$ would be inconsiderable in comparison with the second $\sinh\frac{2}{3}(b^{\frac{2}{3}}-a^{\frac{2}{3}})$. The converse would happen if the amplitude of a were above π .

The second term $a^{-\frac{1}{3}}b^{-\frac{1}{3}} \sinh\frac{2}{3}(b^{\frac{2}{3}}-a^{\frac{2}{3}})$ in the expression for $\omega(b, a)$ is the same in form as the original expression which we found by supposing a, b negative. The other term $a^{-\frac{1}{3}}b^{-\frac{1}{3}} \sinh\frac{2}{3}(b^{\frac{2}{3}}+a^{\frac{2}{3}})$ can be derived from the same expression by putting $be^{2\pi i}$ for b , that is, by travelling from a to b through the third and fourth quadrants instead of through the second and first. Hence the result may be stated in the form of a rule; take the two paths from a to b one above the origin, the other below, apply the formula $a^{-\frac{1}{3}}b^{-\frac{1}{3}} \sinh\frac{2}{3}(b^{\frac{2}{3}}-a^{\frac{2}{3}})$ to each of the two paths and add the results: the sum is the approximate value

for $\omega(b, a)$. There is a similar rule, similarly proved, for the derivatives of $\omega(b, a)$. The expressions are

$$\omega'(b, a) = a^{-\frac{1}{2}} b^{\frac{1}{2}} \{ \iota \cosh \frac{2}{3}(b^{\frac{2}{3}} + a^{\frac{2}{3}}) + \cosh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) \},$$

$$\omega'(b, a) = a^{\frac{1}{2}} b^{-\frac{1}{2}} \{ \iota \cosh \frac{2}{3}(b^{\frac{2}{3}} + a^{\frac{2}{3}}) - \cosh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) \},$$

$$\omega''(b, a) = a^{\frac{1}{2}} b^{\frac{1}{2}} \{ \iota \sinh \frac{2}{3}(b^{\frac{2}{3}} + a^{\frac{2}{3}}) - \sinh \frac{2}{3}(b^{\frac{2}{3}} - a^{\frac{2}{3}}) \}.$$

It is found without difficulty that, when a, b are both $\parallel k^{\frac{2}{3}}$, these approximations are normal in the sense of § 7.

When a or b is of a lower order of magnitude than $k^{\frac{2}{3}}$, the approximations are not so good, and when either a or b has a finite value, not increasing indefinitely with k , the error is of the same order of magnitude as the estimated value, so that we can only say that the estimated value is of the right order of magnitude. We can truly say so much in general, for, when a is finite, $\omega(a, 0)$ and its derivatives are finite, and the hyperbolic functions of $\frac{2}{3}(b^{\frac{2}{3}} \pm a^{\frac{2}{3}})$ are $\parallel$ those of $\frac{2}{3}b^{\frac{2}{3}}$; also the formula

$$\omega(b, a) = \omega(a, 0) \omega'(b, 0) - \omega'(a, 0) \omega(b, 0)$$

shews that $\omega(b, a)$, in general, $\parallel \omega'(b, 0)$ and $\omega(b, 0)$. It must be added that the factor $a^{\pm \frac{1}{2}}$ or $b^{\pm \frac{1}{2}}$ is to be dropped when a or b approaches 0, say when $|a|$ or $|b| < 1$.

Application to ϕ .

15. We are to apply these results to the problem of finding an approximate value for $\phi(b, a)$ and, in fact, finding the possibility of error when $f(b, a)$ is put for $\phi(b, a)$, where $f(x, x_0) = (w'w_0)^{-\frac{1}{2}} \omega(w, w_0)$ as in § 8.

Now $\frac{2}{3}w^{\frac{2}{3}} = \int P^{\frac{1}{2}} dx$, vanishing with P , and the lateral part of P is of constant sign, say $+\iota$, but of lower order than k^2 , while the real part vanishes and changes sign at a point of the path of x , but is generally of the order of k^2 : suppose the change to be from $-$ to $+$ as x increases. Before the change the amplitude of $P^{\frac{1}{2}}$ is slightly less than $\frac{1}{2}\pi$; after the change it is slightly greater than 0. In both parts of the path the real part of $w^{\frac{2}{3}}$ increases with x and, in fact, the point $w^{\frac{2}{3}}$ comes up to the neighbourhood of the origin along a path in the third quadrant which lies near the imaginary axis, then crosses the real axis, describes a small arc in the second quadrant, and comes out into the first quadrant, where it follows a path slightly above the real axis. The point w describes a path slightly above the real axis which begins far on the left in the second quadrant and ends far on the right in the first. We have found approximate values for $\omega(b, a)$ and its derivatives when ab is such a path, and we can also approximate to $\omega(b, c)$, $\omega(c, a)$ when c is any point on the path.

We have

$$f(x, x_0) = (w'w_0)^{-\frac{1}{2}} \omega(w, w_0) = (w'w_0)^{-\frac{1}{2}} \Sigma (ww_0)^{-\frac{1}{2}} \sinh \frac{2}{3}(w^{\frac{2}{3}} - w_0^{\frac{2}{3}})$$

nearly, the Σ referring to the two possible paths from w_0 to w .

[The notation has been changed, w_0, w taking the place of a, b , and x_0, x being the corresponding values of the original variable x in (1).]

But $w'^2 w = P$, $\frac{2}{3}w^{\frac{3}{2}} = \int P^{\frac{1}{2}} dx$, so that

$$f(x, x_0) = \Sigma (PP_0)^{-\frac{1}{2}} \sinh \int_{x_0}^x P^{\frac{1}{2}} dx \text{ nearly.}$$

Let η be the greatest absolute value of

$$\delta f(x_1, x_2) / (P_1 P_2)^{-\frac{1}{2}} \exp \int_{x_2}^{x_1} P^{\frac{1}{2}} dx$$

for any positions of x_1, x_2 on the path $x_0 x$, such that x_0, x_2, x_1, x occur in this order. Also, let γ be the greatest absolute value of

$$f(x_1, x_2) / (P_1 P_2)^{-\frac{1}{2}} \exp \int_{x_2}^{x_1} P^{\frac{1}{2}} dx.$$

Then we know that γ does not exceed a fixed finite limit, however great k may be, since there is a finite limit to the absolute value of

$$\omega(w_1, w_2) / (w_1 w_2)^{-\frac{1}{2}} \exp \frac{2}{3} (w_1^{\frac{3}{2}} - w_2^{\frac{3}{2}})$$

whether w_1, w_2 are small, finite, or great, provided that the real part of $w^{\frac{3}{2}}$ constantly increases along the path $w_2 w_1$.

Now $\phi = f + \delta f$, and thus $|\phi(x_1, x_2)|$ is not greater than

$$(\gamma + \eta) |P_1 P_2|^{-\frac{1}{2}} \left| \exp \int_{x_2}^{x_1} P^{\frac{1}{2}} dx \right|.$$

By the Lemma (7),

$$\begin{aligned} |\delta f(x, x_0)| &= \left| \int_{x_0}^x f(x, x_1) \phi(x_1, x_0) \delta Q_1 dx_1 \right| \\ &\leq \int_{x_0}^x \gamma(\gamma + \eta) |PP_1^2 P_0|^{-\frac{1}{2}} \left| \exp \int_{x_1}^x P^{\frac{1}{2}} dx \right| \left| \exp \int_{x_0}^{x_1} P^{\frac{1}{2}} dx \right| |\delta Q_1 dx_1|, \end{aligned}$$

and in this any two points on the path may be put for x_0, x , so that

$$\eta \leq \gamma(\gamma + \eta) \int_{x_0}^x |P^{-\frac{1}{2}} \delta Q dx|.$$

Now δQ is finite; suppose m to be its greatest value. $P^{-\frac{1}{2}} dx = 2d(P^{\frac{1}{2}})/P'$, and P' always $\parallel k^2$; suppose $|P'|$ always $> pk^2$, where p is a fixed finite number. Thus

$$\eta \leq 2\gamma(\gamma + \eta) \frac{m}{pk^2} \int_{x_0}^{x_1} |d(P^{\frac{1}{2}})|,$$

and the last factor is the length of path described by the point $P^{\frac{1}{2}}$, which $\parallel k$ and $< kq$, say, where q is finite.

Thus

$$\eta < 2\gamma(\gamma + \eta)mq/pk.$$

Hence $\eta \parallel 1/k$, that is, $f(x, x_0)$ is a normal approximation to $\phi(x, x_0)$.

Since our approximation to $\omega(b, a)$ was normal when a, b were $\parallel k^{\frac{3}{2}}$, it follows that

$$(PP_0)^{-\frac{1}{2}} \Sigma \sinh \int_{x_0}^x P^{\frac{1}{2}} dx$$

is also a normal approximation to $\phi(x, x_0)$ unless, as k increases, x or x_0 tends

to the value where $P = 0$. Similar results hold for the derivatives, only differing in form from those of § 7 in the presence of Σ .

Case when P Vanishes near a Terminus.

16. When the path x_0x begins or ends in the neighbourhood of the point where $P = 0$, the Riccati function ω still gives a normal approximation, but the hyperbolic function has a greater possibility of error, and, in fact, it gives a wrong order of magnitude when w or w_0 is small.

Since $w^2w = P$, a finite value of w corresponds to a value of P of the order of $k^{\frac{2}{3}}$; when P is of a lower order than this, w is small and the factor w^{-1} is to be dropped in the expression for $\omega(w, w_0)$. Hence, when $|P|$ is less than $k^{\frac{2}{3}}$, we are to substitute $k^{-\frac{1}{3}}$ for $P^{-\frac{1}{2}}$ in the expressions for $\phi(x, x_0)$, $\phi'(x, x_0)$. The results will give the true order of magnitude, but will not be normal approximations. Similarly, in the expressions for $\phi'(x, x_0)$, $\phi''(x, x_0)$, we must substitute $k^{\frac{1}{3}}$ for $P^{\frac{1}{2}}$. When $|P| \geq k^{\frac{2}{3}}$ these substitutions are not to be made, and the ordinary expressions give the true order of magnitude.

Case when P Vanishes more than Once.

17. We may combine the results for ϕ and its derivatives in the statement that the value of the matrix $W(x, x_0)$ (§ 2) is nearly $\Sigma Y(x, x_0)$, where

$$Y(x, x_0) = \begin{vmatrix} P^{-1}P_0^{\frac{1}{2}} \cosh \int_{x_0}^x P^{\frac{1}{2}} dx, & P^{-1}P_0^{-\frac{1}{2}} \sinh \int_{x_0}^x P^{\frac{1}{2}} dx \\ P^{\frac{1}{2}}P_0^{\frac{1}{2}} \sinh \int_{x_0}^x P^{\frac{1}{2}} dx, & P^{\frac{1}{2}}P_0^{-\frac{1}{2}} \cosh \int_{x_0}^x P^{\frac{1}{2}} dx \end{vmatrix},$$

the approximation being normal except as noted. It is readily verified that

$$Y(x_1, x) Y(x, x_0) = Y(x_1, x_0),$$

and that this product is again a normal approximation to $W(x_1, x) W(x, x_0)$, or $W(x_1, x_0)$, if the real part of $P^{\frac{1}{2}}dx$ has a constant sign along the whole path x_0xx_1 , and if the factors in the one product are normal approximations to those in the other.

We also have, since the multiplication of matrices is distributive,

$$\Sigma Y(x_1, x) \Sigma Y(x, x_0) = \Sigma Y(x_1, x_0),$$

where, if the first summation refers to α paths from x to x_1 , and the second to β paths from x_0 to x , the third refers to the $\alpha\beta$ combined paths x_0xx_1 .

Hence, when on the path ab the real part of P vanishes (simply) at several points, say three, c_1, c_3, c_3 in order, we have

$$W(b, a) = W\left(b, \frac{c_2+c_3}{2}\right) W\left(\frac{c_2+c_3}{2}, \frac{c_1+c_2}{2}\right) W\left(\frac{c_1+c_2}{2}, a\right).$$

Each factor refers to a part of the path in which there is only one of the points c_1, c_2, c_3 , so that the approximation by means of the Riccati function can be used.

The result is $W(b, a) = \Sigma Y(b, a)$, nearly,

where the summation refers to the eight paths which are possible when one may go either above or below each one of the zeros of P . The approximation is normal unless a or b tends to a zero of P when k increases.

Can the Terms in ΣY destroy each other?

18. It is important to prove that the order of magnitude of the constituents of $W(b, a)$ is not reduced by cancelling among the terms in ΣY .

We may write

$$\phi(b, a) = \frac{1}{2} \Sigma (Pa/Pb)^{\frac{1}{2}} (Pa)^{-\frac{1}{2}} \exp \int_a^b P^{\frac{1}{2}} dx, \text{ nearly,} \quad (12)$$

where the summation now refers to the different paths ab and also to the two initial values of $P^{\frac{1}{2}}$. Each of the paths may be deformed so as to pass through the zeros of P , and thus all the paths may be taken to coincide; each is divided into sections by the zeros of P and the distinction between any two paths consists in a difference in the sign of $P^{\frac{1}{2}}$ on one or more of the sections.

The straight path for x and corresponding path of $z = \int P^{\frac{1}{2}} dx$ are exemplified in a figure (Fig. 1). The lateral part of P is supposed to be always of the sign +

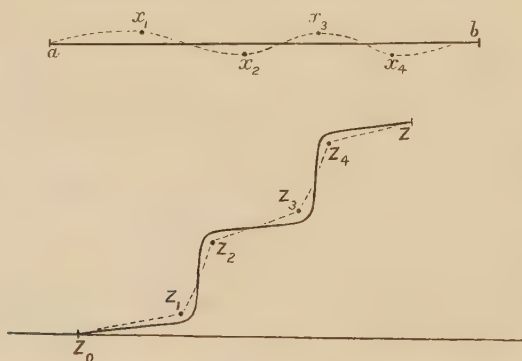


FIG. 1.

when x is real, so that x_1, x_2, x_3, x_4 , the zeros of P , lie on the two sides of the straight path alternately. The path of z is zigzag with rounded corners, and $z_0, z_1, z_2, z_3, z_4, z$ correspond to a, x_1, x_2, x_3, x_4, b respectively. The dotted line shows the deformed path for x , and its division into sections: in the z plane the deformed path follows the dotted line if x originally follows the straight path, so that the real part of $P^{\frac{1}{2}}$ is of constant sign. If the x -path passes to the other side, say of x_1 , then in the z -plane the sections of the dotted path from z_1 onwards are to be reversed. The sections of the z -path are nearly horizontal and nearly vertical alternately. If we write p for $\int P^{\frac{1}{2}} dx$ on a nearly horizontal section, and

q for $\int P^{\frac{1}{2}} dx$ on a nearly vertical one, the real part in each being taken positive, then on the whole path

$$\int_a^b P^{\frac{1}{2}} dx = \Sigma (\pm p) + \Sigma (\pm q),$$

and all possible combinations of the signs in this expression are to be taken into account in the summation in (12).

The number of sections is limited, being at most two more than the number of stationary values of S/T . Those terms in which one or more of the p 's or q 's are negative are comparatively small, and may be neglected, if the real part of $\int P^{\frac{1}{2}} dx$ is great on each section.

Now we are supposing the real parts of λ, μ and therefore generally of P , to be $\parallel k^2$; let the lateral parts be $\parallel k k_1$, so that k_1/k is small. Then the real and lateral parts of $P^{\frac{1}{2}}$ are $\parallel k, k_1$ on a p -section and $\parallel k_1, k$ on a q -section, and therefore we need to have k_1 great. If k_1 increases indefinitely with k , then the approximate value of $\phi(b, a)$ is

$$\frac{1}{2} (Pa Pb)^{-\frac{1}{2}} \exp (\Sigma p + \Sigma q) \quad \text{or} \quad \frac{1}{2} (Pa Pb)^{-\frac{1}{2}} \exp \int_a^b \sqrt{P} dx,$$

as in the simpler case when the hyperbolic approximation is valid and the real part of $\int \sqrt{P} dx$ is great; so that the terms of ΣY do not cancel.

This result holds if we are able to use the hyperbolic value for the Riccati function on each section of the path, but the proof fails when the path begins or ends so near a zero of P that the argument of the Riccati function involved is finite or small. This cannot happen when k_1 is great in comparison with $k^{\frac{1}{2}}$, and the error will not vitiate the proof if the argument of the Riccati function exceeds a certain finite limit, that is, if $k_1 k^{-\frac{1}{2}}$ exceeds a certain finite number: when this is not so, further investigation is needed.

Another Method.

19. The following method of combining the sections is slightly different, and may be used when the path ab begins or ends near a zero of P . Let $x_1, x_2, \dots, x_n$ be the zeros of P , occurring on the two sides of the real path ab alternately, then

$$W(b, a) = W(b, x_n) \dots W(x_2, x_1) W(x_1, a).$$

Suppose to fix the ideas that x_1 is on the upper side of ab , then $x_1 x_2$ is a q -section, $x_2 x_3$ a p -section, and so on. We shall take the amplitude of P to be nearly 0 on a p -section, and nearly π on a q -section; the amplitudes of $-P'_1, P'_2, -P'_3, \dots$ will be nearly 0.

When x is between x_1 and x_3 , but not near either of them, we have the approximation

$$\phi(x, x_2) = (w' w'_2)^{-\frac{1}{2}} w(w, 0),$$

where $\frac{2}{3} w^{\frac{3}{2}} = \int_{x_2}^x P^{\frac{1}{2}} dx$, $w' = dw/dx$, and w'_2 is the value of w' when $x = x_2$, namely $P'_2^{\frac{1}{2}}$.

Thus on the section $x_1 x_2$ we have, by the approximation to ω

$$\phi(x, x_2) = A(w'w'_2)^{-\frac{1}{2}} w^{-\frac{1}{2}} \sinh\left(\frac{2}{3}w^{\frac{3}{2}} + \alpha\right) = AP_2'^{-\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\alpha - \int_x^{x_2} P^{\frac{1}{2}} dx\right),$$

and also

$$\phi'(x, x_2) = B(w'_2/w')^{\frac{1}{2}} w^{-\frac{1}{2}} \sinh\left(\frac{2}{3}w^{\frac{3}{2}} + \beta\right) = BP_2'^{\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\beta - \int_x^{x_2} P^{\frac{1}{2}} dx\right),$$

where A, B, α, β have the meanings of § 10. On the section $x_2 x_3$,

$$\phi(x, x_3) = A_1(w'w'_2)^{-\frac{1}{2}} w^{-\frac{1}{2}} \sinh\left(\frac{2}{3}w^{\frac{3}{2}} - \alpha_1\right) = A_1 P_2'^{-\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\int_x^{x_3} P^{\frac{1}{2}} dx - \alpha_1\right),$$

$$\phi'(x, x_3) = B_1 P_2'^{\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\int_{x_2}^x P^{\frac{1}{2}} dx - \beta_1\right),$$

where, since the distinction between $\sinh$ and $\cosh$ disappears, we may put

$$\alpha_1 = \beta_1 = 0, \quad A_1 = Ae^{i\pi}, \quad B_1 = -Be^{i\pi};$$

thus A_1, B_1 are real. These approximate equations may be differentiated with respect to x ; there are similar expressions in the domains of $x_4, x_5, \dots$

When x is between x_2 and x_4 , we use an auxiliary variable w which decreases as x increases, and we have

$$\phi(x, x_3) = (w'w'_3)^{-\frac{1}{2}} \omega(w, 0),$$

where now $\frac{2}{3}w^{\frac{3}{2}} = \int_x^{x_3} P^{\frac{1}{2}} dx$, $w' = dw/dx$, and w'_3 is the value of w' when $x = x_3$, that is, $-(-P'_3)^{\frac{1}{2}}$.

Hence on the section $x_2 x_3$,* approximately,

$$\phi(x, x_3) = -A_1(-P'_3)^{-\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\int_x^{x_3} P^{\frac{1}{2}} dx - \alpha_1\right),$$

$$\phi'(x, x_3) = B_1(-P'_3)^{\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\int_x^{x_3} P^{\frac{1}{2}} dx - \beta_1\right),$$

and on the section $x_3 x_4$,

$$\phi(x, x_3) = -A(-P'_3)^{-\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\alpha - \int_{x_3}^x P^{\frac{1}{2}} dx\right),$$

$$\phi'(x, x_3) = B(-P'_3)^{\frac{1}{2}} P^{-\frac{1}{2}} \sinh\left(\beta - \int_{x_3}^x P^{\frac{1}{2}} dx\right),$$

with similar expressions for the domains of $x_1, x_5, \dots$

Now $W(x_2, x_1) = W(x_2, x)W(x, x_1)$, where x is on $x_1 x_2$:

$$W(x_2, x) = \begin{pmatrix} AP_2'^{-\frac{1}{2}} & 0 \\ 0 & BP_2'^{\frac{1}{2}} \end{pmatrix} \begin{vmatrix} \cosh(z_2 - \alpha) & \sinh(z_2 - \alpha) \\ \cosh(z_2 - \beta) & \sinh(z_2 - \beta) \end{vmatrix} \begin{pmatrix} P^{\frac{1}{2}} & 0 \\ 0 & P^{-\frac{1}{2}} \end{pmatrix},$$

* There is a difficulty in getting the signs right in these expressions; it may be overcome by taking a particular form for P , say $-m^2(x-x_3)$ where m is positive.

where

$$z_2 = \int_x^{x_2} P^{\frac{1}{2}} dx,$$

$$\text{and } W(x, x_1) = \begin{pmatrix} P^{-\frac{1}{2}}, & 0 \\ 0, & P^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \sinh(z_1 - \beta), & \sinh(z_1 - \alpha) \\ \cosh(z_1 - \beta), & \cosh(z_1 - \alpha) \end{pmatrix} \begin{pmatrix} B(-P_1')^{\frac{1}{2}}, & 0 \\ 0, & A(-P_1')^{-\frac{1}{2}} \end{pmatrix},$$

where

$$z_1 = \int_{x_1}^x P^{\frac{1}{2}} dx;$$

thus

$$W(x_2, x_1) = \begin{pmatrix} AP_2'^{-\frac{1}{2}}, & 0 \\ 0, & BP_2'^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \sinh(z_1 + z_2 - \alpha - \beta), & \sinh(z_1 + z_2 - 2\alpha) \\ \sinh(z_1 + z_2 - 2\beta), & \sinh(z_1 + z_2 - \alpha - \beta) \end{pmatrix} \\ \times \begin{pmatrix} B(-P_1')^{\frac{1}{2}}, & 0 \\ 0, & A(-P_1')^{-\frac{1}{2}} \end{pmatrix},$$

where

$$z_1 + z_2 = \int_{x_1}^{x_2} P^{\frac{1}{2}} dx.$$

This is the typical expression for a q -section; if $x_1 x_2$ had been a p -section we should have had P_1' for $-P_1'$ and $-P_2'$ for P_2' , and also $A_1, -B_1, \alpha_1, \beta_1$ for A, B, α, β .

Since $x_2 x_3$ is a p -section, the product $W(x_3, x_2) W(x_2, x_1)$ is now resolved into six factors of which the two middle ones are

$$\begin{pmatrix} -B_1 P_2'^{\frac{1}{2}}, & 0 \\ 0, & A_1 P_2'^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} AP_2'^{-\frac{1}{2}}, & 0 \\ 0, & BP_2'^{\frac{1}{2}} \end{pmatrix},$$

whose combined effect is simply a multiplication by $A_1 B$, since $AB_1 = -A_1 B$. All such factors may therefore be put outside the product.

Moreover in a p -section, such as $x_2 x_3$, the middle factor is simply

$$\begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix},$$

multiplied by the common factor $\sinh \int_{x_2}^{x_3} P^{\frac{1}{2}} dx$, which may also be put outside.

Thus now, apart from factors at the two ends, every alternate matrix in the sequence is of the form

$$\begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix}.$$

But

$$\begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix} \begin{pmatrix} a, & b \\ c, & d \end{pmatrix} \begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix} = (a+b+c+d) \begin{pmatrix} 1, & 1 \\ 1, & 1 \end{pmatrix},$$

and thus if a q -section lies between two p -sections we may strike out two matrices from the sequence, putting outside the factor

$$2 \sinh(q - \alpha - \beta) + \sinh(q - 2\alpha) + \sinh(q - 2\beta) \\ = 4 \sinh(q - \alpha - \beta) \cosh^2 \frac{1}{2}(\alpha - \beta) = 4i \cosh q / (-A^2 B^2) \\ = 4 \cosh q / A_1^2 B^2.$$

This is consistent with the former result, if we take account of the factors A, B and $\sinh p$ that have been put outside already, and substitute $\exp q, \exp p$ for $2 \cosh q, 2 \sinh p$.

Case of Terminal Short Sections.

20. The factors corresponding to the terminal sections are also found on examination to agree with those already found, when those sections are of finite length, so that the approximation to ω is available; this approximation cannot, however, be used for a section so short that the argument of the corresponding ω does not become infinite with k . Denoting a short section by s , we have to consider the following possible arrangements:—

$$sp, ps, sq, qs, sps, sqs, spq, pqs, sqp, qps, spqs, sqps, sqpq, qpqs, sqpps;$$

here spq , for instance, means a short section ax_1 followed by a p -section x_1x_2 , and then by a q -section x_2b : we have shewn that the sequence ppp can be replaced by p and an outside factor, so that any possible sequence can be reduced to one of those written, so far as concerns the matrices involved. The matrix corresponding to p is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with an outside factor. There is no reduction in the

order of magnitude on account of these outside factors, as has been pointed out.

When a p -section is between two others we have a product of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

or

$$\begin{pmatrix} (a+b)(\alpha+\gamma) & (a+b)(\beta+\delta) \\ (c+d)(\alpha+\gamma) & (c+d)(\beta+\delta) \end{pmatrix}.$$

Here the orders of magnitude are found at once if we know those of $a+b, c+d, \alpha+\gamma, \beta+\delta$ which are the coefficients in the separate products

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Hence the consideration of the different arrangements is simplified, for they all depend on the following:—

$$sqs, sp, sq, sqp,$$

and the last three reversed, ps, qs, pqs .

For an arrangement ending in qs (or, reversed, beginning with sq) we may make use of the formulæ of §§ 3, 4, for instance,

$$\phi(b, a) \phi'_1(b, a) - \phi_1(b, a) \phi'(b, a) = \int_a^b (P_1 - P) \phi(x, a) \phi_1(x, a) dx,$$

where the suffix 1 now indicates the change of λ, μ into the conjugate quantities.

Here $P_1 - P$ is of constant sign and always $\parallel k k_1$. Also $\phi(x, a) \phi_1(x, a)$ is always positive but may oscillate about a mean value $\parallel$

$$K^2 \{P(x) P_1(x) + k^{\frac{3}{2}}\}^{-\frac{1}{2}} \exp \int_a^x (\sqrt{P} + \sqrt{P_1}) dx,$$

where K does not depend on x , and the term $k^{\frac{3}{2}}$ in the bracket is added to provide for the case when P falls below the order of $k^{\frac{1}{2}}$.

There is a zero of P near to b , suppose at x_n . The period of oscillation of $\omega(x, 0)$ is variable but finite so long as x is finite, and hence in the neighbourhood of x_n the period of oscillation of ϕ is greater than in the body of the q -segment and reaches the order of magnitude of $k^{-\frac{3}{2}}$, which in the present case is not higher than $1/k_1^2$.

In the neighbourhood of x_n , $P \parallel k^2 |x - x_n|$, and the lateral part of $P \parallel k k_1$ on the path, so that on the path the real part of $\sqrt{P}$ is $\parallel k \sqrt{|x - x_n|}$ on the short p -section and $\parallel k_1 / \sqrt{|x - x_n|}$ on the q -section; the integral of $\sqrt{P} + \sqrt{P_1}$ is therefore finite on the p -section so long as $k(x - x_n)^{\frac{3}{2}}$ is finite, that is, $x - x_n \parallel k^{-\frac{2}{3}}$, and on the q -section so long as $k_1(x - x_n)^{\frac{1}{2}}$ is finite, that is, $x - x_n \parallel k_1^{-2}$.

Hence in $\int_a^b (P_1 - P) \phi(x, a) \phi_1(x, a) dx$, the argument of the exponential factor in $\phi \phi_1$ is only reduced from its maximum by a finite quantity if x travels back from the upper limit b through a distance of the order of k_1^{-2} , that is, the order of magnitude of the exponential factor is at its maximum through such an interval; such an interval can also be made to include complete periods of oscillation, so that we may neglect the oscillation* in estimating the order of magnitude of the integral: we shall also neglect the whole path ab except this interval at the upper limit.†

The integral is therefore at least $\parallel$

$$k k_1 K^2 \exp \int_a^b (\sqrt{P} + \sqrt{P_1}) dx \int \{P(x) P_1(x) + k^{\frac{3}{2}}\}^{-\frac{1}{2}} dx,$$

the integration in the last factor being taken over the interval at the upper limit. The last factor is then $\parallel$

$$\int \{k^4 |x - x_n|^2 + k^{\frac{3}{2}}\}^{-\frac{1}{2}} dx \quad \text{or} \quad k^{-1} \int \{|x - x_n|^2 + k^{-\frac{3}{2}}\}^{-\frac{1}{2}} dx,$$

* On the same principle, although $\sin^2 kx$ is a violently oscillating function, $\int_a^b \sin^2 kx dx$ is $\parallel \beta - \alpha$, that is, $\parallel$ the range multiplied by the mean value, unless the range is a small fraction of the period of oscillation and is near to a zero of the subject of integration.

† Similarly in estimating the mere order of magnitude of $\int_0^1 e^{kx} dx$ one may take the upper part $\int_{1-k^{-1}}^1 e^{kx} dx$, and neglect the rest. The subject is now always $> e^k/e$, and therefore $\parallel e^k$, while the range is $1/k$. Thus the integral is at least $\parallel e^k/k$.

which comes out $\parallel (kk_1)^{-1}$. The result is therefore that

$$\phi(b, a) \phi'_1(b, a) - \phi_1(b, a) \phi'(b, a)$$

is at least $\parallel K^2 \exp \int_a^b (\sqrt{P} + \sqrt{P_1}) dx,$

and, since $\phi\phi'_1$ and $\phi_1\phi'$ are conjugate imaginaries, each of them must be of this order.

But ϕ, ϕ_1 are at most $\parallel Kk^{-\frac{1}{2}} \exp \int_a^b \sqrt{P} dx,$

and ϕ', ϕ'_1 are at most $\parallel Kk^{\frac{1}{2}} \exp \int_a^b \sqrt{P} dx.$

Hence we must have

$$\phi, \phi_1 \parallel Kk^{-\frac{1}{2}} \exp \int_a^b \sqrt{P} dx,$$

$$\phi', \phi'_1 \parallel Kk^{\frac{1}{2}} \exp \int_a^b \sqrt{P} dx,$$

that is, the oscillation does not affect the order of magnitude, when the arrangement begins with sq or ends with qs .

21. An arrangement ending with ps may be treated as follows. The p -segment gives a factor $\begin{pmatrix} 1, 1 \\ 1, 1 \end{pmatrix}$ to whose left are the matrices

$$\begin{pmatrix} w'_{n+1}{}^{-\frac{1}{2}} & 0 \\ 0 & w'_{n+1}{}^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} -\omega(w_{n+1}, 0), & \omega(w_{n+1}, 0) \\ -\omega^\wedge(w_{n+1}, 0), & \omega^\wedge(w_{n+1}, 0) \end{pmatrix} \begin{pmatrix} w_n^{\frac{1}{2}} & 0 \\ 0 & w_n'^{-\frac{1}{2}} \end{pmatrix} \\ \times \begin{pmatrix} A_1(-P'_n)^{-\frac{1}{2}} & 0 \\ 0 & -B_1(-P'_n)^{\frac{1}{2}} \end{pmatrix},$$

where $\frac{2}{3}w_{n+1}^{\frac{2}{3}} = \int_b^n P^{\frac{1}{2}} dx$ and $w'_n = -(-P'_n)^{\frac{1}{2}}$.

The product of these five matrices is

$$\begin{pmatrix} w'_{n+1}{}^{-\frac{1}{2}} & 0 \\ 0 & w'_{n+1}{}^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} -A_1\omega' + B_1\omega & -A_1\omega' + B_1\omega \\ -A_1\omega^\wedge + B_1\omega', & -A_1\omega^\wedge + B_1\omega' \end{pmatrix},$$

where the constituents in the right-hand factor are constant multiples of $\chi(w_{n+1})$ and $\chi'(w_{n+1})$ (§ 13). Now in the present case w_{n+1} is small or finite and thus χ, χ' are not great, nor will either be small except when w_{n+1} is near one of its zeros: it has been proved that these zeros are real, whereas w_{n+1} will be found

to have a lateral part at least $\parallel k_1 k^{-\frac{1}{2}}$. Hence the order of $\chi(w_{n+1})$ or $\chi'(w_{n+1})$ is at least $k_1 k^{-\frac{1}{2}}$.

Thus in the case of a short terminal section there is no reduction in order of magnitude when the next section is of the q -kind, but there may be a reduction by a factor $\parallel k_1 k^{-\frac{1}{2}}$ when the next section is of the p -kind; this may, in fact, happen at both ends of the path simultaneously.

The short section may be either forwards or backwards, so that the discussion serves even when x_n is near the terminus a or b but on the far side.

Stationary Values of S/T . Riccati case.

22. When S/T has a stationary value, say p , there appear to be further complications when μ/λ is nearly equal to p , but, in fact, the foregoing approximations give all that is wanted for our purpose.

Let c be the value of x for which $S-pT=0$, and m the order of multiplicity of the factor $x-c$. Let $\lambda = \xi + i\eta$, $\mu = \zeta + i\omega$, then

$$P = \alpha(x-c)^m + \beta + i\gamma,$$

where

$$\alpha(x-c)^m = \xi(-S+pT),$$

$$\beta = (-p\xi + \zeta)T - X,$$

$$\gamma = -\eta S + \omega T.$$

Thus when $|x-c|$ is below a certain finite value, α is of constant sign and of the order of k^2 , γ is of constant sign and of the order of kk_1 , β is either finite, and therefore of lower order than γ , or else is of constant sign, namely that of $-p\xi + \zeta$: suppose k_2 then to be the order of magnitude of $\beta + i\gamma$.

P vanishes at a cluster of m points which are, approximately, placed symmetrically round a circle with centre c ; the path of integration passes diametrically across the circle; let x_1 be the nearest of the m points where P vanishes to one end of this diameter, and take the Riccati approximation with origin at x_1 , that is, put

$$\frac{2}{3}w^{\frac{3}{2}} = \int_{x_1}^x \sqrt{P} dx.$$

The relative magnitude of the error is (§§ 8, 15) that of

$$\int |\{w, x\} dx/w' \sqrt{w}|.$$

* The method which was applied to the other kind of short section would also give this result, the only change being that the interval at the upper limit of which account is taken in

$$\int_a^b (P_1 - P) \phi(x, a) \phi_1(x, a) dx$$

is now $\parallel k^{-\frac{1}{2}}$ instead of $\parallel k_1^{-\frac{1}{2}}$, so that the integral is now $\parallel$

$$k_1 k^{-\frac{1}{2}} K^2 \exp \int_a^b (\sqrt{P} + \sqrt{P_1}) dx.$$

Thus the lowering of order is possible; the method of the text has the advantage of shewing when it happens.

The distances between the m points of the cluster and their distances from c are $\parallel (k_2/k^2)^{1/m}$. Suppose $x-x_1$ not to be above this order, and the distance of x from c or any other point of the cluster not to be below it. Then P' is $\parallel \alpha(x-c)^{m-1}$, that is $\parallel k_2(k^2/k_2)^{1/m}$;

$$P \parallel k_2(k^2/k_2)^{1/m}(x-x_1),$$

$$w \parallel k_2^{\frac{1}{2}}(k^2/k_2)^{1/3m}(x-x_1),$$

$$w' \parallel k_2^{\frac{1}{2}}(k^2/k_2)^{1/3m},$$

since $\frac{2}{3}w^{\frac{3}{2}} = \int_{x_1}^x \sqrt{P} dx$ and $w^2 w' = P$: also since w , as a function of x , has no singularity in a domain whose linear dimensions $\parallel (k_2/k^2)^{1/m}$ and is of the order just given, the orders of w'' , w''' are at most $k_2^{\frac{1}{2}}(k^2/k_2)^{4/3m}$, $k_2^{\frac{1}{2}}(k^2/k_2)^{7/3m}$.

Thus w'''/w' and $(w''/w')^2$, and therefore $\{w, x\}$, are not of higher order than $(k^2/k_2)^{2/3m}$.

The other factor under the integral sign is

$$dx/w' \sqrt{P} = dx/\sqrt{P} = \frac{2}{P'} d\sqrt{P},$$

and thus the integral on the part of the path now in question is $\parallel$

$$(k^2/k_2)^{2/3m} \cdot k_2^{-1} (k^2/k_2)^{-1/m} \cdot k_2^{\frac{1}{2}},$$

the first factor being given by $\{w, x\}$, the second by $2/P'$, and the third by the length of path of the point $\sqrt{P}$. The product is $k^{2/m} k_2^{-\frac{1}{2}-1/m}$, and this is therefore the relative order of error when all points of the path are at distances not below the order of $(k_2/k^2)^{1/m}$ from c and from all the points of the cluster but x_1 , and not above this order from x_1 .

Hyperbolic Case.

23. To combine the section thus limited with other parts of the given path we can use the hyperbolic approximation, the error in which (§ 7) is comparatively small on any section, if $\left| \int P^{-\frac{1}{2}} \frac{d^2}{dx^2} P^{-\frac{1}{2}} dx \right|$ is small on that section. This integral does not exceed the product of the greatest value of $|P^{-\frac{1}{2}}|$ and the length of path of the point $d/dx P^{-\frac{1}{2}}$ or of $P'/4P^{\frac{3}{2}}$.

Now, when the real part of P vanishes, $P \parallel k k_1$, which may or may not be lower than k_2 .

If $k k_1 \parallel k_2$, or if the section does not extend to any point where P is below the order of k_2 , the greatest value of $|P^{-\frac{1}{2}}|$ is $\parallel k_2^{-\frac{1}{2}}$, and the length of path of $P'/4P^{\frac{3}{2}}$ is $\parallel$ the greatest value of $|P'/P^{\frac{3}{2}}|$, that is, $\parallel k_2^{-\frac{1}{2}}(k^2/k_2)^{1/m}$. Hence under these restrictions the relative magnitude of error for the section is $k_2^{-\frac{1}{2}-1/m} k^{2/m}$.

If the section is not thus restricted, but includes the point where the real part of P vanishes, the greatest value of $|P^{-\frac{1}{2}}|$ is $\parallel (k k_1)^{-\frac{1}{2}}$ and the length of path is $\parallel (k k_1)^{-\frac{1}{2}} k_2 (k^2/k_2)^{1/m}$, so that the relative order of possible error is raised to $(k k_1)^{-\frac{1}{2}} k_2^{1-1/m} k^{2/m}$.

Even this tends to zero as k increases, if $k_2 \leq k k_1^2$, and if k_1 increases indefinitely with k , for it may be written $(k_2/k k_1^2)^{1-1/m} (k/k_1)^{-\frac{1}{2}+1/m} k_1^{-1/m}$ and $m = 2$, at least.

Combination of the Two Cases.

24. These results are to be applied as follows, when ξ/ξ is nearly equal to p . Let $\xi - p\xi = k_3$, and $\eta = k k_1$, where k_1 increases indefinitely with k ; the value of ω is such that $-\eta S + \omega T$ is of constant sign, and always $\parallel k k_1$.

If $k_3 \leq k k_1^2$, the hyperbolic approximation is valid. If $k_3 > k k_1^2$, so that $k_2 = k_3$, let $r = |x_1 - c|$, the radius of the circle on which the m zeros of P approximately lie. Divide the path into sections at the points $c \pm \frac{3}{2}r$, $c \pm \frac{1}{2}r$. Between $c \pm \frac{1}{2}r$, and outside $c \pm \frac{3}{2}r$, the hyperbolic approximation is taken, since P does not fall below the order of k_3 . Between $c - \frac{3}{2}r$ and $c - \frac{1}{2}r$ or $c + \frac{1}{2}r$ and $c + \frac{3}{2}r$ a Riccati approximation may be used, with origin at the nearest zero of P in either case. The matrices corresponding to the different sections are then to be compounded, and the error in the result is relatively $\parallel$

$$k^{2/m} k^{-\frac{1}{2}-1/m} \quad \text{or} \quad (k_3/k)^{-\frac{1}{2}-1/m} k^{-\frac{1}{2}+1/m},$$

which tends to zero as k increases, since k_3/k increases indefinitely with k and $m \nless 2$.

For the Riccati functions we may substitute their approximate values in terms of hyperbolic functions, since the arguments are great, as will appear from the following discussion of the present nature of the sections into which the path was divided in § 18. Then the resulting approximate formulæ will be the same as in the simpler case, being formed in exactly the same way.

If m is even and the circumference of a circle is divided into m equal parts the points of division are diametrically opposite in pairs. Thus the origins in the two Riccati approximations are almost diametrically opposite, and the diameter between them forms a section which may be of either kind, p or q , but is short. It has to be shewn that the real part of $\int \sqrt{P} dx$ on this section is great.

If m is odd the points of division of the circle are not diametrically opposite. If at one end of the diameter the Riccati approximation must be used on account of its origin coming so near the path that P falls below the order of k_2 on the path, then at the other end of the diameter this does not happen, and the hyperbolic approximation will serve: hence there is no short section when m is odd.

There will be a terminal short section for any value of m when c falls at one of the termini of the path, but as c does not depend on k it cannot tend to one of the termini as a limit, and some complication is thus saved.

Short Sections.

25. When $x = c$, $P = \beta + \gamma$, and there are two cases according as β is great and positive or great and negative. When β is finite, or indeed below

the order of kk_1^2 , there is no short section, for we use the hyperbolic approximation.

In the other two cases we need to discuss the order of magnitude of the real part of $\int_c^{x_1} \sqrt{P} dx$.

Now, so long as $x-c \parallel (k_2/k^2)^{1/m}$ or less, we have P'' at most $\parallel a(x-c)^{m-2} \parallel k^2 (k_2/k^2)^{1-2/m}$. Thus, by two integrations, since $P=0$ at x_1 ,

$$P-P'_1(x-x_1) \parallel k^2 (k_2/k^2)^{1-2/m} (x-x_1)^2,$$

where P'_1 is the value of P' at x_1 .

Hence, since $P'_1 \parallel k^2 (k_2/k^2)^{1-1/m}$, we have

$$\sqrt{P}-\sqrt{P'_1}(x-x_1)^{\frac{1}{2}} \parallel k (k_2/k^2)^{\frac{1}{2}-3/2m} (x-x_1)^{\frac{3}{2}},$$

$$\text{and} \quad \int_{x_1}^x \sqrt{P} dx - \frac{2}{3} \sqrt{P'_1} (x-x_1)^{\frac{3}{2}} \parallel k (k_2/k^2)^{\frac{1}{2}-3/2m} (x-x_1)^{\frac{5}{2}}.$$

Now, if we take a triangle x_1ef , having its base ef on the real axis, its vertical angle equal to $\frac{2}{3}\pi$ and neither of its base angles less, say, than $\frac{1}{12}\pi$, the amplitudes of $\frac{2}{3}\sqrt{P'_1}(x-x_1)^{\frac{3}{2}}$ at e, f differ by π , and cannot in all positions of e, f be $\pm \frac{1}{2}\pi$: thus we may choose positions of e, f where $\frac{2}{3}\sqrt{P'_1}(x-x_1)^{\frac{3}{2}}$ has substantial real parts with opposite signs. Since at e, f , $x-x_1 \parallel$ the lateral part of x_1-c , that is $\parallel kk_1/k_2 (k_2/k^2)^{1/m}$, the absolute value and real part of $\frac{2}{3}\sqrt{P'_1}(x-x_1)^{\frac{3}{2}}$ are $\parallel k (k_2/k^2)^{\frac{1}{2}-1/2m} (kk_1/k_2)^{\frac{3}{2}} (k_2/k^2)^{3/2m}$, that is, $k (k_2/k^2)^{\frac{1}{2}+1/m} (kk_1/k_2)^{\frac{3}{2}}$, while $\int_{x_1}^x \sqrt{P} dx$ differs from $\frac{2}{3}\sqrt{P'_1}(x-x_1)^{\frac{3}{2}}$ by a quantity $\parallel k (k_2/k^2)^{\frac{1}{2}+1/m} (kk_1/k_2)^{\frac{5}{2}}$, which may be neglected since k_2/kk_1 increases indefinitely with k . Thus the real part of $\int_{x_1}^x \sqrt{P} dx$ has opposite signs at e, f and must vanish at some intermediate point, say g . Also the distances ce, cf are equal to r in the limit. Thus we have real part of

$$\int_{c+\frac{1}{2}r}^{x_1} \sqrt{P} dx = \text{R.P. of } \int_{c+\frac{1}{2}r}^g \sqrt{P} dx > \text{R.P. of } \int_{c+\frac{1}{2}r}^{c+\frac{3}{2}r} \sqrt{P} dx.$$

In this integral the range $\parallel (k_2/k^2)^{1/m}$ and the real part of the subject is $\parallel \sqrt{k_2}$ if β is positive and $\parallel kk_1/\sqrt{k_2}$ if β is negative. The real part of the integral is therefore $\parallel k_2^{\frac{1}{2}+1/m} k^{-2/m}$ and $k_1 k_2^{-\frac{1}{2}+1/m} k^{1-2/m}$ in the two cases respectively: in each case it increases indefinitely when $k_1, k_2/k$, and k^2/k_2 do so.

The limits in the integral have been taken to be $c+\frac{1}{2}r$ and x_1 ; the same method would have applied if they had been x_1 and $c+\frac{3}{2}r$, or c and x_1 , or if we had taken the other Riccati origin instead of x_1 .

Thus the arguments in the Riccati approximations must be great, since the corresponding values of $\int \sqrt{P} dx$ are great: we are therefore entitled to use the hyperbolic approximate values of the Riccati functions.

Also, if we go back to the form (§ 18),

$$\frac{1}{2} \Sigma \exp \{ \Sigma (\pm p) + \Sigma (\pm q) \} (PaPb)^{-\frac{1}{2}},$$

the real part in each section, p or q , increases indefinitely with k in all cases that come within the scope of the last investigation (§§ 24, 25). Thus the whole expression is $\parallel$ its greatest term

$$\frac{1}{2} (PaPb)^{-\frac{1}{2}} \exp \int_a^b \sqrt{P} dx$$

in which all the upper signs are taken, that is, where the integration is along the straight path, the real axis.

Approximate Values of Θ and Φ .

26. Using the hyperbolic formulæ of § 7, we have as the approximate value of Θ ,

$$\begin{aligned} & [E_1 \{P(0)P(1)\}^{-\frac{1}{2}} - E_4 \{P(0)P(1)\}^{\frac{1}{2}}] \sinh \int_0^1 P^{\frac{1}{2}} dx \\ & + [E_2 \{P(1)/P(0)\}^{\frac{1}{2}} - E_3 \{P(0)/P(1)\}^{\frac{1}{2}}] \cosh \int_0^1 P^{\frac{1}{2}} dx - 2E. \end{aligned}$$

Suppose $E_4 \neq 0$. Then the predominant term in Θ is the one that contains E_4 , the other terms merge in the error, and thus the approximation to Θ is $-E_4 \{P(0)P(1)\}^{\frac{1}{2}} \sinh \int_0^1 P^{\frac{1}{2}} dx$ under the restrictions stated in § 7 as necessary for the validity of the hyperbolic approximation.

If the real part of P , and consequently that of $P^{\frac{1}{2}}$, is positive and great along the whole x -path, the hyperbolic sine and cosine of $\int_0^1 P^{\frac{1}{2}} dx$ are great also, and thus Θ cannot vanish.

Also, if λ, μ are such that the lateral part of $\int_0^1 P^{\frac{1}{2}} dx =$ an odd multiple of $\frac{1}{2}\pi$, the absolute value of the hyperbolic sine is not less than 1, and Θ cannot vanish for values of λ, μ in the neighbourhood.

Thus the formulæ of § 7 enable us to say that in two important cases Θ cannot vanish. In conjunction with the results of §§ 8-25, they also enable us to form an estimate of the orders of magnitude of the factors $\Phi(x, t)/\Theta$ and $\Psi(y, u)/Z$ of the subject of integration in § 6.

The approximate values found in § 15 by means of the Riccati functions were substantially the same as those of § 7, unless P/k^2 was small at an end of the path; it was assumed that the values of λ, μ were such that the lateral part of P was $\parallel kk_1$ and of constant sign, k_1 increasing indefinitely with k .

If P/k^2 is small, say when $x = 0$, the same expression gives at least the true order of magnitude, provided when P/k^2 is small we substitute $k^{\pm\frac{1}{2}}$ for $P(0)^{\pm\frac{1}{2}}$ in the formulæ, except that (§ 21) when the next section is of the p -kind there may be a loss in order of magnitude by a factor $\parallel k_1 k^{-\frac{1}{2}}$, either in the pair $\phi(1, 0)$, $\phi'(1, 0)$ or the pair $\phi(1, 0)$, $\phi''(1, 0)$. These changes do not affect the predominance of the term $E_4 \phi''(1, 0)$.

In $\Phi(x, t)$ the predominant term may be $E_4 \phi'(1, t) \phi(x, 0)$ or $E\phi(t, x)$ according to the positions of x, t on the path 0, 1. The latter would, for instance, be the most important if we had $x = 0$, $t = 1$, P being great and positive. When the E_4 term predominates in Φ , we have

$$\Phi(x, t) = -E_4 \{P(0)P(1)/P(t)P(x)\}^{\frac{1}{2}} \cosh \int_0^x P^{\frac{1}{2}} dx \cosh \int_t^1 P^{\frac{1}{2}} dx$$

approximately in general, so that

$$\Theta^{-1} \cdot \Phi(x, t) \parallel \{P(t)P(x)\}^{-\frac{1}{2}} \exp \left(-\int_x^t P^{\frac{1}{2}} dx \right),$$

which is small if $t > x$ and the real part of $P^{\frac{1}{2}}$ is positive and great along the whole path.

The statement as to the order of magnitude is not affected even when $P(0)/k^2$ is small, for it will be seen that the changes to be made in Θ are accompanied by similar changes in Φ , and the ratio of the two is not affected. Similarly when $P(1)/k^2$ is small. When x or t is near to a zero of P there may be modifications in the order of magnitude of Φ , but these are always in the way of reduction.

When the term $E\phi(t, x)$ predominates in Φ , it is also easy to see in a general way that Φ/Θ is small under the restrictions that have been assumed.

The approximate values of $\partial\Phi/\partial x$, $\partial\Phi/\partial t$, and $\partial^2\Phi/\partial x \partial t$ differ from those of Φ , when the real part of $P^{\frac{1}{2}}$ is great throughout, by the factors $\sqrt{P(x)}$, $-\sqrt{P(t)}$, $-\sqrt{[P(x)P(t)]}$.

When the real part of $P^{\frac{1}{2}}$ is great and not of constant sign we cannot conclude that Φ/Θ is small even when $t > x$. A case of importance is when the real part of P is always negative and $\parallel k^2$, while the lateral part is of variable sign and $\parallel k \log \log k$, in fact

$$\eta = k \log \log k.$$

The real part of $P^{\frac{1}{2}}$ is $\parallel \log \log k$, and so therefore is $\int_0^1 |\bar{d}\rho|$. Thus $\exp \int_0^1 |\bar{d}\rho|$ is at most $\parallel$ a finite power of $\log k$, say $(\log k)^\delta$, and the error is relatively small since $(\log k)^\delta/k$ is small.

Also

$$\{P(x)P(t)\}^{\frac{1}{2}}\Phi(x,t)/\Theta \text{ is } \parallel \cosh \int_0^x d\rho \cosh \int_t^1 d\rho / \sinh \int_0^1 d\rho$$

$$\left(\text{or else } \{P(0)P(1)\}^{-\frac{1}{2}} \sinh \int_x^t d\rho / \sinh \int_0^1 d\rho \right),$$

and this cannot be more than $\parallel \exp \int_0^1 |d\rho|$ or $(\log k)^\delta$.

The same method can of course be applied to Z and Ψ ; we shall write R for $\lambda U - \mu V - Y$.

The same number δ will serve for both x and y if it is not too small.

Distribution of the Values of μ for which $\Theta = 0$.

27. Let p_1, p_2 be the greatest and least values of S/T , q_1, q_2 those of U/V , and take four positive constants r_1, r_2, s_1, s_2 , so that

$$r_1 > p_1 > p_2 > r_2 > s_1 > q_1 > q_2 > s_2.$$

Now $P = -X - \lambda S + \mu T = (\xi T - \xi S - X) + \iota(\omega T - \eta S),$

where $\lambda = \xi + \iota\eta$, $\mu = \xi + \iota\omega$ as in § 22, and therefore if $\xi - p_2\xi$, $\xi - p_1\xi$ are both positive and great, the real part of P is positive and great along the straight path for x from 0 to 1: the real part of $\int_0^1 P^{\frac{1}{2}} dx$ is positive and great; hence Θ cannot vanish (§ 26). Thus when λ is fixed and Θ vanishes, a superior limit to ξ is $r_1(\xi + \tau)$ when ξ is positive, $r_2(\xi + \tau)$ when ξ is negative, where τ may be zero when ξ is of the order of k^2 , but is a positive quantity of the order of k^2 when ξ falls below that order.

We also know from § 4 that when $\Theta = 0$, ω must lie between $p_1\eta$ and $p_2\eta$. Hence when λ has a given value and $\Theta = 0$, μ must lie in a region C bounded on the right by a vertical line, and above and below by two straight lines at constant distances $p_1\eta$, $p_2\eta$ from the real axis; on the left C is unbounded. C also contains the values of μ for which $P = 0$, whatever the value of x .

The values of μ for which $Z = 0$ are similarly all to be found in a region D bounded by two parallels to the real axis at distances $q_1\eta$, $q_2\eta$ from it, and bounded on the left by the vertical line $\xi = s_2(\xi - \tau)$, or when ξ is negative $\xi = s_1(\xi - \tau)$; D contains also the values of μ for which R vanishes.

We shall also speak of regions C' , D' having the same vertical boundaries as C , D respectively, but bounded horizontally, C' by the lines $\omega = r_1\eta$, $\omega = r_2\eta$, and D' by $\omega = s_1\eta$, $\omega = s_2\eta$.

When λ is real and positive, C, D are simply segments of the real axis, and overlap. When λ is complex, C, D have no common point, nor have C', D' . When ξ is negative and of the order of k^2 , the vertical boundaries of C, D are $\xi = r_2 \xi, \xi = s_1 \xi$; hence the regions are separated by a horizontal distance $(r_2 - s_1) \xi$, which is of the order of k^2 . In particular this holds when $\eta = 0$, and C, D are again segments of the real axis.

Thus generally, except when λ is real and positive, we can make a contour include zeros of Z and not of Θ by making it cross D once and avoid C .

The Field of Integration.

28. For the starting point of the field of integration, when E_4, G_4 do not vanish, such real positive values of λ, μ will be taken that $\int_0^1 P^{\frac{1}{2}} dx$ and $\int_0^1 R^{\frac{1}{2}} dy$ are nearly equal to odd multiples of $\frac{1}{2}\pi$: we must prove that such values exist. Now the ratio of

$$\int_0^1 (\lambda S - \mu T)^{\frac{1}{2}} dx \quad \text{to} \quad \int_0^1 (\mu V - \lambda U)^{\frac{1}{2}} dy$$

depends only on that of $\lambda : \mu$, and decreases continually as μ increases from $s_1 \lambda$ to $r_2 \lambda$: take a fraction p/q , whose numerator and denominator are odd, and whose value is a mean among the values of

$$\int_0^1 (\lambda S - \mu T)^{\frac{1}{2}} dx \bigg/ \int_0^1 (\mu V - \lambda U)^{\frac{1}{2}} dy,$$

when $s_1 \lambda < \mu < r_2 \lambda$. Then the equations

$$\int_0^1 (\lambda S - \mu T)^{\frac{1}{2}} dx = \frac{1}{2} k p \pi,$$

$$\int_0^1 (\mu V - \lambda U)^{\frac{1}{2}} dy = \frac{1}{2} k q \pi,$$

determine uniquely first the ratio and then the separate values of λ, μ , and if k is great $\int_0^1 P^{\frac{1}{2}} dx$ and $\int_0^1 R^{\frac{1}{2}} dy$ will differ from $\frac{1}{2} k p \pi$ and $\frac{1}{2} k q \pi$ respectively by small quantities of the order of $1/k$, since

$$P^{\frac{1}{2}} - \iota (\lambda S - \mu T)^{\frac{1}{2}} \quad \text{and} \quad R^{\frac{1}{2}} - \iota (\mu V - \lambda U)^{\frac{1}{2}}$$

are, throughout the range, of that order. Let g, h be the values of λ, μ thus determined.

We shall take k to be a great odd integer, and choose g, h for the initial values of λ, μ , which are real and positive, and such that Θ, Z do not vanish, as was pointed out in § 26. The conditions of validity of the hyperbolic approximation (§ 7) are fulfilled by both P and R .

29. Let the values of μ for which $\Theta = 0$ when λ has any particular real value be $m_0, m_1, m_2, \dots$ in descending order of magnitude, and let those for which $Z = 0$ be $\mu_0, \mu_1, \mu_2, \dots$ in ascending order. Then, when λ has its initial value g , no member of either series is equal to h , the initial value of μ . Suppose h to be between m_a and m_{a+1} , also between μ_β and $\mu_{\beta+1}$.

Again, when $\lambda = g$ and μ is zero or negative $\int_0^1 P^{\frac{1}{2}} dx$ is purely imaginary; let h' be the lowest negative value of μ for which $\int_0^1 (gS - \mu T)^{\frac{1}{2}} dx$ is equal to an odd multiple of $\frac{1}{2}k\pi$, and let this multiple be $\frac{1}{2}kr\pi$. Then the circle on hh' as diameter will be chosen for the μ -path when $\lambda = g$. R is constantly positive when $\lambda = g, \mu = h'$, and hence $\mu_0, \mu_1, \dots$ are all greater than h' ; also h' cannot be equal to any member of the series $m_0, m_1, \dots$ (§ 26). If we suppose h' to lie between m_γ and $m_{\gamma+1}$, the μ -path will enclose $\mu_0, \mu_1, \dots, \mu_\beta$, and also $m_{a+1}, m_{a+2}, \dots, m_\gamma$, but no others from either series: it will also include the point where $R = 0$, and exclude that for which $P = 0$, for any values of y, x .

The path followed by λ will consist of the circle on gg' as diameter where g' is a negative quantity of the order of k^2 (see § 31 below). The μ -path may vary continuously with λ , but must never be crossed by any of the points where Θ, Z, P , or R vanishes.

Now when λ moves away from the real axis, the regions C', D' do the same, and also widen out and separate from each other, their widths and distances from the axis and each other being proportional to the lateral part of λ . Their vertical boundaries depend only on the real part of λ , and when λ moves parallel to the real axis the two regions have a motion of translation in the same direction, but with velocities respectively r_1 and s_2 times the velocity of λ (or r_2 and s_1 times when λ has a negative real part). Hence, as the real part of λ changes from g to g' , C' moves to the left faster than D' until there is a clear space between them in the horizontal direction. Then λ moves down again to the real axis at g' , and C', D' also move down to the real axis, but there is a gap between them; the process is reversed as λ travels from g' to g along the lower half of its path.

The μ -path must cross D' once, between μ_β and $\mu_{\beta+1}$, and C' twice, between m_α and $m_{\alpha+1}$ and also between m_γ and $m_{\gamma+1}$, as shewn in the figure (Fig. 2) by the dotted line. When the two regions are well apart

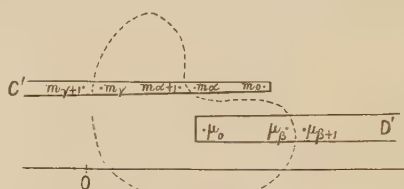


FIG. 2.

the path may be reduced to a pair of ovals, one including $\mu_0 \dots \mu_\beta$, and the other $m_{\alpha+1} \dots m_\gamma$, joined by a line which is travelled in both directions and may be left out. Then, when λ is in the lower part of its path, the ovals may be joined again, so that the final μ -path will coincide with the initial one, which it would not do without some such transition.

30. In the region of integration, if $t > x$, $u > y$, the subject $\Phi(x, t) \Psi(y, u) / \Theta Z$ is generally small in comparison with any negative power of k , but there are parts of the region where this will not always be the case, and these occur first near the initial values g, h , and secondly when μ is in the region C or D , λ having any value.

First, as to values near g, h ; let η_1 denote for a moment the greater of $|\eta|$ and $|\omega|$ and let η_1 be $< k^{\frac{1}{2}}$. Then, since the path of λ is circular with diameter gg' , the real part of $\lambda - g$ is of no higher order than η_1^2 / k^2 , and we shall subject μ to a like restriction, so that $\xi - g, \zeta - h$ will lie between $\pm e\eta_1^2 / k^2$ where e is finite and fixed. If, further, $\eta_1 < k^{\frac{1}{2}}$, the lateral parts of $\int_0^1 P^{\frac{1}{2}} dx$ and $\int_0^1 R^{\frac{1}{2}} dy$ will differ from $\frac{1}{2}kp\pi i$ and $\frac{1}{2}kq\pi i$ by quantities $\parallel k^{-\frac{1}{2}}$. Denoting by K this part of the region, where $\eta_1 < k^{\frac{1}{2}}$, we have that Θ and Z do not nearly vanish within K .

Outside K , so long as μ is not in C', D' the real parts of $P^{\frac{1}{2}}, R^{\frac{1}{2}}$ are of constant sign and great, so that the subject of integration is small when $t > x, u > y$. It does not matter how this part of the μ -path is drawn, so long as (1) the terminals are rightly placed and the condition just stated is satisfied near g, h ; (2) P, R are always $\parallel k^2$ for all values of x, y ; (3) the length of the path is $\parallel k^2$, as will naturally follow from (2).

31. It remains to define the path of μ within C' or D' , and it will in fact be necessary to vary λ, μ together.

When ξ , η , ω have given values such that $\omega = r_1\eta$ (or $r_2\eta$) the lateral part of $\lambda S - \mu T$ is of constant sign, and thus the real part of $\sqrt{\lambda S - \mu T}$ is of constant sign, say positive: it is increased by decreasing ξ and conversely. Hence, by varying ξ , we can make the real part of $\int_0^1 \sqrt{\lambda S - \mu T} dx$ have any positive value from 0, when ξ is $+\infty$, to ∞ , when ξ is $-\infty$. We may therefore prescribe that, when μ in its path is on the upper or lower boundary of C' ($\omega = r_1\eta$ or $r_2\eta$) at the first crossing, the real part of $\int_0^1 \sqrt{\lambda S - \mu T} dx$ shall be equal to $\frac{1}{2}kp\pi^*$ as at the starting point $\lambda = g$, $\mu = h$: in the passage between these two points we shall make λ vary as well as μ , keeping ξ constant and the real part of $\int_0^1 \sqrt{\lambda S - \mu T} dx$ equal to $\frac{1}{2}kp\pi$, but varying η , ω as follows.

If $\eta \nless \theta k$, where θ is $\log \log k$, let η be kept constant and ω increased continuously from $r_2\eta$ to $r_1\eta$. If $\eta > \theta k$, then diminish ω , η together, so that ω/η keeps the constant value r_2 until $\eta = \theta k$; then keep η constant and increase ω from $r_2\eta$ to $r_1\eta$ (this is the actual crossing), and then increase ω , η together, keeping ω/η constantly $= r_1$ till the original value of λ is regained.

We have proved that ξ can be so chosen that the real part of $\int_0^1 \sqrt{\lambda S - \mu T} dx$ shall have the value $\frac{1}{2}kp\pi$ when $\omega = r_1\eta$ or $r_2\eta$, but the proof does not apply in the actual crossing when $r_1\eta > \omega > r_2\eta$. However, as long as $r_1\xi - \xi$, $r_2\xi - \xi$ are both positive and $\parallel k^2$, the real part of $\lambda S - \mu T$ is positive and of the same order, and, since η , ω are now only $\parallel \theta k$, the real part of $\int_0^1 \sqrt{\lambda S - \mu T} dx$ differs from $\int_0^1 \sqrt{(\xi S - \xi T)} dx$ by a small quantity $\parallel \theta^2/k$ (§ 7, note): thus as a first approximation we give ξ such a value that $\int_0^1 \sqrt{(\xi S - \xi T)} dx = \frac{1}{2}kp\pi$, and a change $\parallel \theta^2$ in ξ will give the true solution, provided always that we do not make $r_1\xi - \xi$ or $r_2\xi - \xi$ negative. Now the greatest value of ξ is its initial one, g , and we then have $\xi = h$, which is less than r_2g . If ξ decreases and $\int_0^1 \sqrt{(\xi S - \xi T)} dx$ is constant, ξ must decrease too, and so also must ξ/ξ , until ξ vanishes, ξ being still of the order of k^2 . Then ξ decreases further and ξ also vanishes and becomes negative. When both are negative and decreasing

* This condition fixes two of the terminals of the path outside C' , D' , and in a way consistent with the restriction that near g , h , when $\eta_1 < k^2$, $\xi - g$ and $\xi - h$ shall be $\parallel \eta_1^2/k^2$: within C' , D' this restriction may be ignored. The path outside C' , D' is in three detached parts, and its other four terminals are fixed in the same manner at the second crossing of C' and the crossing of D' .

algebraically, the fraction ξ/ξ must decrease, beginning at $+\infty$, and it reaches the value r_1 when

$$\xi = - \left\{ \frac{1}{2}kp\pi \left/ \int_0^1 \sqrt{(r_1 T - S)} dx \right. \right\}^2.$$

If g' is algebraically greater than this quantity, $r_1\xi - \xi$ and $r_2\xi - \xi$ are always positive: and thus the value of ξ is satisfactorily given by the prescribed condition.

There is a second crossing of C' in the opposite direction, where the process will be the same reversed and $\frac{1}{2}kr\pi$ will take the place of $\frac{1}{2}kp\pi$. Similarly for the crossing of D' , where the values of ω on the upper and lower bounding lines are $s_1\eta$, $s_2\eta$, and ξ is made to vary so that the real part of $\int_0^1 \sqrt{(\mu V - \lambda U)} dy = \frac{1}{2}kq\pi$. From each crossing there is derived an inferior limit for g' : only the highest of the three need be attended to. We have supposed η positive, but there would be no essential change if it were negative, and, in fact, the whole region of integration will be supposed symmetrical about the real axes.

The advantages of this way of crossing C' , D' are as follows; in the case of the first crossing of C' , for example:

(1) Since the lateral part of $\int_0^1 P^{\frac{1}{2}} dx$ is nearly $\frac{1}{2}kp\pi i$, its hyperbolic sine cannot nearly vanish.

(2) Since the real part of $P^{\frac{1}{2}}$ is $\parallel \theta$ during the crossing, the hyperbolic approximation may be used for ϕ ; and so Θ cannot nearly vanish.

(3) We do not need to trace the variations of the points $\mu_0, \mu_1, \dots, m_0, m_1, \dots$ for complex values of λ in order to make the μ -path avoid these points. Difficulties that might be caused by branch points are thus evaded.

(4) $\Phi(x, t)$ ($t > x$) never becomes too great in comparison with Θ as it sometimes would if $P^{\frac{1}{2}}$ had a real part great, but not of constant sign, that is, if μ were in the region C when η was of the order of k^2 .

(5) The results of §§ 22-25 may be applied to ψ when necessary, and $k_1 \parallel \log \log k$ at least.

The region of integration is now finally fixed; let it be called Q . The general idea is that λ travels round a circle of great radius, and that μ describes a simple closed path varying with λ ; the region so generated differs from Q in that Q has suffered deformation, but no essential change, for the sake of the advantages just now enumerated.

An effect of the deformation should be noted, namely, that $|\lambda|$ is brought down below the order of k^2 in certain parts of Q . When this happens, ξ and $|\mu|$ are not brought down, but are still of the same order

as k^2 . Hence, of P and R , both do not at the same time fall below the order of k^2 .

Supplement to § 6b.

32. To complete the discussion of § 6b, we shall subject Q to a deformation as follows:—For every point $(\xi + i\eta, \zeta + i\omega)$ belonging to Q write $(\xi + i\epsilon\eta, \zeta + i\epsilon\omega)$, and let ϵ be changed continuously from 1 to a very small positive value. This will bring the whole region very near to the real axes, but without affecting that relation between μ and λ , which consists in μ lying within C' or D' or on the edge of either region. Make a further deformation by keeping λ, ζ constant while μ crosses C' or D' , that is, by abandoning the process of § 31: the complications introduced by this process are unnecessary when ω, η are small, but the conditions as to the real parts of $\int_0^1 \sqrt{(\lambda S - \mu T)} dx$ and $\int_0^1 \sqrt{(\mu V - \lambda U)} dy$ are to be maintained in C', D' respectively.

In the whole course of the deformation the subject of integration, $\Phi\Psi/\Theta Z$, never becomes infinite, and thus the value of the integral is unaffected, while now λ describes a path L which is, in fact, a long loop between g, g' , and to each point of L there corresponds a certain path for μ , say M .

Initially M excludes $m_0, m_1, \dots, m_a$, and for all positions of λ within L these points are separated from $m_{a+1}, m_{a+2}, \dots$ by the line drawn across C' on which ζ has such a value that

$$\int_0^1 \sqrt{(\xi S - \zeta T)} dx = \frac{1}{2} k p \pi,$$

and on which, therefore, Θ cannot vanish. Similarly the points $m_{\gamma+1}, m_{\gamma+2}, \dots$ are always separated from $m_0, m_1, \dots, m_\gamma$, and $\mu_0, \mu_1, \dots, \mu_\beta$ from $\mu_{\beta+1}, \mu_{\beta+2}, \dots$. Hence, when L is gradually reduced to a point there are no coincidences, whether with branching or otherwise, between the included and excluded members in either one of the series

$$m_0, m_1, \dots; \quad \mu_0, \mu_1, \dots$$

Thus we have only to discuss coincidences between the two series.

Now, if λ moves along the real axis with unit velocity, the velocity of any one of the points $m_0, m_1, \dots$ is a value of $d\mu/d\lambda$ given by the equation $\Theta = 0$, and is therefore a mean among the values of S/T (§ 5); the velocity of any one of the points $\mu_0, \mu_1, \dots$ is similarly a mean among the values of U/V . Hence each of the points $m_0, m_1, \dots$ moves relatively to any one of the points $\mu_0, \mu_1, \dots$ with a velocity at least equal to

$p_2 - q_1$. If, then, λ starts at g and moves to the left, each of the points $m_0, m_1, \dots, m_a$ passes ultimately from right to left of each of the points $\mu_0, \dots, \mu_\beta$, the process being complete before λ arrives at g' , since before that the regions C, D have drawn clear of each other.

Let λ_{rs}, μ_{rs} be the values of λ, μ when $m_r = \mu_s$,* so that when $\lambda = \lambda_{rs}$ and $\mu = \mu_{rs}$, Θ, Z both vanish; let Ω_{rs} be the corresponding value of Ω . Then

$$\iint_Q \frac{\Phi(x, t) \Psi(y, u)}{\Theta Z} d\mu d\lambda = -4\pi^2 \sum_{r=0}^{\alpha} \sum_{s=0}^{\beta} \Omega_{rs}.$$

Since Ω_{rs} is symmetrical in x, t and also in y, u , there are three other expressions for the sum of the double series on the right. It will be our object to choose, out of the four, one in which the subject of integration is always small.

Expansion of an Arbitrary Analytical Function.

33. Suppose $f(x, y)$ to be an analytical function of x, y throughout a rectangular area H , included in the square $0 \leq x \leq 1, 0 \leq y \leq 1$, and having its sides $x = x_0, x = x_1, y = y_0, y = y_1$ parallel to those of the square. Then we have

$$\begin{aligned} -4\pi^2 \sum_{r=0}^{\alpha} \sum_{s=0}^{\beta} \iint_H \Omega(x, t, y, u, \lambda_{rs}, \mu_{rs}) f(t, u) dt du \\ = \iint_Q \iint_H \Theta^{-1} Z^{-1} \Phi \Psi f(t, u) dt du d\mu d\lambda, \end{aligned}$$

where Φ will stand for $\Phi(t, x)$ when $t < x$ and $\Phi(x, t)$ when $t > x$, Ψ for $\Psi(u, y)$ when $u < y$ and $\Psi(y, u)$ when $u > y$.

* The suffixes r, s have a definite meaning. When $\mu = m_r$, for instance,

$$E_2 \phi(x, 0) + E_4 \phi'(x, 0)$$

vanishes for exactly r values of x between 0 and 1, unless E_2, E_4 both vanish. If $E_2 = E_4 = 0$, then $E = 0$ also, and it is $E_1 \phi(x, 0) + E_3 \phi'(x, 0)$ that vanishes for r values of x between 0 and 1, as well as when $x = 1$. (See Klein, *Math. Annalen*, Vol. xviii., pp. 410-427; Bôcher, *Ueber die Reihenentwickelungen der Potentialtheorie*, Teubner, 1894.) These things follow from the considerations that the roots of $E_2 \phi(1, 0) + E_4 \phi'(1, 0) = 0$ and of $\Theta = 0$ separate each other, and that if μ, x vary in such a way that $E_2 \phi(x, 0) + E_4 \phi'(x, 0) = 0$, then they must both increase or both decrease, since the derivatives of this expression with respect to μ, x have opposite signs when the expression itself vanishes, for

$$\begin{aligned} \{E_2 \phi'(x, 0) + E_4 \phi''(x, 0)\} \frac{\partial}{\partial \mu} \{E_2 \phi(x, 0) + E_4 \phi'(x, 0)\} \\ - \{E_2 \phi(x, 0) + E_4 \phi'(x, 0)\} \frac{\partial}{\partial \mu} \{E_2 \phi'(x, 0) + E_4 \phi''(x, 0)\} \\ = - \int_0^x T \{E_2 \phi(x, 0) + E_4 \phi'(x, 0)\}^2 dx, \end{aligned}$$

a negative quantity. Thus it will be found that in our case $\alpha = \frac{1}{2}(kp-1)$, $\beta = \frac{1}{2}(kq-1)$, $\gamma = \frac{1}{2}(kr-1)$, and in the limit α/β has the value p/q , which is restricted to a certain finite range of values (§ 28).

In this equation integrate by parts on the right with respect to t and again with respect to u . Since

$$\partial^2 \Phi / \partial t^2 = P(t) \Phi, \quad \partial^2 \Psi / \partial u^2 = R(u) \Psi,$$

$$\begin{aligned} \text{we have } \iint_H \Phi \Psi f(t, u) dt du &= \iint_H \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial u} \frac{\partial^2}{\partial t \partial u} \frac{f(t, u)}{P(t) R(u)} dt du \\ &+ \int \left\{ \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial u} \frac{\partial}{\partial t} \frac{f(t, u)}{P(t) R(u)} dt + \frac{\partial \Phi}{\partial t} \Psi \frac{f(t, u)}{P(t)} du \right\}, \end{aligned}$$

the single integral being taken along the boundary of H and also along the two sides of the lines $t = x$, $u = y$ so far as they lie within H . The line $u = y$, when $y_1 > y > y_0$, thus contributes

$$- \int_{x_0}^{x_1} \frac{\partial \Phi}{\partial t} Z \frac{\partial}{\partial t} \frac{f(t, y)}{P(t) R(y)} dt,$$

since Ψ has different meanings on the two sides of the line, and

$$\frac{\partial}{\partial u} \{ \Psi(u, y) - \Psi(y, u) \} = Z,$$

when $u = y$. The contribution of the line $t = x$ is similarly, when $x_1 > x > x_0$,

$$\int_{y_0}^{y_1} \Theta \Psi \frac{f(x, u)}{P(x)} du,$$

that is

$$\frac{\Theta}{P(x)} \left[\frac{\partial \Psi}{\partial u} \frac{f(x, u)}{R(u)} \right]_{y_0}^y + \frac{\Theta}{P(x)} \left[\frac{\partial \Psi}{\partial u} \frac{f(x, u)}{R(u)} \right]_y^{y_1} - \frac{\Theta}{P(x)} \int_{y_0}^{y_1} \frac{\partial \Psi}{\partial u} \frac{\partial}{\partial u} \frac{f(x, u)}{R(u)} du.$$

On account of the discontinuity in $\partial \Psi / \partial u$, when $u = y$, this expression, when (x, y) lies within H , contains the term

$$\Theta Z f(x, y) / P(x) R(y),$$

besides terms at the actual limits y_0 , y_1 , and the residual integral. The contribution to the original quadruple integral on account of the same discontinuity is therefore

$$\iint_Q \frac{f(x, y)}{P(x) R(y)} d\mu d\lambda$$

in which, for the purpose of integration, x, y are constants and $P(x)$, $R(y)$ linear functions of λ, μ . The μ -path encloses the zero of R and not that of P , while the λ -path encloses the real finite value of λ for which these two coincide; hence the value of this contribution is, by the theory of § 6,

$$-4\pi^2 f(x, y) / (SV - UT).$$

When (x, y) lies outside H , this term does not present itself.

It is to be proved that the rest of the original quadruple integral tends to zero when k is increased without limit: this remainder includes quadruple, triple, and double integrals, and, in general, the subjects of integration are small on account of an exponential factor of great negative index: when, however, λ and μ are nearly real this factor becomes less important, and thus different parts of the field Q have to be discussed separately.

The Remainder.—A Lemma.

34. The terms in the remainder are as follows, without regard to sign:—

$$\begin{aligned}
 (1) \quad & \iint_Q \iint_H \frac{1}{\Theta Z} \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial u} \frac{\partial^2}{\partial t \partial u} \frac{f(t, u)}{P(t) R(u)} dt du d\mu d\lambda, \\
 (2) \quad & \iint_Q \int_{x_0}^{x_1} \frac{1}{\Theta Z} \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial u} \frac{\partial}{\partial t} \frac{f(t, u)}{P(t) R(u)} dt d\mu d\lambda \quad (u = y_0, y_1), \\
 (3) \quad & \iint_Q \int_{y_0}^{y_1} \frac{1}{\Theta Z} \frac{\partial \Phi}{\partial t} \Psi \frac{f(t, u)}{P(t)} du d\mu d\lambda \quad (t = x_0, x_1), \\
 (4) \quad & \iint_Q \frac{1}{ZP(x)} \frac{\partial \Psi}{\partial u} \frac{f(x, u)}{R(u)} d\mu d\lambda \quad (u = y_0, y_1), \\
 (5) \quad & \iint_Q \int_{y_0}^{y_1} \frac{1}{ZP(x)} \frac{\partial \Psi}{\partial u} \frac{\partial}{\partial u} \frac{f(x, u)}{R(u)} du d\mu d\lambda, \\
 (6) \quad & \iint_Q \int_{x_0}^{x_1} \frac{1}{\Theta R(y)} \frac{\partial \Phi}{\partial t} \frac{\partial}{\partial t} \frac{f(t, y)}{P(t)} dt d\mu d\lambda.
 \end{aligned}$$

When $x < x_0$ or $> x_1$, the terms (4), (5) are absent, and when $y < y_0$ or $> y_1$, (6) is absent.

The absolute value of a single integral does not exceed the product of the length of the path of integration and the greatest absolute value of the subject. When we have to deal with a double integral $\iint J d\mu d\lambda$, it must be supposed that λ, μ are differentiable functions of two real parameters p, q , and that the point whose Cartesian co-ordinates are (p, q) takes all positions in a certain continuum s of two dimensions. The region of integration is defined by s and by the relations which give λ, μ in terms of p, q , and the integral is therefore

$$\iint_s J \frac{\partial(\mu, \lambda)}{\partial(p, q)} dp dq.$$

Its absolute value does not exceed the greatest absolute value of J multiplied by the area of the region of integration, if we define this area as

$$\iint_s \left| \frac{\partial(\mu, \lambda)}{\partial(p, q)} \right| dp dq,$$

a quantity which does not depend on the particular choice of parameters p, q . This area, moreover, is less than the sum of the four integrals $\iint |d\xi d\xi|, \iint |d\omega d\xi|, \iint |d\xi d\eta|, \iint |d\omega d\eta|$ taken over the same region.

In the case of the region Q , the range of values of each of the real variables ξ, η, ζ, ω is of the order of k^2 , and thus the area is of the order of k^4 .

35. Let us take first those parts of Q in which the lateral parts of λ, μ are not both less than $k^{\frac{1}{2}}$. The lateral parts of $P^{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ are of constant sign, since we do not enter the regions C, D ; also, either P or R is constantly of the order of k^2 (§ 31) and the other does not fall below that of $k^{\frac{1}{2}}$.

We have, approximately, when $t < x$, as at § 26,*

$$\Theta^{-1} \frac{\partial}{\partial t} \Phi(t, x) = \frac{1}{2} \{P(t)/P(x)\}^{\frac{1}{2}} \exp \left\{ - \int_t^x P^{\frac{1}{2}} dx \right\},$$

and the absolute value of the exponential factor is not greater than $\exp K_1(t-x)$, where K_1 is the least value of the real part of $P^{\frac{1}{2}}$, and is therefore at least of the order of $k^{\frac{3}{2}}$.

Similarly, when $u < y$,

$$Z^{-1} \frac{\partial}{\partial u} \Psi(u, y) = \frac{1}{2} \{R(u)/R(y)\}^{\frac{1}{2}} \exp \left\{ - \int_u^y R^{\frac{1}{2}} dy \right\},$$

where the exponential is not greater absolutely than $\exp K_2(u-y)$, K_2 being also at least of the order of $k^{\frac{3}{2}}$.

Also

$$\{P(t)R(u)/P(x)R(y)\}^{\frac{1}{2}} \frac{\partial^2}{\partial t \partial u} \frac{f(t, u)}{P(t)R(u)}$$

is not of higher order than $k^{-\frac{1}{2}}$, since f and its derivatives are finite, P', R' of the order of k^2 , and PR of that of $k^{\frac{1}{2}}$.

Again,

$$\int_0^x e^{K_1(t-x)} dt < 1/K_1, \quad \int_0^y e^{K_2(u-y)} du < 1/K_2.$$

Thus in $\iiint \int_{y_0}^y \int_{x_0}^x \Theta^{-1} Z^{-1} \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial u} \frac{\partial^2}{\partial t \partial u} \frac{f(t, u)}{P(t)R(u)} dt du d\mu d\lambda$

* When the positions of t, x are such that the E term in $\Phi(t, x)$ is important, we ought to add to this expression for $\Theta^{-1}(\partial\Phi/\partial t)$ the term

$$E \{P(t)/P(x)\}^{\frac{1}{2}} \exp \int_t^x P^{\frac{1}{2}} dx / E_4 \{P(1)P(0)\}^{\frac{1}{2}} \exp \int_0^1 P^{\frac{1}{2}} dx,$$

which here is small as compared with any negative power of k , except when t, x have their extreme values (0, 1). (Compare I., p. 98.)

the integrations with respect to t, u give a result of the order of $1/K_1 K_2 k^{\frac{1}{2}}$ or k^{-5} , and, since the whole area of Q is of the order of k^4 , the quadruple integral is of the order of $1/k$. The same holds for the other parts of H , that is, when t runs from x to x_1 or u from y to y_1 or both.

Hence, on this first part of Q , the quadruple integral (1) tends to zero when k is made great.

One of the triple integrals is (6),

$$- \iiint_{x_0}^{x_1} \Theta^{-1} \frac{\partial \Phi}{\partial t} \frac{\partial}{\partial t} \frac{f(t, y)}{P(t) R(y)} dt d\mu d\lambda,$$

which comes out in like manner to be of the order of $k^{-\frac{1}{2}}$ at most, and the same holds for the other triple integral (5),

$$- \iiint_{y_0}^{y_1} Z^{-1} \frac{\partial \Psi}{\partial u} \frac{\partial}{\partial u} \frac{f(x, u)}{P(x) R(u)} du d\mu d\lambda.$$

In the other parts (2), (3), (4) of the expression, only points on the boundary of H are concerned, and the exponential factor, in Φ/Θ or Ψ/Z or both, ensures that the integrals tend to the limit zero.

36. Next consider that part of Q in which, while the lateral parts of λ, μ are both less than $k^{\frac{1}{2}}$, the real parts of P, R are numerically greater than $k^{\frac{1}{2}}$ for all values of x, y .

We may still use the hyperbolic approximation (§ 7), for P, R do not fall to the order of $k^{\frac{1}{2}}$, and when the real part of $P^{\frac{1}{2}}$ or $R^{\frac{1}{2}}$ is not of constant sign, it is at most $\parallel \log \log k$ (§ 31).

The approximate value of $(\partial \Phi / \partial t) / \Theta$ contains the factor

$$\sinh \int_0^t P^{\frac{1}{2}} dx \cosh \int_x^1 P^{\frac{1}{2}} dx / \sinh \int_0^1 P^{\frac{1}{2}} dx \quad (t < x)$$

or, say, $\sinh (u_1 + iv_1) \cosh (u_2 + iv_2) / \sinh (u_3 + iv_3)$,

in which $|u_1| + |u_2| - |u_3|$ is negative unless the real part of $P^{\frac{1}{2}}$ is zero or changes sign on the x -path 0, 1; only when this happens can the factor increase indefinitely with k , and, since $|u_1| + |u_2| - |u_3|$ is at most $\parallel \log \log k$, while the argument $u_3 + iv_3$ in the denominator is nearly $\frac{1}{2} k \rho i \pi$ or $\frac{1}{2} k \nu i \pi$, this factor and the similar factor in $(\partial \Psi / \partial u) / Z$ are at most $\parallel (\log k)^{\delta}$ if not small (§ 26), and, in fact, only one of the two can increase indefinitely with k .

The other factor in the subject of integration of the quadruple integral (1) is

$$\{P(t) R(u) / P(x) R(y)\}^{\frac{1}{2}} \frac{\partial^2}{\partial t \partial u} \frac{f(t, u)}{P(t) R(u)},$$

the order of which is at most $k^{-\frac{3}{2}}$; it remains to find the order of the area of the region of integration.

Now the range of variation is $\parallel k^2$ in the case of ξ or ζ , and $\parallel k^{\frac{1}{2}}$ in that of ω or η ; but, when ξ is constant, the range of ζ is restricted either by the conditions near g , h (§ 30) or those at the crossings of C' , D' (§ 31), to the order of $k^{\frac{1}{2}}$, and so the order of magnitude of the area is that of $\iint |d\xi d\omega|$ or $\iint |d\eta d\zeta|$, namely, $k^2 \cdot k^{\frac{1}{2}}$. Hence the order of this part of the quadruple integral is $(\log k)^{\delta} k^{-\frac{3}{2}}$; the same is true of the triple integrals (5), (6), (2) contributed by the lines $u = y$, $t = x$ and by the lines $u = y_0$, $u = y_1$. The terms (4) at the limits ($t = x$, $u = y_0$, y_1) give a quantity $\parallel (\log k)^{\delta} k^{-\frac{3}{2}}$. We have left (3),

$$\iiint_{y_0}^{y_1} \Theta^{-1} Z^{-1} \frac{\partial \Phi}{\partial t} \Psi \frac{f(t, u)}{P(t)} du d\mu d\lambda \quad (t = x_0, x_1),$$

which, on integration by parts, gives

$$\begin{aligned} \iint \Theta^{-1} Z^{-1} \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial u} \frac{f(t, u)}{P(t) R(u)} d\mu d\lambda \quad (\text{between limits of } u) \\ - \iiint_{y_0}^{y_1} \Theta^{-1} Z^{-1} \frac{\partial \Phi}{\partial t} \frac{\partial \Psi}{\partial u} \frac{\partial}{\partial u} \frac{f(t, u)}{P(t) R(u)} du d\mu d\lambda, \end{aligned}$$

the two terms of which are respectively $\parallel (\log k)^{\delta} k^{-\frac{3}{2}}$, $(\log k)^{\delta} k^{-\frac{3}{2}}$ again.

37. In what is left of the field Q , the crossing of one of the regions C , D happens comparatively near to the vertical boundary of the other. Near the vertical boundary of C , P falls below the order of k^2 for some value of x and the Riccati approximation may be used for ϕ . The real part of P falls, in fact, below $k^{\frac{3}{2}}$, and generally vanishes and changes sign; the lateral part is of constant sign, since C is not entered, and of some order between $k \log \log k$ and $k^{\frac{1}{2}}$, in consequence of the provision made in § 31; when P has a cluster of zeros, the results of §§ 22–25 are applicable, since $k_1 \parallel \log \log k$ at least.

R , on the other hand, at the crossing of D , has a real part $\parallel k^2$ for all values of y , and there is nothing new in the behaviour of ψ ; so that the factor $(\partial \Psi / \partial u) / Z$ is at most $\parallel (\log k)^{\delta}$. Leaving it out for the present and dropping also the factor $1/R$ or R'/R^2 , which $\parallel k^{-2}$, we have two types of integral to consider, namely:

$$(a) \quad \iiint \left| \frac{1}{\Theta} \frac{\partial \Phi}{\partial t} \frac{\partial}{\partial t} \frac{\Gamma}{P(t)} dt d\mu d\lambda \right|$$

in (1), (2), (6),

$$(b) \quad \iint \left| \frac{1}{\Theta} \frac{\partial \Phi}{\partial t} \frac{\Gamma}{P(t)} \partial \mu \partial \lambda \right| \quad (t = x, x_0, x_1)$$

in the last form (§ 36) of (3), and also in (4), (5) before simplification, where Γ may be $f(t, u)$ or $\partial f/\partial u$.

Here $\Theta^{-1}\partial\Phi/\partial t$ is of no higher order than the greater of

$$\{P(t)/P(x)\}^{\frac{1}{2}} \quad \text{and} \quad k^{\frac{1}{2}}\{P(x)\}^{-\frac{1}{2}}.$$

In (a) we may put

$$\{P(t)\}^{\frac{1}{2}} \frac{\partial}{\partial t} \frac{\Gamma}{P(t)} = \frac{4}{3}\Gamma^{\frac{1}{2}} \frac{\partial}{\partial t} \left\{ \frac{\Gamma}{P(t)} \right\}^{\frac{3}{2}},$$

and, since Γ is finite throughout, while $P(t)$ falls to the order of ω , the integration in (a) with respect to t , when $\omega > k^{\frac{1}{2}}$, gives a result ||

$$\{P(x)\}^{-\frac{1}{2}} \times \text{length of path of the point } \{\Gamma/P(t)\}^{-\frac{3}{2}},$$

that is, || $\{P(x)\}^{-\frac{1}{2}}\omega^{-\frac{3}{2}}$. When $\omega < k^{\frac{1}{2}}$, the result is ||

$$k^{\frac{1}{2}}\{P(x)\}^{-\frac{1}{2}} \times \text{length of path of } \Gamma/P(t),$$

that is, || $k^{\frac{1}{2}}\{P(x)\}^{-\frac{1}{2}}/\omega$. The type (a) is thus reduced to the least favourable case of the type (b).

The further integration is with respect to λ, μ , which vary as explained in § 31. Let us take ω, ξ as independent variables and η, ζ as depending on them.

As ω, ξ vary, η is either constant (θk) or equal to ω/s_1 or ω/s_2 . The condition determining ζ is that the real part of

$$\int_0^1 \sqrt{(\mu V - \lambda U)} dy = \frac{1}{2} k q \pi.$$

Thus, when ω, η are constant, the rate of change of ζ as compared with ξ must be a mean among the values of U/V , *i.e.*, must lie between q_1 and q_2 ; for, if $\mu V - \lambda U$ has its real part increased for all values of y , the real part of the integral will be increased and cannot be constant.

Also, when ξ is constant, it is found without difficulty that $\partial\zeta/\partial\omega$ || ω/k^2 , and is therefore small. Thus

$$\frac{\partial(\lambda, \mu)}{\partial(\xi, \omega)} = \frac{\partial\xi}{\partial\omega} + \iota + \iota \frac{\partial(\eta, \xi)}{\partial(\xi, \omega)} - \frac{\partial\eta}{\partial\xi} = \frac{\partial\xi}{\partial\omega} + \iota - \iota \frac{\partial\xi}{\partial\xi} \frac{\partial\eta}{\partial\omega} \quad \left(\text{since } \frac{\partial\eta}{\partial\xi} = 0 \right)$$

and is a finite quantity, for the first term is small and the last lies between $-iq_1/s_2$ and $-iq_2/s_1$ when it is not zero.

Also

$$P = -\lambda S + \mu T - X,$$

and thus

$$\frac{\partial P}{\partial\xi} = -S + T \frac{\partial\xi}{\partial\xi},$$

which lies between the negative finite values $q_1 T - S$ and $q_2 T - S$, and cannot approach zero.

Hence, over the part of the field where $\omega > k^{\frac{1}{2}}$, the integrals (a), (b) are at most ||

$$\iint |P^{-\frac{1}{2}} \omega^{-\frac{3}{2}} d\xi d\omega|,$$

which ||

$$\iint \left| P^{-\frac{1}{2}} \frac{\partial P}{\partial \xi} \omega^{-\frac{3}{2}} d\xi d\omega \right|,$$

which || length of path of $P^{\frac{1}{2}}$, when ω is constant $\times$ length of path of $\omega^{\frac{1}{2}}$.

Now P has a range of values that || k^2 , and ω ranges from $k^{\frac{1}{2}}$ to $k^{\frac{1}{2}}$. Thus the integrals (a), (b) over this part of the field are at most || $k^{\frac{1}{2}}.k^{\frac{1}{2}}$, that is, $k^{\frac{1}{2}}$.

Over the part of the field where $\omega < k^{\frac{1}{2}}$, we have the subject changed by a factor $k^{\frac{1}{2}} \omega^{-\frac{1}{2}}$, and thus the integrals are || length of path of $P^{\frac{1}{2}} \times$ length of path of $\log \omega.k^{\frac{1}{2}}$, that is, ||

$$k^{\frac{1}{2}}.\log k.k^{\frac{1}{2}} \quad \text{or} \quad k^{\frac{1}{2}} \log k.$$

In each case the result is small when multiplied by the omitted factor || $(\log k)^{\delta}/k^2$.

The same results follow when it is R instead of P that falls below the order of k^2 .

Hence the whole remainder tends to zero when k is increased without limit.

Statement of the Result.

38. Thus, writing $F(x, y)$ for $f(x, y)/(SV-TU)$ we have

$$F(x, y) = \lim_{k=\infty} \sum_{r=0}^{\alpha} \sum_{s=0}^{\beta} \iint_H \Omega(x, t, y, u, \lambda_{rs}, \mu_{rs}) [S(t) V(u) - T(t) U(u)] \\ \times F(t, u) dt du.$$

Each term in the double summation is the sum of four parts, each of the form $\phi(x) \psi(y)$ multiplied by a constant (see § 6a), and $F(x, y)$ is therefore expanded in a series of such terms. λ_{rs}, μ_{rs} are such values of λ, μ that Θ, Z both vanish; the meaning of the suffixes r, s was discussed in § 32 n. α, β tend to infinity in a ratio that lies between certain limits: the function F can be differentiated as to x and to y successively at least once: the expansion holds good for any point (x, y) within the rectangular area H , over which the integration takes place; if (x, y) lies outside H the sum of the doubly infinite series is zero.

The theorem can easily be extended, by addition, to areas that lie

within the unit square and are bounded wholly by lines parallel to its sides, and to functions which have lines of discontinuity parallel to those sides.

Case when E_4 or $G_4 = 0$.

39. Hitherto it has been assumed that $E_4 \neq 0$, $G_4 \neq 0$, but only a few changes in detail are needed when $E_4 = 0$ or $G_4 = 0$.

When E_4 , E_2 , E_3 , E are all zero, Θ , Φ reduce to single terms, and the work is exactly the same as when $E_4 \neq 0$. The factor $\{P(0)P(1)\}^{\frac{1}{2}}$ is replaced by its reciprocal, but as it is common to the approximate values of Θ , Φ , this makes no practical difference.

When E_2 , E_3 do not both vanish it is important to notice that they cannot have the same sign, since $E_1 E_4 - E_2 E_3 - E^2$ must be positive and now $E_4 = 0$. The predominant terms (§ 26) in Θ are apparently

$$E_2 \phi'(1, 0) + E_3 \phi'(1, 0),$$

but this may conceivably not be true if there is a terminal short section, or if

$$E_2 \{P(1)/P(0)\}^{\frac{1}{2}} - E_3 \{P(0)/P(1)\}^{\frac{1}{2}}$$

is small, in which latter case the amplitude of the complex quantity P must increase by nearly an odd multiple of 2π as x travels from 0 to 1. This is impossible if either the real or the lateral part of P keeps a constant sign, which is always the case in the above work.

In the case of a terminal short section there is no fresh feature if either E_2 or $E_3 = 0$, but if this is not so we must examine the possibility of the two terms destroying each other. If there is one short section, say at the end 1, the factors $P(1)^{\pm \frac{1}{2}}$ are to be replaced by $k^{\pm \frac{1}{2}}$, and this makes the E_3 term predominate, unless the factor $\chi(w_{n+1})$ occurs and is small, and in fact $\parallel k^{-\frac{1}{2}}$ at most. This does not happen, for k_1 is great and $\chi(w_{n+1})$ is at least $\parallel k_1 k^{-\frac{1}{2}}$.

If there are short sections at both ends, and the neighbouring section in each case is of the q -kind, the formula (5) of § 4 enables us to see that Θ is of its full order of magnitude, that is, that the E_2 , E_3 terms do not destroy each other, the reasoning being the same as in § 20. When the sections neighbouring the short ones are of different kinds, suppose that near the end 1 to be of the p -kind: then the approximations to $\phi'(1, 0)$, $\phi'(1, 0)$ contain the factors $\chi'(w_{n+1})$, $\chi(w_{n+1})$ respectively, and one term or the other may suffer a loss through this factor being small, but only one of the two χ , χ' can be small at once: suppose then that χ is not small.

From (5) in § 4 we have

$$\begin{aligned} & \{E_2 \phi'(1, 0) + E_3 \phi''(1, 0)\} \phi_1(1, 0) - \{E_2 \phi'_1(1, 0) + E_3 \phi''_1(1, 0)\} \phi(1, 0) \\ &= \int_0^1 [E_3 \phi(x, 1) \phi_1(x, 1) - E_2 \phi(x, 0) \phi_1(x, 0)] (P_1 - P) dx, \end{aligned}$$

and the reasoning of § 20 applied to this shews that

$$E_2 \phi'(1, 0) + E_3 \phi''(1, 0)$$

cannot suffer loss of order, since in the first term in the square brackets χ enters and not χ' .

If χ were small we could use similarly the equation

$$\begin{aligned} & \{E_2 \phi'(1, 0) + E_3 \phi''(1, 0)\} \phi''_1(1, 0) - \{E_2 \phi'_1(1, 0) + E_3 \phi''_1(1, 0)\} \phi''(1, 0) \\ &= \int_0^1 [E_2 \phi'(1, x) \phi''_1(1, x) - E_3 \phi'(x, 0) \phi''_1(x, 0)] (P_1 - P) dx. \end{aligned}$$

When, lastly, the neighbouring section is in each case of the p -kind the two terms contain respectively the factors*

$$E_2 w'_{n+1} \chi'(w_{n+1}) \chi(w_0), \quad E_3 w'_0 \chi'(w_0) \chi(w_{n+1}),$$

their other factors being common to both, and if there is a loss of order of magnitude in Θ it will be because w_0, w_{n+1} are such that

$$E_2 w'_{n+1} \chi'(w_{n+1}) \chi(w_0) - E_3 w'_0 \chi'(w_0) \chi(w_{n+1})$$

is comparatively small, that is, below the order of $k^{\frac{2}{3}}$. Now the equation (5) of § 4 shews that any such reduction does not exceed that due to a factor $\parallel k_1 k^{-\frac{1}{3}}$ (as in the final footnote to § 21). This will not affect the predominance of the E_2 and E_3 terms, and moreover Φ will suffer a similar reduction, since its approximate expression also contains the factor in question. Thus the order of magnitude of Φ/Θ is not affected.

The only other special point to be attended to when $E_4 = 0$ relates to the integer p of § 28, which is better taken even, unless E_2, E_3 vanish also. The two fractions $E_3/E, -E_2/E$ have the same sign; if they are positive, $\frac{1}{2}p$ should be even, to make the E_2, E_3, E terms in Θ all of the same sign at the starting point: if they are negative, $\frac{1}{2}p$ should be odd (see I., p. 99). Similarly for r (§ 29) and when $G_4 = 0$ for q .

* Here w_0 and w'_0 denote the value and derivative of the Riccati argument corresponding to $x = 0$, just as w_{n+1}, w'_{n+1} are similar quantities at the other end of the path.

Double Solutions of the Equations $\Theta = 0$, $Z = 0$.

40. We have supposed so far that all the solutions of the equations $\Theta = 0$, $Z = 0$ for λ , μ are simple. If this is not the case the only effect is a modification in the form of the series of residues $\Sigma\Omega$.

Suppose, for instance, that the curve $\Theta = 0$ has a node, and that $Z = 0$ passes once through this node. From the conditions for a node found in § 5 it follows that $\Phi = 0$ there also for all values of x , t (see I., p. 89). If the μ -path encloses the zero of Z but not those of Θ , the residue in the μ -integration is

$$\Phi\Psi/\Theta \frac{\partial Z}{\partial \mu}.$$

In the λ -integration we are to treat μ as a function of λ defined by the relation $Z = 0$, and thus at the point in question the numerator $\Phi\Psi$ has a simple zero and the denominator a double one. The contribution to the value of the double integral comes out to be

$$-4\pi^2\Psi \frac{\partial(\Phi, Z)}{\partial(\lambda, \mu)} \Big/ \frac{1}{2} \left[\frac{\partial^2\Theta}{\partial\lambda^2} \left(\frac{\partial Z}{\partial\mu} \right)^2 - 2 \frac{\partial^2\Theta}{\partial\lambda\partial\mu} \frac{\partial Z}{\partial\lambda} \frac{\partial Z}{\partial\mu} + \frac{\partial^2\Theta}{\partial\mu^2} \left(\frac{\partial Z}{\partial\lambda} \right)^2 \right]$$

if the point is enclosed once by the λ -path. By the formulæ of § 5, the denominator here

$$= -\frac{1}{2}E \int_0^1 \int_0^1 [\phi(x_1, x)]^2 \left\{ S \frac{\partial Z}{\partial \mu} + T \frac{\partial Z}{\partial \lambda} \right\} \left\{ S_1 \frac{\partial Z}{\partial \mu} + T_1 \frac{\partial Z}{\partial \lambda} \right\} dx_1 dx,$$

and since
$$\delta Z = \int_0^1 \Psi(y, y)(U\delta\lambda - V\delta\mu) dy,$$

the denominator

$$= -\frac{1}{2}E \int_0^1 \int_0^1 \int_0^1 [\phi(x_1, x)]^2 \Psi(y, y) \Psi(y_1, y_1) (SV - TU) (S_1 V_1 - T_1 U_1) \\ \times dx_1 dx dy_1 dy.$$

The factor $\frac{\partial(\Phi, Z)}{\partial(\lambda, \mu)}$ reduces similarly (see § 6*n*) to

$$-E \int_0^1 \int_0^1 \phi(x_1, x) \phi(x_1, t) \Psi(y_1, y_1) (S_1 V_1 - T_1 U_1) dx_1 dy_1,$$

and so we have two terms of the series $\Sigma\Omega$ replaced by

$$\frac{\Psi(y, u) \int_0^1 \int_0^1 \phi(x_1, x) \phi(x_1, t) \Psi(y_1, y_1) (S_1 V_1 - T_1 U_1) dx_1 dy_1}{\frac{1}{2} \int_0^1 \int_0^1 \int_0^1 [\phi(x_1, x)]^2 \Psi(y, y) \Psi(y_1, y_1) (SV - TU) (S_1 V_1 - T_1 U_1) dx_1 dx dy_1 dy}.$$

(50)

The expression in the case when $Z = 0$ has a node which lies on $\Theta = 0$ is of the same nature.

When the curves have a common node, and the μ -path encloses the zeros of Z and not those of Θ , the value of λ at the node being enclosed by the λ -path, the expression is found, after some calculation, to be

$$\frac{(2i\pi)^2 [2a_1a_2(H_2B_1 - H_1B_2) + (a_1b_2 + a_2b_1)(A_1B_2 - A_2B_1) + 2b_1b_2(H_1A_2 - H_2A_1)]}{\frac{1}{4}(A_1B_2 + A_2B_1 - 2H_1H_2)^2 - (A_1B_1 - H_1^2)(A_2B_2 - H_2^2)}$$

where a_1, b_1 are the first derivatives of Φ , a_2, b_2 those of Ψ , A_1, H_1, B_1 the second derivatives of Θ , and A_2, H_2, B_2 those of Z with respect to λ, μ . On substitution of the values given in §§ 5, 6, the numerator becomes the product of $(2i\pi)^2 EG$, and

$$\int_0^1 \int_0^1 \phi(x_1, x) \dot{\phi}(x_1, t) \psi(y_1, y) \psi(y_1, u) [-2S_1U_1(H_2B_1 - H_1B_2) \\ + (S_1V_1 + T_1U_1)(A_1B_2 - A_2B_1) - 2T_1V_1(H_1A_2 - H_2A_1)] dy_1 dx_1.$$

The expression in square brackets

$$= EG \int_0^1 \int_0^1 \int_0^1 \int_0^1 \{ \phi(x_2, x_3) \psi(y_2, y_3) \}^2 [-2S_1U_1(-U_2V_3 \cdot T_2T_3 + V_2V_3 \cdot S_2T_3) \\ + (S_1V_1 + T_1U_1)(S_2S_3V_2V_3 - U_2U_3T_2T_3) \\ + 2T_1V_1(S_3T_2 \cdot U_2U_3 - U_3V_2S_2S_3)] dx_3 dx_2 dy_3 dy_2,$$

which is not affected by our adding to the factor in square brackets the terms

$$(S_1V_1 + T_1U_1)(S_2T_3U_3V_2 - S_3T_2U_2V_3),$$

which change that factor into

$$2(S_2V_2 - T_2U_2)(S_1V_3 - T_1U_3)(S_3V_1 - T_3U_1) \\ - (S_2V_2 - T_2U_2)(S_1V_1 - T_1U_1)(S_3V_3 - T_3U_3).$$

Thus by an interchange of suffixes in the first part of this expression, we get the numerator in the form

$$(2i\pi EG)^2 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 (S_1V_1 - T_1U_1)(S_2V_2 - T_2U_2)(S_3V_3 - T_3U_3) \\ \times \psi(y_1, y) \psi(y_1, u) [\psi(y_2, y_3)]^2 \\ \times \{ 2\phi(x_3, x) \phi(x_3, t) [\phi(x_2, x_1)]^2 - \phi(x_1, x) \phi(x_1, t) [\phi(x_2, x_3)]^2 \} \\ \times dx_3 dx_2 dx_1 dy_3 dy_2 dy_1.$$

For the factors depending on ϕ , ψ in the subject we may put

$$\frac{1}{3} [\Sigma \psi(y_1, y) \psi(y_1, u) \phi(x_3, x) \phi(x_3, t) \{\psi(y_2, y_3) \phi(x_1, x_2)\}^2 \\ - \Sigma \psi(y_1, y) \psi(y_1, u) \phi(x_1, x) \phi(x_1, t) \{\psi(y_2, y_3) \phi(x_2, x_3)\}^2],$$

the summations referring to permutations of the suffixes 1, 2, 3; this expression is found to be

$$\frac{1}{6} \{\Delta(x, y) \Delta(t, u) + \Delta(x, u) \Delta(t, y)\},$$

where $\Delta(x, y)$ is the determinant whose first row is

$$\phi_1 x \psi_1 y, \quad \phi_1 x \psi_2 y, \quad \phi_2 x \psi_1 y, \quad \phi_2 x \psi_2 y,$$

and whose second, third, fourth rows are formed by putting the suffixes 1, 2, 3 to x and y in the first, where ϕ_1, ϕ_2 are two functions satisfying the equation (1), and such that $\phi_1 \phi'_2 - \phi_2 \phi'_1 = 1$; ψ_1, ψ_2 are similar functions with regard to (2).

The denominator is equal to

$$\frac{1}{4} \begin{vmatrix} A_1, & 2H_1, & B_1, & 0 \\ 0, & A_1, & 2H_1, & B_1 \\ A_2, & 2H_2, & B_2, & 0 \\ 0, & A_2, & 2H_2, & B_2 \end{vmatrix},$$

which is the product of $\frac{1}{4}E^2G^2$ and

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \{ \phi(x_1, x) \phi(x_2, x_3) \psi(y_1, y) \psi(y_2, y_3) \}^2 \\ \times \begin{vmatrix} SS_1, & ST_1 + S_1 T, & TT_1, & 0 \\ 0, & S_2 S_3, & S_2 T_3 + S_3 T_2, & T_2 T_3 \\ UU_1, & UV_1 + U_1 V, & VV_1, & 0 \\ 0, & U_2 U_3, & U_2 V_3 + U_3 V_2, & V_2 V_3 \end{vmatrix} \Pi(dx dy).$$

The determinant differs from

$$2(S_1 V_3 - U_3 T_1)(S_3 V_1 - U_1 T_3)(SV - UT)(S_2 V_2 - U_2 T_2) \\ - (SV - UT)(S_1 V_1 - U_1 T_1)(S_2 V_2 - U_2 T_2)(S_3 V_3 - U_3 T_3)$$

by terms which contribute nothing to the result, since they can be made to cancel by allowable permutations of the suffixes, and the whole expression then reduces to the eightfold integral of

$$\frac{1}{12} E^2 G^2 [\Delta(x, y)]^2 (SV - TU)(S_1 V_1 - T_1 U_1)(S_2 V_2 - T_2 U_2)(S_3 V_3 - T_3 U_3),$$

by the same process as was applied to the numerator. Thus we derive denominator from numerator by putting $t = x$, $u = y$, multiplying by $-(SV-TU)/16\pi^2$ and integrating with respect to x and y from 0 to 1.

Form of the General Term in the Expansion.

41. The equation $\Theta = 0$ may be understood as the condition that the multipliers of the linear transformation

$$\begin{pmatrix} -E_3 & -E_4 \\ E_1 & E_2 \end{pmatrix} \begin{pmatrix} -\phi'(1, 0) & \phi(1, 0) \\ -\phi'(1, 0) & \phi'(1, 0) \end{pmatrix},$$

in which the right hand factor is $W(1, 0)$, shall be the roots of

$$\theta^2 - 2E\theta + E_1E_4 - E_2E_3 = 0,$$

that is, fixed complex quantities, say θ_1, θ_2 .

If we put $\phi(x) = A\phi(x, 0) + B\phi'(x, 0)$,

we have $\{\phi(1), \phi'(1)\} = W(1, 0) \{\phi(0), \phi'(0)\}$,

and thus it is possible to choose A_1, B_1 values of A, B , such that if ϕ_1 is the corresponding function ϕ ,

$$-E_3\phi_1(1) - E_4\phi_1'(1) = \theta_1\phi_1(0),$$

$$E_1\phi_1(1) + E_2\phi_1'(1) = \theta_1\phi_1'(0).$$

If A_2, B_2, ϕ_2 are the complex quantities conjugate to A_1, B_1, ϕ_1 , then similar equations hold with ϕ_2, θ_2 in the place of ϕ_1, θ_1 .

It is readily verified that $\phi_1(x)$ can only differ by a constant factor from

$$E_3\phi(1, x) + E_4\phi'(1, x) - \theta_1\phi(x, 0)$$

$$\text{or } E_1\phi(1, x) + E_2\phi'(1, x) + \theta_1\phi'(x, 0)$$

$$\text{or } E_1\phi(x, 0) + E_3\phi'(x, 0) + \theta_2\phi'(x, 1)$$

$$\text{or } E_2\phi(x, 0) + E_4\phi'(x, 0) - \theta_2\phi(x, 1).$$

The constant factors in the first two cases are

$$\{E_4 - \theta_1\phi(1, 0)\} / \phi_1 1, \quad \{E_2 + \theta_1\phi'(1, 0)\} / \phi_1 1,$$

and thus

$$\begin{aligned} \Phi(x, t)\phi_1 1 &= \{E_2\phi_1 t + \theta_1\phi'(1, 0)\phi_1 t - \theta_1\phi'(t, 0)\phi_1 1\}\phi(x, 0) \\ &\quad + \{E_4\phi_1 t - \theta_1\phi(1, 0)\phi_1 t + \theta_1\phi(t, 0)\phi_1 1\}\phi'(x, 0) + E\phi(t, x)\phi_1 1 \\ &= \phi_1 t \{E_2\phi(x, 0) + E_4\phi'(x, 0) - \theta_1\phi(x, 1)\} + (E - \theta_1)\phi(t, x)\phi_1 1, \end{aligned}$$

after some reduction. Thus

$$\Phi(x, t) \phi_1 1 \phi_2 1 = \phi_1 t \phi_2 x \{E_2 \phi(1, 0) + E_4 \phi'(1, 0)\} + (E - \theta_1) \phi(t, x) \phi_1 1 \phi_2 1,$$

and by proper choice of the arbitrary constants in ϕ_1, ϕ_2 ,

$$\Phi(x, t) = \phi_1 t \phi_2 x + \phi_2 t \phi_1 x,$$

since the same equation holds with the interchange of the suffixes 1, 2, and

$$E - \theta_1 = -(E - \theta_2).$$

Also
$$\frac{\partial \Theta}{\partial \lambda} = - \int_0^1 \Phi(x, x) S dx = -2 \int_0^1 \phi_1 x \phi_2 x S dx,$$

and so for the other derivatives.

Hence the typical term in the expansion of $F(x, y)$ consists of four parts, which are respectively multiples of

$$\phi_1 x \psi_1 y, \quad \phi_1 x \psi_2 y, \quad \phi_2 x \psi_1 y, \quad \phi_2 x \psi_2 y,$$

and the coefficient of $\phi_r x \psi_s y$ ($r, s = 1, 2$) is*

$$\frac{1}{4} \int_0^1 \int_0^1 F(x, y) \phi_{3-r} x \psi_{3-s} y (SV - TU) dy dx$$

$$\bigg/ \int_0^1 \int_0^1 \phi_1 x \phi_2 x \psi_1 y \psi_2 y (SV - TU) dy dx.$$

The expression is simplified when $E_1 E_4 - E_2 E_3 - E^2 = 0$, for then $\theta_1 = \theta_2 = E$, and ϕ_1, ϕ_2 are the same function; the typical term is then the sum of two parts instead of four: when $G_1 G_4 - G_2 G_3 - G^2 = 0$ also, there is further simplification, the coefficient of $\phi_1 x \psi_1 y$ being

$$\int_0^1 \int_0^1 F(x, y) \phi_1 x \psi_1 y (SV - TU) dy dx \bigg/ \int_0^1 \int_0^1 (\phi_1 x \psi_1 y)^2 (SV - TU) dy dx,$$

the familiar form.

* So in expanding a function of one variable, as, for instance, if we take $T = 0$ and expand $F(x)$ in terms of the form ϕx which satisfy the condition $\Theta = 0$, the typical term consists of two parts, multiples of $\phi_1 x, \phi_2 x$ respectively, and the coefficient of $\phi_r x$ ($r = 1, 2$) is

$$\frac{1}{2} \int_0^1 S F(x) \phi_{3-r} x dx \bigg/ \int_0^1 S \phi_1 x \phi_2 x dx. \quad (\text{See I.})$$

It is curious that the expansion so given of a function that could be expressed in the form $A_1 \phi_1 x + A_2 \phi_2 x$ never coincides with this form unless $E_1 E_4 - E_2 E_3 - E^2 = 0$.

42. In the special case when the curve $Z = 0$ passes through a node on $\Theta = 0$, it follows from the results of § 5 that the linear transformation

$$\begin{pmatrix} -E_3 & -E_4 \\ E_1 & E_2 \end{pmatrix} W(1, 0)$$

amounts to a simple multiplication by E , so that the relations

$$-E_3 \phi(1) - E_4 \phi'(1) = E \phi(0),$$

$$E_1 \phi(1) + E_2 \phi'(1) = E \phi'(0)$$

hold for all functions ϕ that satisfy (1). Let ϕ_1, ϕ_2 be two such functions in which the arbitrary constants are so determined that

$$\phi_1 \phi_2' - \phi_2 \phi_1' = 1$$

and
$$\int_0^1 \int_0^1 \phi_1(x) \phi_2(x) \Psi(y, y) (SV - TU) dy dx = 0.$$

Then
$$\phi(x_1, x) = \phi_1 x \phi_2 x_1 - \phi_1 x_1 \phi_2 x,$$

and thus in the fraction (50) the denominator can be simplified, and, in fact, is equal to

$$\int_0^1 \int_0^1 (\phi_1 x)^2 \Psi(y, y) (SV - TU) dy dx \int_0^1 \int_0^1 (\phi_2 x)^2 \Psi(y, y) (SV - TU) dy dx.$$

The numerator can be similarly transformed and the fraction breaks up into two of the customary form, namely,*

$$\begin{aligned} \Psi(y, u) \phi_1 x \phi_1 t / \int_0^1 \int_0^1 (\phi_1 x)^2 \Psi(y, y) (SV - TU) dy dx \\ + \Psi(y, u) \phi_2 x \phi_2 t / \int_0^1 \int_0^1 (\phi_2 x)^2 \Psi(y, y) (SV - TU) dy dx. \end{aligned}$$

The same kind of thing happens when the curves $\Theta = 0, Z = 0$ have a node in common: the corresponding term in $\Sigma\Omega$ has been discussed (§ 40), and to simplify it we go on as follows. If ϕ_1, ϕ_2 are two inde-

* Similarly, if $F(x)$ is expanded in multiples of functions ϕ (last note), when the equation $\Theta = 0$ gives two equal values of λ the expansion has the corresponding terms

$$\phi_1 x \int_0^1 S F(x) \phi_1 x dx / \int_0^1 S (\phi_1 x)^2 dx + \phi_2 x \int_0^1 S F(x) \phi_2 x dx / \int_0^1 S (\phi_2 x)^2 dx.$$

The trigonometrical series in sines and cosines is, in fact, a case of this, where $S = 1, X = 0, E_1 = E_4 = 0, E_2 = -E_3 = E$, so that

$$\Theta = 2 (\cos \sqrt{\lambda} - 1),$$

and $\lambda = (2n\pi)^2$ is a double solution of $\Theta = 0$, corresponding values of ϕ_1, ϕ_2 being

$$\cos 2n\pi x, \quad \frac{1}{2n\pi} \sin 2n\pi x.$$

pendent solutions of (1), and ψ_1, ψ_2 the like for (2), and

$$(\phi\psi)_r = A_r\phi_1\psi_1 + B_r\phi_1\psi_2 + C_r\phi_2\psi_1 + D_r\phi_2\psi_2,$$

then

$$\int_0^1 \int_0^1 (\phi\psi)_r (\phi\psi)_s (SV - TU) dy dx$$

is a symmetrical bilinear function of the two sets of coefficients A, B, C, D , say

$$(abcd fghlmn) \chi (A_r B_r C_r D_r) (A_s B_s C_s D_s),$$

where, for instance,

$$a = \int_0^1 \int_0^1 \phi_1^2 \psi_1^2 (SV - TU) dy dx,$$

$$f = l = \int_0^1 \int_0^1 \phi_1 \phi_2 \psi_1 \psi_2 (SV - TU) dy dx.$$

Another relation among the coefficients can be found, for any such expression as

$$\int_0^1 \int_0^1 \int_0^1 S_1 T_2 T_3 \phi(x_2, x_3) \phi(x_3, x_1) \phi(x_1, x_2) dx_3 dx_2 dx_1$$

vanishes, since its sign is reversed by at least one permutation of the suffixes 1, 2, 3. Hence

$$\begin{aligned} & \int_0^1 \int_0^1 \int_0^1 \int_0^1 \int_0^1 \prod_{r=1}^3 (S_r V_r - T_r U_r) (\phi_1 x_2 \phi_2 x_3 - \phi_2 x_2 \phi_1 x_3) (\phi_1 x_3 \phi_2 x_1 - \phi_2 x_3 \phi_1 x_1) \\ & \times (\phi_1 x_1 \phi_2 x_2 - \phi_2 x_1 \phi_1 x_2) (\psi_1 y_1 \psi_2 y_3)^2 \psi_1 y_2 \psi_2 y_2 \Pi dx dy = 0, \end{aligned}$$

which, on expansion, gives

$$-afd - chm - bgn + bcl + ghd + amn = 0$$

or

$$\begin{vmatrix} a & h & l \\ g & f & n \\ l & m & d \end{vmatrix} - \begin{vmatrix} h & b & f \\ g & f & c \\ l & m & n \end{vmatrix} = 0.$$

Hence the two quadrics $(abcd fghlmn) \chi (ABCD)^2$,

and

$$AD - BC$$

are such that both the invariants Θ, Θ' (Salmon, *Solid Geometry of Three Dimensions*, Fourth Edition, p. 173) vanish. Hence four sets of quantities

$$A_r, B_r, C_r, D_r \quad (r = 1, 2, 3, 4)$$

can be found such that $A_r D_r - B_r C_r = 0$,

and $(abcd fghlmn \times A_r B_r C_r D_r)(A_s B_s C_s D_s) = 0 \quad (r \neq s).$

Now $A_r D_r - B_r C_r = 0$ is the condition that $(\phi\psi)_r$ should be the product of factors, say $\phi_r \psi_r$, and hence four products

$$\phi_1 \psi_1, \quad \phi_2 \psi_2, \quad \phi_3 \psi_3, \quad \phi_4 \psi_4$$

can be found such that

$$\int_0^1 \int_0^1 \phi_r \psi_r \phi_s \psi_s (SV - TU) dy dx = 0 \quad (r \neq s).$$

The rows in $\Delta(x, y)$ may now be taken as of the form

$$\phi_1 \psi_1, \quad \phi_2 \psi_2, \quad \phi_3 \psi_3, \quad \phi_4 \psi_4,$$

and the denominator of the term in $\Sigma\Omega$ is a constant multiple of

$$\prod_{r=1}^4 \int_0^1 \int_0^1 (SV - TU) \phi_r^2 \psi_r^2 dy dx,$$

since, when $\Delta(x, y)$ is expanded and squared, all the double product terms contribute nothing to the integral.

Treating the numerator similarly we find that the term in $\Sigma\Omega$ takes the form

$$\sum_{r=1}^4 \phi_r x \phi_r t \psi_r y \psi_r u \bigg/ \int_0^1 \int_0^1 (SV - TU) \phi_r^2 \psi_r^2 dy dx,$$

which is the sum of four terms of the usual type.

Hence, if the curves $\Theta = 0$, $Z = 0$ meet in a point which is a node on either or both, the form of expansion is not affected, but for two or four terms in the expansion the values of λ , μ are the same.

Another special case, which includes most of those in which the expansions are commonly used, is when

$$E_1 E_4 = E_2 E_3,$$

and, of course, $E = 0.$

Thus $\theta_1 = \theta_2 = 0$, and the general term is a multiple of $\phi_1 x$, a function satisfying (1) and the further conditions

$$E_1 \phi_1(1) + E_2 \phi_1'(1) = 0,$$

$$E_1 \phi_1(0) + E_3 \phi_1'(0) = 0,$$

so that at each limit the ratio of the function to its derivative has a prescribed value, which may of course be zero or infinite.

43. The expansion is not generally valid at the boundary of H ; but when that boundary coincides, wholly or in part, with the boundary of the unit square there are some special cases (compare I., pp. 101-2).

The expansion holds when $y = 0, 1$, if

$$(1) \quad F(x, 1) = F(x, 0) = 0,$$

or (2) G_1, G_2, G_3, G vanish,

or (3) U, V, Y have the same values at the two limits and $G_1 = 0$,
 $G_2 = mG = -m^2G_3, \quad F(x, 1) = mF(x, 0).$

Similarly when $x = 0, 1$.

In case (3), taking $m = 1$, we have expansions in periodic functions. The present proof does not apply to Lamé's expansions on a complete ellipsoid, because in Lamé's case $SV - UT$ vanishes for one pair of values of x, y . The expansions moreover are of different kinds, the array of terms in this paper being rectangular, whereas in Lamé's formulæ it is triangular, the complete n -th term including $2n+1$ partial terms. It might be possible to prove the Lamé expansions by taking a quadratic form in Θ, Z , instead of their simple product, as denominator in the subject of integration.

ON THE NUMBERS OF REPRESENTATIONS OF A NUMBER
AS A SUM OF $2r$ SQUARES, WHERE $2r$ DOES NOT
EXCEED EIGHTEEN

By J. W. L. GLAISHER.

[Received and Read June 13th, 1907.]

1. In a series of papers which are in course of publication in the *Quarterly Journal of Mathematics** I have investigated systematically, by means of formulæ derived from elliptic functions, the numbers of representations of any number as a sum of 2, 4, 6, 8, 10, 12, 14, 16, and 18 squares, determining also in many cases the numbers of representations when the numbers of uneven and even squares in the representations are assigned.

The papers are somewhat lengthy, and the main results are partially obscured by the investigation of subsidiary formulæ, of recurring series required for the calculation of new functions introduced, and other collateral matters, so that I have thought I might be justified in submitting to the Society this short paper which contains a statement of the principal formulæ and results obtained relating to numbers of representations, separated from the mass of detail by which they are surrounded in the original papers.

*Formulæ for the Numbers of Representations and Definitions of the
Arithmetical Functions involved.*

2. Using $R^{(t)}(n)$ to denote the number of representations of n as a sum of t squares, the following formulæ express the values of $R^{(t)}(n)$ for even

* These papers are :—

(1) "On the Representations of a Number as the Sum of two, four, six, eight, ten, and twelve Squares," Vol. xxxviii., pp. 1-62.

(2) "On the Representations of a Number as the Sum of fourteen and sixteen Squares," *ib.*, pp. 178-236.

(3) "On the Representations of a Number as the Sum of eighteen Squares," *ib.*, pp. 289-351.

(4) "On Elliptic Function Expansions in which the Coefficients are Powers of the Complex Numbers having n as Norm," Vol. xxxix.

Connected also with the same subject are two other papers which had previously appeared in the *Quarterly Journal*, viz. : (i.) "On the Representations of a Number as a Sum of four Squares and on some Allied Arithmetical Functions," Vol. xxxvi., pp. 305-358 ; (ii.) "The Arithmetical Functions $P(m)$, $Q(m)$, $\Omega(m)$," Vol. xxxvii., pp. 36-48.

values of t up to $t = 18$:—

- (i.) $R^{(2)}(n) = 4E_0(n),$
- (ii.) $R^{(4)}(n) = (-1)^{n-1}8\xi_1(n),$
- (iii.) $R^{(6)}(n) = 4 \{4E'_2(n) - E_2(n)\},$
- (iv.) $R^{(8)}(n) = (-1)^{n-1}16\xi_3(n),$
- (v.) $R^{(10)}(n) = \frac{4}{5} \{E_4(n) + 16E'_4(n) + 8\chi_4(n)\},$
- (vi.) $\begin{cases} R^{(12)}(2n) = -8\xi_5(2n), \\ R^{(12)}(2n+1) = 8 \{\Delta_5(2n+1) + 2\Omega(2n+1)\}, \end{cases}$
- (vii.) $R^{(14)}(n) = \frac{4}{61} \{64E'_6(n) - E_6(n) + 364W(n)\},$
- (viii.) $R^{(16)}(n) = (-1)^{n-1}\frac{32}{17} \{\xi_7(n) + 16\Theta(n)\},$
- (ix.) $R^{(18)}(n) = \frac{4}{18065} \{13E_8(n) + 3328E'_8(n) + 97504\chi_8(n) \\ - 61200G(n) - 6120G(2n)\}.$

3. Five of the arithmetical functions which occur in the expression of these results, viz., $E_r(n)$, $E'_r(n)$, $\Delta_r(n)$, $\xi_r(n)$, $\xi'_r(n)$, depend upon the real divisors of n ; one, $\chi_8(n)$, depends upon certain complex divisors of n , and, for the other four, I have obtained no useful arithmetical definition. Two of them, $\Omega(n)$ and $\Theta(n)$, may be defined by means of the representations of n as a sum of 4 squares; but, in the case of $W(n)$ and $G(n)$, I have had to content myself with their definitions as coefficients.

The functions $E_r(n)$, ... which depend upon the real divisors of n (1 and n included) are defined as follows :—

(i.) $E_r(n)$ denotes the sum of the r -th powers of the divisors of n which are of the form $4k+1$ diminished by the sum of the r -th powers of the divisors of n which are of the form $4k+3$.

Thus $E_0(n)$ denotes the number of divisors of n of the form $4k+1$ diminished by the number of those of the form $4k+3$.

(ii.) $E'_r(n)$ denotes the sum of the r -th powers of the divisors of n whose conjugates are of the form $4k+1$ diminished by the sum of the r -th powers of those whose conjugates are of the form $4k+3$.

Thus, if n is of the form $4k+1$, $E'_r(n) = E_r(n)$; and, if n is of the form $4k+3$, $E'_r(n) = -E_r(n)$.

(iii.) $\Delta_r(n)$ denotes the sum of the r -th powers of the uneven divisors of n .

(iv.) $\xi_r(n)$ denotes the sum of the r -th powers of the uneven divisors of n diminished by the sum of the r -th powers of the even divisors of n .

(v.) $\xi_r(n)$ denotes the sum of the r -th powers of the even divisors of n whose conjugates are even and of the uneven divisors of n whose conjugates are uneven diminished by the sum of the r -th powers of the even divisors of n whose conjugates are uneven and of the uneven divisors of n whose conjugates are even.

Thus, if d is any divisor of n , and $dd' = n$, then

$$\xi_r(n) = \Sigma(-1)^{d-1} d^r, \quad \xi_r(n) = \Sigma(-1)^{d+d'} d^r.$$

When n is uneven, $\Delta_r(n)$, $\xi_r(n)$, and $\xi_r(n)$ become equal.

The definition of the function $\chi_r(n)$ is as follows:—

(vi.) $\chi_r(m)$, m being any uneven number,* denotes the sum of the r -th powers of the primary complex numbers having m as norm.

When r is a multiple of 4 (as is the case with both the χ 's used in the expression of the representations by 10 and 18 squares), we may define $\chi_r(n)$, n being any number, as one-fourth part of the sum of the r -th powers of *all* the complex numbers having m as norm.

The functions $\Omega(n)$, $W(n)$, $\Theta(n)$, $G(n)$ may be defined as coefficients by the elliptic function expansions—

$$(vii.) \quad k^2 k'^2 \rho^6 = 16 \Sigma_1^\infty \Omega(n) q^n,$$

$$(viii.) \quad k^2 k'^2 \rho^7 = 16 \Sigma_1^\infty W(n) q^n,$$

$$(ix.) \quad k^2 k'^4 \rho^8 = 16 \Sigma_1^\infty \Theta(n) q^n,$$

$$(x.) \quad k^2 k'^2 (k'^2 - k^2) \rho^9 = 16 \Sigma_1^\infty G(n) q^n,$$

where ρ denotes $2K/\pi$.

The functions $\Omega(n)$ and $\Theta(n)$ may be defined arithmetically as follows:—

If $a^2 + b^2 + c^2 + d^2$ is any representation of n as a sum of 4 squares, then

$$\Omega(n) = \frac{1}{8} \Sigma^{(r)} \{[a^4] - 2[a^2 b^2]\},$$

$$\Theta(n) = e \Sigma^{(r)} \{[a^6] - 5[a^4 b^2] + 30[a^2 b^2 c^2]\},$$

where e is $\frac{1}{8}$ or $\frac{1}{24}$ according as n is uneven or even. The square brackets signify that the sum of all the quantities of the type indicated is to be taken: *e.g.*, $[a^4]$ denotes $a^4 + b^4 + c^4 + d^4$; and $\Sigma^{(r)}$ refers to all the representations of n as a sum of 4 squares. $\Omega(m)$ and $\Theta(m)$ can also

* Throughout this paper, unless otherwise stated, n denotes any number and m any uneven number.

be defined by means of the compositions* of $4m$ as a sum of 4 uneven squares.

Representations by 2, 4, 6, and 8 Squares.

4. The first four formulæ of § 2 lead to the following theorems:—

(i.) The number of representations of any number n as a sum of 2 squares is equal to four times the excess of the number of divisors of n which are of the form $4k+1$ over the number of divisors which are of the form $4k+3$.

(ii.) The number of representations of n as a sum of 4 squares is equal to $(-1)^{n-1}8\xi_1(n)$, and is therefore equal to eight times the sum of the divisors of n , if n is uneven, and to twenty-four times the sum of the uneven divisors of n , if n is even.

(iii.) The number of representations of n as a sum of 6 squares is equal to $4\{4E'_2(n) - E_2(n)\}$.

It follows from this result that the number of representations of an uneven number m as a sum of 6 squares is equal to $12E_2(m)$, if m is of the form $4k+1$, and is equal to $-20E_2(m)$, if m is of the form $4k+3$.

(iv.) The number of representations of n as a sum of 8 squares is equal to $(-1)^{n-1}16\xi_3(n)$, and is therefore equal to sixteen times the sum of the cubes of the divisors of n , if n is uneven, and to sixteen times the excess of the sum of the cubes of the even divisors over the sum of the cubes of the uneven divisors, if n is even.

The first of these theorems is a well known result due originally to Jacobi. The other three were given by Jacobi or Eisenstein in the case of n uneven, and by H. J. S. Smith in their generality.

Representations by 10 Squares.

5. The formula (v.) of § 2 shows that the number of representations of n as a sum of 10 squares is equal to $\frac{4}{5}\{E_4(n) + 16E'_4(n) + 8\chi_4(n)\}$. Now $\chi_4(n)$ vanishes when n is of the form $4k+3$, and, more generally, whenever n is divisible by a factor of the form $4k+3$, which does not occur raised to an even power.

* Each method of expressing n as a sum of squares, when the order of the squares is taken into account, is called a composition of n . If, in addition, account is taken of the signs of the roots of the squares, so that a^2 may be either $(+a)^2$ or $(-a)^2$ but 0^2 has only one root, then each arrangement is called a representation of n (*Quarterly Journal*, Vol. xxxvi., p. 305). Thus the number of representations of a number as a sum of r uneven squares is equal to $2^r \times$ the the number of compositions.

Taking the case of n uneven, the formula shows that, if m is of the form $4k+1$, the number of representations of m is $\frac{4}{5} \{17E_4(m)+8\chi_4(m)\}$, and that, if m is of the form $4k+3$, the number of representations is $-12E_4(m)$.

The second portion of this theorem (which relates to the case when m is of the form $4k+3$) is due to Eisenstein, who stated that he had found (by arithmetical methods) that "the number of representations of a number of the form $4n+3$ by 10 squares is equal to twelve times the difference between the sum of the fourth powers of those factors which have the form $4n+3$ and the sum of the fourth powers of those which have the form $4n+1$; but," he adds, "for numbers of the form $4n+1$ no similar theorem exists."*

The formula $\frac{4}{5} \{17E_4(m)+8\chi_4(m)\}$ shows that, for numbers of the form $4k+1$, the number of representations depends also upon the sum of the fourth powers of the complex divisors of m which have m as norm. For some values of m of the form $4k+1$, the number of representations depends upon real divisors only, viz., when m is divisible by a factor of the form $4k+3$ which is not raised to an even power as, *e.g.*, for

$$m = 21, 33, 57, 69, 77, 93, \dots$$

The number of representations of an even number will depend only upon real divisors of n whenever the largest uneven divisor of n is of the form $4k+3$, or, more generally, whenever n is divisible by any factor of the form $4k+3$ which is not raised to an even power.

It seems to me a very interesting result that the number of representations of n by 10 squares should depend upon the complex divisors of n , which, regarded as arithmetical quantities, are nearly as fundamental and as easy to determine as the real divisors.

6. It was not till after the completion of the series of papers (and after this notice of them was written) that, on looking through Liouville's writings, I found that he had given in 1866† a formula for the number of representations of any number n as a sum of 10 squares. His result is that, if $n = 2^a m$, m being uneven, the number of representations is

$$\frac{4}{5} \{16^{a+1} + (-1)^{\frac{1}{2}(m-1)}\} \lambda + \frac{8}{5} n^2 \mu - \frac{64}{5} \nu,$$

where λ is the excess (taken positively) of the sum of the fourth powers of the divisors of n of the form $4k+1$ over the sum of the fourth powers

* *Crelle's Journal*, Vol. xxxv., p. 135.

† *Liouville's Journal*, Ser. 2, Vol. xi., pp. 1-8.

of the divisors of n of the form $4k+3$; μ is the number of solutions of the equation $n = s^2 + s'^2$, and ν is the sum of the products $s^2 s'^2$ for all the solutions.

It is easy to identify the first two terms with $\frac{4}{5} \{E_4(n) + 16E'_4(n)\}$ and the last two with $\frac{3 \cdot 2}{5} \chi_4(n)$. Liouville writes these last two terms also in the form $\frac{1 \cdot 6}{5} \Sigma (s^4 - 3s^2 s'^2)$, but he does not appear to have noticed the connection of this quantity with the complex divisors of n which have n as norm.

I found also that Liouville had given, in 1864, an expression for the number of representations of an even number as a sum of 12 squares. (See § 8.)

Representations by 12 Squares.

7. From the formula (vi.) of § 2 we see that the number of representations of an even number $2n$ as a sum of 12 squares is equal to $-8\hat{\xi}_5(2n)$, that is, to $8\Sigma(-1)^{d+d'-1}d^5$, where d is any divisor of $2n$ and $dd' = 2n$. We thus have the remarkable result that the number of representations of an even number as a sum of 12 squares can be expressed by means of real divisors only.

In the case of an uneven number m , the number of representations of m as a sum of 12 squares is equal to $8\{\Delta_5(m) + 2\Omega(m)\}$. The function $\Omega(m)$ does not vanish for any form of m , so that, unless $\Omega(m)$ for some form of m is dependent upon real divisors of m , it would seem that the number of representations of an uneven number m as a sum of 12 squares cannot for any form of m be expressed by real divisors only.

8. Liouville's formula, referred to in § 6, for the number of representations of an even number $2^a m$ as a sum of 12 squares is

$$\frac{2 \cdot 4}{3 \cdot 1} (21 + 2^{5a+1} 5) \Delta_5(m),^*$$

which can be readily identified with $-8\hat{\xi}_5(2n)$ by means of the general formula that, if $n = 2^s m$,

$$\hat{\xi}_t(n) = - \frac{2^{ts+t} - 2^{ts+1} + 2^{t+1} - 1}{2^t - 1} \Delta_t(n).^\dagger$$

* *Liouville's Journal*, Ser. 2, Vol. IX., p. 296. Liouville uses the notation $\xi_5(m)$ for what I have denoted by $\Delta_5(m)$.

† *Messenger of Mathematics*, Vol. XVIII., p. 47. In the statement of arithmetical formulæ relating to an even number n , $= 2^s m$, I have always endeavoured to avoid the use of the exponent s , expressing the result by means of divisors of n .

Representations by 14, 16, 18 Squares.

9. The formulæ (vii.), (viii.), (ix.) of § 2 give the numbers of representations of any number n as a sum of 14, 16, and 18 squares. I do not find that the functions $W(n)$, $\Theta(n)$, $G(n)$ vanish for any form of n , so that (unless for some form of n any of these functions depend upon real divisors only) it would seem that for no form of n can the number of representations of n by 14, 16, or 18 squares be expressed by means of real divisors only.

Compositions by Uneven Squares.

10. When the squares are all uneven we may conveniently use compositions instead of representations. Denoting by $C_r(n)$ the number of compositions of n as a sum of r uneven squares, the formulæ for $C_r(n)$, for $r = 2, 4, \dots, 18$, are

$$C_2(8n+2) = E_0(4n+1),$$

$$C_4(8n+4) = \Delta(2n+1),$$

$$C_6(8n+6) = -\frac{1}{8}E_2(4n+3),$$

$$C_8(8n) = \Delta'_8(n),$$

$$C_{10}(8n+2) = \frac{1}{640} \{E_4(4n+1) - \chi_4(4n+1)\},$$

$$C_{12}(8n+4) = \frac{1}{256} \{\Delta_5(2n+1) - \Omega(2n+1)\},$$

$$C_{14}(8n+6) = \frac{1}{12480} \{-E_6(4n+3) + 91W(4n+3)\},$$

$$C_{16}(8n) = \frac{1}{136} \{\Delta'_7(n) - \Theta(n)\},$$

$$C_{18}(8n+2) = \frac{1}{2^{15} \cdot 18005} \{13E_8(4n+1) + 3047\chi_8(4n+1) - 3060G(4n+1)\}.$$

In all these formulæ the argument is of the most general form possible, as a sum of r uneven squares must be of the form $8k+r$.

The function $\Delta'_r(n)$ denotes the sum of the r -th powers of those divisors of n whose conjugates are uneven.

The formulæ in the case of 2 and 4 squares are due to Jacobi, and those for 6 and 8 squares were given by H. J. S. Smith. The others I believe to be new.

We may pass from compositions to representations by multiplying by 2^r .

Classification of the Representations by 2, 4, 6, and 8 Squares.

11. I have obtained expressions for the numbers of representations of any number n by r uneven and $t-r$ even squares for $r = 0, 1, 2, 3, \dots, t$

and $t = 2, 4, 6, 8, 10, 12$, and also for some values of r , when $t = 14, 16, 18$.

These results would occupy too much space if quoted in their entirety,* and so I confine myself here to the cases $t = 2, 4, 6, 8$.

Denoting by $R_{r,s}(n)$ the number of representations of n as a sum of r uneven and s even squares, the formulæ are

$$R_{0,2}(4n) = 4E_0(n),$$

$$R_{1,1}(4n+1) = 4E_0(4n+1),$$

$$R_{2,0}(8n+2) = 4E_0(4n+1);$$

$$R_{0,4}(4n) = (-1)^{n-1} 8\hat{\zeta}_1(n),$$

$$R_{1,3}(4n+1) = 8\Delta(4n+1),$$

$$R_{2,2}(4n+2) = 24\Delta(4n+1),$$

$$R_{3,1}(4n+3) = 8\Delta(4n+3),$$

$$R_{4,0}(8n+4) = 16\Delta(2n+1);$$

$$R_{0,6}(4n) = 4 \{ 4E'_2(n) - E_2(n) \},$$

$$R_{1,5}(4n+1) = 6 \{ E_2(4n+1) + \chi_2(4n+1) \},$$

$$R_{2,4}(4n+2) = (-1)^n 60E_2(2n+1),$$

$$R_{3,3}(4n+3) = -20E_2(4n+3),$$

$$R_{4,2}(4n) = 240E'_2(n),$$

$$R_{5,1}(4n+1) = 6 \{ E_2(4n+1) - \chi_2(4n+1) \},$$

$$R_{6,0}(8n+6) = -8E_2(4n+3);$$

$$R_{0,8}(4n) = (-1)^{n-1} 16\hat{\zeta}_3(n),$$

$$R_{1,7}(4n+1) = 2 \{ \Delta_3(2n+1) + 7P(4n+1) \},$$

$$R_{2,6}(4n+2) = 56 \{ \Delta_3(2n+1) + P(4n+1) \},$$

$$R_{3,5}(4n+3) = 14 \{ \Delta_3(4n+3) - P(4n+3) \},$$

$$R_{4,4}(4n) = 1120\Delta'_3(n),$$

$$R_{5,3}(4n+1) = 14 \{ \Delta_3(4n+1) - P(4n+1) \},$$

$$R_{6,2}(4n+2) = 56 \{ \Delta_3(2n+1) - P(2n+1) \},$$

$$R_{7,1}(4n+3) = 2 \{ \Delta_3(4n+3) + 7P(4n+3) \},$$

$$R_{8,0}(8n) = 256\Delta'_3(n).$$

* The complete lists up to $t = 16$ are given in the *Quarterly Journal*, Vol. xxxviii., pp. 206-201.

In all the formulæ the argument on the left-hand side is the most general linear form that is capable of being represented by the quadratic form indicated by the suffixes, *e.g.*, if a number can be represented as the sum of 2 uneven and 6 even squares it must be of the form $4n+2$.

The expression of these results has required a new function $\chi_2(m)$ in the case of 6 squares and a new function $P(m)$ in the case of 8 squares. The function $\chi_2(m)$ is included among those defined in § 3, being the sum of the squares of the primary numbers having m for norm. The function $P(m)$ may be defined by means of the representations of m as a sum of 4 squares, or of $4m$ as a sum of 4 uneven squares.

The results in the case of 2 and 4 squares may be regarded as known: the other classified results are, I believe, new.

It is evident that $R_{0,s}(4n)$, the number of representations of $4n$ as a sum of s even squares, is necessarily equal to $R^{(s)}(n)$, the total number of representations of n , and that $R_{r,0}(n) = 2^r C_r(n)$.

Classification of the Representations by 10, 12, 14, 16, 18 Squares.

12. In the case of 10 squares, the functions $E_4(n)$, $E'_4(n)$, $\chi_4(n)$ suffice to express the values of

$$R_{0,10}(4n), \quad R_{2,8}(4n+2), \quad R_{4,6}(4n), \quad R_{5,5}(4n+1), \quad R_{6,4}(4n+2), \\ R_{8,2}(4n), \quad R_{10,0}(8n+2),$$

but a new function $Q(m)$ is required for

$$R_{1,9}(4n+1), \quad R_{3,7}(4n+3), \quad R_{7,3}(4n+3), \quad R_{9,1}(4n+1).$$

The function $Q(m)$ may be defined by means of the representations of m as a sum of 4 squares or of $4m$ as a sum of 4 uneven squares.

In the case of 12 squares the values of $R_{r,12-r}(n)$ for even values of r can be expressed by means of $\xi_5(n)$, $\Delta_5(m)$, $\Delta'_5(n)$, and $\Omega(m)$, but for the expression of $R_{r,12-r}(n)$ when r is uneven, a new function $\gamma(m)$ is required.

In the cases of 14, 16, and 18 squares, I have given the classified results only so far as they can be expressed without the introduction of additional functions, *i.e.*, the values given are those of $R_{2r,14-2r}(n)$ and $R_{7,7}(n)$, $R_{4r,16-4r}(n)$, $R_{2r,18-2r}(n)$ and $R_{9,9}(n)$ for all integral values of r .

General Remarks upon the Investigation.

13. The most interesting of the results obtained relate to 10 and 12 squares. So long ago as 1884 I had obtained the formulæ for the numbers

of representations of any even and uneven number as a sum of 12 squares, but I deferred the publication of the results in hopes of proving the relation $\Omega(m_1 m_2) = \Omega(m_1) \Omega(m_2)$, m_1 and m_2 being prime to one another, which I had observed to be satisfied by the values of $\Omega(m)$ that I had calculated. I resumed the question in 1889, and then obtained the classified results for 2, 4, 6, and 8 squares, using the functions $\chi_2(m)$ and $P(m)$, but I was still unsuccessful in proving the relation $\Omega(m_1 m_2) = \Omega(m_1) \Omega(m_2)$. The whole subject was then laid aside until the latter part of 1904, when I treated it more systematically, the results of the investigations which I then began being contained in the papers whose titles are given in the note to § 1. It was in 1905 that I worked out the formulæ for 10 squares, and was surprised to find that the number of representations depended upon the sum of the fourth powers of complex divisors. At that time I was not aware that the case of 10 squares had been considered by Eisenstein, and that he had found that the number of representations of a number of the form $4k+3$ could be expressed by real divisors only. The extension of the investigation to 14, 16, and 18 squares produced interesting results, but to some extent they were disappointing, as nothing so striking or unexpected was found as in the case of 10 or 12 squares.

14. The principal difficulty in the investigations was the selection of the new functions to be introduced. It is obvious that the number of representations in any given case could be expressed by means of one or more new functions defined as coefficients and introduced for the purpose. It also appeared on investigation that the number of new functions which it was absolutely necessary to introduce was smaller than might have seemed likely. But my object was not to find the fewest functions which would suffice to express the numbers of representations, but to select suitable functions which had an arithmetical significance. The functions from which the choice had to be made were given in the first instance as coefficients, and this kind of definition throws no light whatever upon the arithmetical properties of the function (except that it might indicate classes of numbers for which the function vanished). The only method available to me for determining the arithmetical nature of a function was to calculate its numerical values and so find what relations it seemed to satisfy. For the calculation of the functions it was necessary to investigate suitable recurring formulæ; such formulæ affording the only convenient method of obtaining the numerical values of a function defined as a coefficient.

15. In all cases I have selected functions which were found (by trial) to satisfy the relation $\phi(m_1 m_2) = \phi(m_1) \phi(m_2)$, m_1 and m_2 being uneven numbers prime to each other, or a similar relation of slightly different form.

This relation is satisfied by

$$P(m), \quad \Omega(m), \quad \gamma(m), \quad \Theta(m),$$

and, except when m_1 and m_2 are both of the form $4k+3$, also by $Q(m)$, $W(m)$, and $G(m)$.*

The property $\phi(m_1 m_2) = \phi(m_1) \phi(m_2)$ would seem to indicate a product structure of some kind for the function $\phi(m)$. In the paper numbered (ii.) in the note to § 1, an imperfect proof† of this property is given in the case of the functions $P(m)$, $Q(m)$, $\Omega(m)$; and in the paper (4) expressions are obtained for these three functions as sums of the second, third, and fourth powers of certain complex numbers derived from the representations of m as a sum of 4 squares.

When I first found that $\Omega(m)$ satisfied the property

$$\Omega(m_1 m_2) = \Omega(m_1) \Omega(m_2),$$

it seemed to me a very special property of the function, but, as has just been mentioned, I have since found that it is shared by many other functions which, like $\Omega(m)$, have no apparent factorial structure.

16. In the series of papers to which this notice relates I have endeavoured to elicit the principal information connected with the numbers of representations of a number as a sum of even and uneven squares (up to 18) that can be derived from the formulæ of elliptic functions. The only complete and effective method of research in such investigations is afforded by the processes of the theory of numbers, and any method dependent upon elliptic functions or other algebraical processes is necessarily partial and inadequate. Still in spite of this incompleteness it seemed to me to be useful to investigate with some completeness the results afforded by elliptic functions, thereby at the same time

* If m_1 and m_2 are both of the form $4k+3$,

$$Q(m_1 m_2) = -3Q(m_1)Q(m_2), \quad W(m_1 m_2) = -15W(m_1)W(m_2), \quad G(m_1 m_2) = -\frac{3}{2}G(m_1)G(m_2).$$

† The proof is imperfect because it depends upon an assumption which I was unable to prove, although I think there can be no doubt of its truth. The "proof" at all events affords a reason for the existence of the property.

ascertaining the limitations of this method of research. It seems fairly safe to conclude that when algebraical methods do yield results in the theory of numbers, they are so obtained in the simplest manner: and it is quite likely that the reduction to these forms from the state in which they are left by the theory of numbers might not be easy. It can also happen that results which are expressed in their generality by delicate and complicated rules may contain no indication of the cases in which they admit of simplification.

INDEX.

INDEX OF PROCEEDINGS.

[The numbers refer to pages.]

- Admissions of Members, iv, vi, viii.
- Annual General Meeting (November, 1906), iv.
- Auditor's Report, v.
- Council and Officers for the Session 1906-1907, iv.
- Elections of Members, v, vi, vii, viii.
- Exchanges of Publications, iv, x.
- Honorary Members, ix.
- International Catalogue, xii.
- Number of Members, iv.
- Presents to Library, xi.
- Treasurer's Report, iv, v.

INDEX OF PAPERS READ AND COMMUNICATIONS MADE
DURING THE SESSION NOVEMBER, 1906—JUNE, 1907.

[The names of Authors are given and the numbers of the pages on which the titles of the papers, &c.,
will be found.]

- | | |
|-----------------------------|----------------------------------|
| Baker, H. F., ix. | Hilton, H., v, viii. |
| Barnes, E. W., v, vi. | Hobson, E. W., v, vi, vii, viii. |
| Basset, A. B., ix. | Jackson, C. S., v. |
| Bateman, H., v, viii, viii. | Lamb, H., viii. |
| Brill, J., vii. | Littlewood, J. E., vi. |
| Bromwich, T. J. I'A., ix. | Love, A. E. H., vii. |
| Burnside, W., ix. | MacMahon, P. A., v. |
| Campbell, J. E., v. | Mercer, J., v. |
| Cunningham, A., vi, vii. | Miller, G. A., vii. |
| Dickson, L. E., vii. | Richardson, L. F., v. |
| Dixon, A. C., v. | Stott, Mrs., vi. |
| Elliott, E. B., vii, ix. | Stuart, T., vi, vii, viii. |
| Evans-Cross, G. W., vii. | Watson, G. N., v. |
| Forsyth, A. R., vi, ix. | Western, A. E., ix. |
| Glaisher, J. W. L., ix. | Whipple, J. F. W., viii. |
| Hardy, G. H., vi, vi. | Wood, P. W., vi. |
| Hill, M. J. M., vii. | |

INDEX OF PAPERS PUBLISHED IN THIS VOLUME ARRANGED ACCORDING TO NAMES OF AUTHORS.

[The numbers refer to pages.]

- Barnes, E. W. The Asymptotic Expansion of Integral Functions defined by Generalized Hypergeometric Series, 59.
- Campbell, J. E. On Bäcklund's Transformation and the Partial Differential Equation $s = F(x, y, z)$, 6.
- Cunningham, A. On Hyper-even Numbers and on Fermat's Numbers, 237.
- Darboux, G. [Obituary Notice of A. Mannheim], xiii.
- Dickson, L. E. Invariants of the General Quadratic Form Modulo 2, 301.
- Dixon, A. C. Harmonic Expansions of Functions of Two Variables, 411.
- Elliott, E. B. On the Projective Geometry of a Binary Quartic and its Hessian, 290.
- Forsyth, A. R. Partial Differential Equations of the Second Order having Integral Systems free from Partial Quadratures, 117.
- [Obituary Notice of E. J. Routh], xiv.
- Glaisher, J. W. L. On the Numbers of Representations of a Number as a Sum of $2r$ Squares, where $2r$ does not exceed Eighteen, 479.
- Hardy, G. H. On the Singularities of Functions defined by Taylor's Series (Remarks in Addition to a Former Paper), 197.
- The Singular Points of certain Classes of Functions of Several Variables, 342.
- Hill, M. J. M. On a Formula for the Sum of a Finite Number of Terms of the Hypergeometric Series when the Fourth Element is Equal to Unity, 335.
- Hilton, H. On Sub-Groups of a Finite Abelian Group, 1.
- [Correction], xxi.
- Hobson, E. W. On Partial Differential Coefficients and on Repeated Limits in General, 225.
- On the Uniform Convergence of Fourier's Series, 275.
- On Repeated Integrals, 325.
- Littlewood, J. E. On the Asymptotic Approximation to Integral Functions of Zero Order, 361.
- MacMahon, P. A. The Diophantine Equation $x^n - Ny^n = z$, 45.
- Mercer, J. On the Limits of Real Variants, 206.
- Wood, P. W. On the Reducibility of Covariants of Binary Quantics of Infinite Order, 177.

INDEX ACCORDING TO SUBJECTS.

[The numbers refer to pages.]

- Asymptotic Expansions and Approximations.
See Functions, Integral.
- Binary Forms: Wood, P. W., 177; Elliott, E. B., 290.
- Biography. Obituary Notice of A. Mannheim, xiii; of E. J. Routh, xiv.
- Conics, Projective Properties of: Elliott, E. B., 290.
- Differential Calculus, Principles of the: Hobson, E. W., 225.
- Equations, Indeterminate: MacMahon, P. A., 45.
- Fourier's Series: Hobson, E. W., 275.
- Functions, Hypergeometric: Barnes, E. W., xxi, 59; Hill, M. J. M., xxi, 335.
- Functions, Integral: Barnes, E. W., 59; Littlewood, J. E., 361.
- Functions, Multiform: Hardy, G. H., 197.
- Functions of Real Variables: Hobson, E. W., 225, 275, 325.
- Functions of several Variables: Hardy, G. H., 342; Dixon, A. C., 411.
- Groups, Abelian: Hilton, H., 1.
- Harmonic Analysis: Dixon, A. C., 411.
- Hyper-Even Numbers. *See* Numbers, Theory of.
- Inequalities, Diophantine. *See* Equations, Indeterminate.
- Integral Calculus, Principles of the: Hobson, E. W., 325.
- Integrals, Multiple: Hobson, E. W., 325.
- Numbers, Theory of. Applications of Elliptic Functions to Arithmetic: Glaisher, J. W. L., 479. Congruences of Order Higher than the First: Cunningham, A., 237. Quadratic Forms: Dickson, L. E., 301. Special Numerical Functions: Glaisher, J. W. L., 479.
- Partial Differential Equations, Methods of Solution of: Campbell, J. E., 6; Forsyth, A. R., 117.
- Representation of a Number as a Sum of Squares. *See* Numbers, Theory of.
- Residuacity. *See* Numbers, Theory of.
- Sequences, Infinite: Mercer, J., 206.
- Series, Finite Summation of: Hill, M. J. M., 335.
- Series, Theory of. Barnes, E. W., 59; Hobson, E. W., 275; Hardy, G. H., 342; Littlewood, J. E., 361.
- Surfaces, Pseudospherical: Campbell, J. E., 6.
- Transformations of Differential Forms: Campbell, J. E., 6.
- Variants. *See* Sequences.



